



A VARIATION ON LACUNARY STATISTICAL QUASI CAUCHY SEQUENCES

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ABSTRACT. In this paper, the concept of a lacunary statistically δ -quasi-Cauchy sequence is investigated. In this investigation, we proved interesting theorems related to lacunary statistically δ -ward continuity, and some other kinds of continuities. A real valued function f defined on a subset A of \mathbb{R} , the set of real numbers, is called lacunary statistically δ ward continuous on A if it preserves lacunary statistically delta quasi-Cauchy sequences of points in A , i.e. $(f(\alpha_k))$ is a lacunary statistically delta quasi-Cauchy sequence whenever (α_k) is a lacunary statistically delta quasi-Cauchy sequence of points in A , where a sequence (α_k) is called lacunary statistically delta quasi-Cauchy if $(\Delta\alpha_k)$ is a lacunary statistically quasi-Cauchy sequence. It turns out that the set of lacunary statistically δ ward continuous functions is a closed subset of the set of continuous functions.

1. INTRODUCTION

The concept of continuity and any concept involving continuity play a very important role not only in pure mathematics but also in other branches of sciences involving mathematics especially in computer science, information theory, economics, and biological science.

Buck [2] introduced Cesaro continuity in 1946. Thereafter, Antoni [1] has studied A -continuity defined by a regular summability matrix A . Öztürk [28] has studied A -continuity for methods of almost convergence or for related methods. Connor and Grosse-Erdman [20] have given sequential definitions of continuity for real functions calling G -continuity instead of A -continuity by means of a sequential method, or a method of sequential convergence, and their results cover the earlier works related to A -continuity where a method of sequential convergence, or briefly a method, is a linear function G defined on a linear subspace of all sequences of points in \mathbb{R} denoted by c_G , into \mathbb{R} . A sequence $\alpha = (\alpha_n)$ is said to be G -convergent to ℓ if $\alpha \in c_G$, then $G(\alpha) = \ell$. In particular, \lim denotes the limit function

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$\lim \alpha = \lim_n \alpha_n$ on the linear space c . On the other hand, Çakalli has introduced a generalization of compactness, a generalization of connectedness via a method of sequential convergence in [5] and [11], respectively.

In recent years, using the same idea, many kinds of continuities were introduced and investigated, not all but some of them we state in the following: slowly oscillating continuity [6], ward continuity [7], δ -ward continuity [8], statistical ward continuity [9], lacunary statistical ward continuity [13], and ρ -statistically ward continuity [12]. Investigation of some of these kinds of continuities lead some authors to find conditions on the domain of a function for some characterizations of uniform continuity of a real function in terms of sequences in the above manner (see [29, Theorem 8], [7, Theorem 7], and [3, Theorem 1]).

A sequence (α_k) of points in \mathbb{R} , the set of real numbers, is called statistically convergent, or *st*-convergent to L , if $\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |\alpha_k - L| \geq \varepsilon\}| = 0$ for every positive real number ε . This is denoted by $st\text{-}lim \alpha_k = L$ (see [21], [22], [26], [17], and [27]). (α_k) is statistically quasi-Cauchy, or *st*-quasi-Cauchy if $(\Delta\alpha_k)$ is a *st*-null sequence, where $\Delta\alpha_k = \alpha_{k+1} - \alpha_k$ for each integer n in \mathbb{N} , the set of positive integers ([9]).

In [23] Fridy and Orhan introduced the concept of lacunary statistically convergence in the sense that a sequence (α_k) of points in \mathbb{R} is called lacunary statistically convergent, or S_θ -convergent, to an element L of \mathbb{R} if $\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |\alpha_k - L| \geq \varepsilon\}| = 0$ for every positive real number ε where $I_r = (k_{r-1}, k_r]$ and $k_0 = 0$, $h_r : k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$ and $\theta = (k_r)$ is an increasing sequence of positive integers (see also [24], and [4]). In this case we write $S_\theta\text{-}lim \alpha_k = L$. The set of lacunary statistically convergent sequences of points in \mathbb{R} is denoted by S_θ . In the sequel, we will always assume that $\lim inf_r q_r > 1$. A sequence (α_k) of points in \mathbb{R} is called to be lacunary statistically quasi-Cauchy if $S_\theta\text{-}lim \Delta\alpha_k = 0$, where $\Delta\alpha_k = \alpha_{k+1} - \alpha_k$ for each positive integer k . The set of lacunary statistically quasi-Cauchy sequences will be denoted by ΔS_θ .

The sequence of Fibonacci numbers has a quite nice property when it is considered as a lacunary sequence. Lacunary sequential method obtained by the sequence of Fibonacci numbers is a regular method, i.e. $\theta = (k_r)$ is the lacunary sequence defined by writing $k_0 = 0$ and $k_r = F_{r+2}$ where (F_r) is the Fibonacci sequence, i.e. $F_1 = 1$, $F_2 = 1$, $F_r = F_{r-1} + F_{r-2}$ for $r \geq 3$. For this lacunary sequence $\theta = (k_r)$, a real valued function defined on a subset of \mathbb{R} is lacunary statistically sequentially continuous if and only if it is ordinary sequentially continuous (see [13]). where a function defined on a subset A of \mathbb{R} is called lacunary statistically continuous or S_θ continuous if it preserves S_θ convergent sequences of points in A , i.e. $(f(\alpha_k))$ is S_θ convergent whenever (α_k) is an S_θ convergent sequence of points in A . Furthermore, a function defined on a subset A of \mathbb{R} is lacunary statistically continuous if and only if it is ordinary continuous. A function defined on a subset A of \mathbb{R} is called lacunary statistically ward continuous or S_θ -ward continuous if it

preserves S_θ -quasi-Cauchy sequences of points in A , i.e. $(f(\alpha_k))$ is S_θ -quasi-Cauchy whenever (α_k) is an S_θ -quasi-Cauchy sequence of points in A (see [13]).

The purpose of this paper is to investigate the notion of lacunary statistically δ ward continuity and prove interesting theorems.

2. LACUNARY STATISTICALLY δ -WARD CONTINUITY

Replacing $(\Delta(\alpha_k))$ with $\Delta^2\alpha_k$ in the definition of an S_θ -quasi-Cauchy sequence we have the following definition.

Definition 1. A sequence (α_k) of points in \mathbb{R} is called lacunary statistically δ quasi Cauchy, or S_θ - δ quasi Cauchy if the sequence $(\Delta\alpha_k)$ is a lacunary statistically quasi-Cauchy sequence, i.e.

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |\Delta^2\alpha_k| \geq \varepsilon\}| = 0$$

for every positive real number ε where $\Delta^2\alpha_k = \alpha_{k+2} - 2\alpha_{k+1} + \alpha_k$ for each positive integer k .

Now we give some interesting examples that show importance of the interest.

Example 1. Let n be a positive integer. In a group of n people, each person selects at random and simultaneously another person of the group. All of the selected persons are then removed from the group, leaving a random number $n_1 < n$ of people which form a new group. The new group then repeats independently the selection and removal thus described, leaving $n_2 < n_1$ persons, and so forth until either one person remains, or no persons remain. Denote by p_n the probability that, at the end of this iteration initiated with a group of n persons, one person remains. Then the sequence $\mathbf{p} = (p_1, p_2, \dots, p_n, \dots)$ is a lacunary statistically delta quasi-Cauchy sequence (see also [30]).

Example 2. In a group of k people, k is a positive integer, each person selects independently and at random one of three subgroups to which to belong, resulting in three groups with random numbers k_1, k_2, k_3 of members; $k_1 + k_2 + k_3 = k$. Each of the subgroups is then partitioned independently in the same manner to form three sub subgroups, and so forth. Subgroups having no members or having only one member are removed from the process. Denote by t_k the expected value of the number of iterations up to complete removal, starting initially with a group of k people. Then the sequence $(t_1, \frac{t_2}{2}, \frac{t_3}{3}, \dots, \frac{t_n}{n}, \dots)$ is a bounded non-convergent lacunary statistically delta quasi-Cauchy sequence (see also [25]).

Now we introduce the definition of lacunary statistically δ ward continuity in the following.

Definition 2. A real valued function f defined on a subset A of \mathbb{R} is called lacunary statistically δ ward continuous, or S_θ - δ ward continuous on A if it preserves lacunary statistically delta quasi-Cauchy sequences of points in A , i.e. $(\Delta f(\alpha_k))$

is a lacunary statistically quasi-Cauchy sequence whenever $(\Delta\alpha_k)$ is a lacunary statistically quasi-Cauchy sequence of points in A .

The set of lacunary statistically δ ward continuous functions on A will be denoted by $\Delta^2 S_\theta(A)$. We note that this definition of continuity cannot be obtained by any A -continuity, i.e., by any summability matrix A , even by the summability matrix $A = (a_{nk})$ defined by $a_{nk} = \frac{1}{h_r}$ for $k = n + 2$, $a_{nk} = -2\frac{1}{h_r}$ for $k = n + 1$, and $a_{nk} = \frac{1}{h_r}$ for $k = n$ and $a_{nk} = 0$ otherwise.

We see in the following that the sum of two lacunary statistically δ ward continuous functions is lacunary statistically δ ward continuous:

Proposition 1. *If $f, g \in \Delta^2 S_\theta(A)$, then $f + g \in \Delta^2 S_\theta(A)$.*

Proof. Let f, g be lacunary statistically δ ward continuous functions on a subset A of \mathbb{R} . To prove that $f + g$ is lacunary statistically δ ward continuous on A , take any lacunary statistically delta quasi-Cauchy sequence (α_k) of points in A . Then $(f(\alpha_k))$ and $(g(\alpha_k))$ are lacunary statistically delta quasi-Cauchy sequences. Let $\varepsilon > 0$ be given. Since $(f(\alpha_k))$ and $(g(\alpha_k))$ are lacunary statistically delta quasi-Cauchy, we have

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_n : |\Delta^2(f(\alpha_k))| \geq \frac{\varepsilon}{2}\}| = 0$$

and

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_n : |\Delta^2(g(\alpha_k))| \geq \frac{\varepsilon}{2}\}| = 0.$$

Hence

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_n : |\Delta^2(f(\alpha_k) + g(\alpha_k))| \geq \varepsilon\}| = 0$$

which follows from the inclusion

$$\{k \in I_n : |\Delta^2(f+g)(\alpha_k)| \geq \varepsilon\} \subseteq \{k \in I_n : |\Delta^2 f(\alpha_k)| \geq \frac{\varepsilon}{2}\} \cup \{k \in I_n : |\Delta^2 g(\alpha_k)| \geq \frac{\varepsilon}{2}\}.$$

This completes the proof. \square

The product of a constant real number and a lacunary statistically δ ward continuous function is lacunary statistically δ ward continuous, so the set of lacunary statistically δ ward continuous functions is a vector space.

In connection with $S_\theta - \delta$ -quasi-Cauchy sequences and convergent sequences the problem arises to investigate the following types of ‘continuity’ of functions on \mathbb{R} :

- $(\delta S_\theta \delta)$: $(\alpha_n) \in \Delta S_\theta(\delta) \Rightarrow (f(\alpha_n)) \in \Delta S_\theta(\delta)$
- $(\delta S_\theta \delta c)$: $(\alpha_n) \in \Delta S_\theta(\delta) \Rightarrow (f(\alpha_n)) \in c$
- (S_θ) : $(\alpha_n) \in S_\theta \Rightarrow (f(\alpha_n)) \in S_\theta$
- (ΔS_θ) : $(\alpha_n) \in \Delta S_\theta \Rightarrow (f(\alpha_n)) \in \Delta S_\theta$
- (c) : $(\alpha_n) \in c \Rightarrow (f(\alpha_n)) \in c$
- $(c \delta S_\theta \delta)$: $(\alpha_n) \in c \Rightarrow (f(\alpha_n)) \in \Delta S_\theta(\delta)$

We see that $(\delta S_\theta \delta)$ is $S_\theta - \delta$ -ward continuity of f , (S_θ) is S_θ -sequential continuity of f , and (c) is the ordinary continuity of f . It is easy to see that $(\delta S_\theta \delta c)$ implies $(\delta S_\theta \delta)$, and $(c \delta S_\theta \delta)$ does not imply $(\delta S_\theta \delta c)$; and $(\delta S_\theta \delta)$ implies $(c \delta S_\theta \delta)$, and $(c \delta S_\theta \delta)$ does not imply $(\delta S_\theta \delta)$; $(\delta S_\theta \delta c)$ implies (c) , and (c) does not imply $(\delta S_\theta \delta c)$.

Now we give the implication $(\delta S_\theta \delta)$ implies (ΔS_θ) , i.e. any $S_\theta - \delta$ -ward continuous function is S_θ -ward continuous.

Theorem 1. *If a real valued function is lacunary statistically δ ward continuous on a subset A of \mathbb{R} , then it is lacunary statistically ward continuous on A .*

Proof. Suppose that f is a lacunary statistically δ ward continuous function on a subset A of \mathbb{R} . Let (α_n) be a lacunary statistically quasi-Cauchy sequence of points in A . Then the sequence

$$(\alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_3, \dots, \alpha_{n-1}, \alpha_{n-1}, \alpha_n, \alpha_n, \dots)$$

is a lacunary statistically quasi-Cauchy sequence, so is a lacunary statistically delta quasi-Cauchy sequence. Since f is lacunary statistically δ ward continuous, the sequence

$$(y_n) = (f(\alpha_1), f(\alpha_1), f(\alpha_2), f(\alpha_2), \dots, f(\alpha_n), f(\alpha_n), \dots)$$

is a lacunary statistically δ -quasi-Cauchy sequence. Then it is easy to see that $(f(\alpha_n))$ is also lacunary statistically quasi-Cauchy sequence. This completes the proof of the theorem. \square

Corollary 1. *If a real valued function is lacunary statistically δ ward continuous on a subset A of \mathbb{R} , then it is ordinary continuous on A .*

Proof. The proof follows immediately from the preceding theorem and [13, Corollary 2], so is omitted. \square

We note that any lacunary statistically δ ward continuous function is statistically continuous ([10]), S_θ -continuous ([13]), I -sequentially continuous for any non-trivial admissible ideal I ([14]), and G -sequentially continuous for any regular subsequential sequential method G (see [5]).

Now we prove the following theorem.

Theorem 2. *If a real valued function f is uniformly continuous on a subset A of \mathbb{R} , then $(f(\alpha_n))$ is lacunary statistically delta quasi-Cauchy whenever (α_n) is a quasi-Cauchy sequence of points in A .*

Proof. Let f be uniformly continuous on A . Take any quasi-Cauchy sequence (α_n) of points in A . Let ε be any positive real number. Since f is uniformly continuous, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. As (α_k) is a quasi-Cauchy sequence, for this δ there exists an $n_0 \in \mathbf{N}$ such that $|\alpha_{k+1} - \alpha_k| < \delta$ for $k \geq n_0$. Therefore $|f(\alpha_{n+1}) - f(\alpha_n)| < \frac{\varepsilon}{2}$ for $n \geq n_0$, so the number of indices k for which $|f(\alpha_{k+1}) - f(\alpha_k)| \geq \frac{\varepsilon}{2}$ is less than n_0 . Hence $\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |\Delta(f(\alpha_{n+1}) - f(\alpha_n))| \geq \varepsilon\}|$

$$\begin{aligned} &\leq \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f(\alpha_{n+2}) - f(\alpha_{n+1})| \geq \frac{\varepsilon}{2}\}| + \\ &+ \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f(\alpha_{n+1}) - f(\alpha_n)| \geq \frac{\varepsilon}{2}\}| \leq \\ &\leq \lim_{r \rightarrow \infty} \frac{n_0}{h_r} + \lim_{r \rightarrow \infty} \frac{n_0}{h_r} = 0 + 0 = 0. \end{aligned}$$

This completes the proof of the theorem. \square

It is a well known result that the uniform limit of a sequence of continuous functions is continuous. This is also true in case of lacunary statistically delta ward continuity, i.e. the uniform limit of a sequence of lacunary statistically δ ward continuous functions is lacunary statistically δ ward continuous.

Theorem 3. *If (f_n) is a sequence of lacunary statistically δ ward continuous functions on a subset A of \mathbb{R} and (f_n) is uniformly convergent to a function f , then f is lacunary statistically δ ward continuous on A .*

Proof. Let (α_k) be any lacunary statistically delta quasi-Cauchy sequence of points in A , and let ε be any positive real number. By the uniform convergence of (f_n) , there exists an $n_1 \in \mathbf{N}$ such that $|f(\alpha) - f_k(\alpha)| < \frac{\varepsilon}{6}$ for $n \geq n_1$ and every $\alpha \in A$. As f_{n_1} is lacunary statistically δ ward continuous on A , it follows that

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f_{n_1}(\alpha_{k+2}) - 2f_{n_1}(\alpha_{k+1}) + f_{n_1}(\alpha_k)| \geq \frac{\varepsilon}{6}\}| = 0.$$

Now

$$\begin{aligned} &\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f(\alpha_{k+2}) - 2f(\alpha_{k+1}) + f(\alpha_k)| \geq \varepsilon\}| = \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : \\ &|f(\alpha_{k+2}) - 2f(\alpha_{k+1}) + f(\alpha_k) - [f_{n_1}(\alpha_{k+2}) - 2f_{n_1}(\alpha_{k+1}) + f_{n_1}(\alpha_k)] + [f_{n_1}(\alpha_{k+2}) - \\ &2f_{n_1}(\alpha_{k+1}) + f_{n_1}(\alpha_k)]| \geq \varepsilon\}| \\ &\leq \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f(\alpha_{k+2}) - f_{n_1}(\alpha_{k+2})| \geq \frac{\varepsilon}{6}\}| + \\ &+ \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |-2f(\alpha_{k+1}) + 2f_{n_1}(\alpha_{k+1})| \geq \frac{\varepsilon}{6}\}| + \\ &+ \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f(\alpha_k) - f_{n_1}(\alpha_k)| \geq \frac{\varepsilon}{6}\}| + \\ &+ \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f_{n_1}(\alpha_{k+2}) - 2f_{n_1}(\alpha_{k+1}) + f_{n_1}(\alpha_k)| \geq \frac{\varepsilon}{6}\}| = 0 + 0 + 0 + 0 = 0. \end{aligned}$$

Thus f preserves lacunary statistically delta quasi-Cauchy sequences. This completes the proof of the theorem. \square

Theorem 4. *The set of lacunary statistically δ ward continuous functions on a subset A of \mathbb{R} is a closed subset of the set of continuous functions on A . i.e. $\overline{\Delta^2 S_\theta(A)} = \Delta^2 S_\theta(A)$ where $\overline{\Delta^2 S_\theta(A)}$ denotes the set of all cluster points of $\Delta^2 S_\theta(A)$.*

Proof. Let f be an element in $\overline{\Delta^2 S_\theta(A)}$. Then there exists a sequence (f_n) of points in $\Delta^2 S_\theta(A)$ such that $\lim_{n \rightarrow \infty} f_n = f$. To show that f is lacunary statistically δ ward continuous, consider a lacunary statistically delta quasi Cauchy-sequence (α_n) of points in A . Since (f_n) converges to f , there exists a positive integer N such that for all $x \in A$ and for all $n \geq N$, $|f_n(x) - f(x)| < \frac{\varepsilon}{6}$. As f_N is lacunary statistically δ ward continuous on A , we have that

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f_N(\alpha_{k+2}) - 2f_N(\alpha_{k+1}) + f_N(\alpha_k)| \geq \frac{\varepsilon}{6}\}| = 0.$$

Now

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f(\alpha_{k+2}) - 2f(\alpha_{k+1}) + f(\alpha_k)| \geq \varepsilon\}| &= \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : \\ |f(\alpha_{k+2}) - 2f(\alpha_{k+1}) + f(\alpha_k) - [f_N(\alpha_{k+2}) - 2f_N(\alpha_{k+1}) + f_N(\alpha_k)] + [f_N(\alpha_{k+2}) - \\ 2f_N(\alpha_{k+1}) + f_N(\alpha_k)]| \geq \varepsilon\}| \\ &\leq \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f(\alpha_{k+2}) - f_N(\alpha_{k+2})| \geq \frac{\varepsilon}{6}\}| + \\ &+ \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |-2f(\alpha_{k+1}) + 2f_N(\alpha_{k+1})| \geq \frac{\varepsilon}{6}\}| + \\ &+ \lim_{r \rightarrow \infty} |\{k \in I_r : |f(\alpha_k) - f_N(\alpha_k)| \geq \frac{\varepsilon}{6}\}| + \\ &+ \lim_{r \rightarrow \infty} |\{k \in I_r : |f_N(\alpha_{k+2}) - 2f_N(\alpha_{k+1}) + f_N(\alpha_k)| \geq \frac{\varepsilon}{6}\}| = 0 + 0 + 0 + 0 = 0. \end{aligned}$$

Thus f preserves lacunary statistically delta quasi-Cauchy sequences. This completes the proof of the theorem. \square

Corollary 2. *The set of lacunary statistically δ ward continuous functions on a subset A of \mathbb{R} is a complete subset of the set of continuous functions on A .*

Theorem 5. *The set of functions on a subset A of \mathbb{R} which map quasi Cauchy sequences to lacunary statistically δ quasi Cauchy sequences is a closed subset of the set of continuous functions on A . i.e. $\overline{Q\Delta^2 S_\theta(A)} = Q\Delta^2 S_\theta(A)$ where $\overline{Q\Delta^2 S_\theta(A)}$ denotes the set of all cluster points of $Q\Delta^2 S_\theta(A)$, and $Q\Delta^2 S_\theta(A)$ is the set of functions on A which map quasi Cauchy sequences to lacunary statistically δ quasi Cauchy sequences.*

Proof. It is easy to see that any function which maps quasi Cauchy sequences to lacunary statistically δ quasi Cauchy sequences is continuous. Let f be an element in $\overline{Q\Delta^2 S_\theta(A)}$. Then there exists a sequence (f_n) of points in $Q\Delta^2 S_\theta(A)$ such that $\lim_{n \rightarrow \infty} f_n = f$. To show that f maps quasi Cauchy sequences to lacunary statistically δ quasi Cauchy sequences, consider a quasi Cauchy-sequence (α_n) of points in A . Since (f_n) converges to f , there exists a positive integer N such that for all $x \in A$ and for all $n \geq N$, $|f_n(x) - f(x)| < \frac{\varepsilon}{6}$. Hence

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f_N(\alpha_{k+2}) - 2f_N(\alpha_{k+1}) + f_N(\alpha_k)| \geq \frac{\varepsilon}{6}\}| = 0.$$

Now

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f(\alpha_{k+2}) - 2f(\alpha_{k+1}) + f(\alpha_k)| \geq \varepsilon\}| &= \\ = \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f(\alpha_{k+2}) - 2f(\alpha_{k+1}) + f(\alpha_k) - [f_N(\alpha_{k+2}) - 2f_N(\alpha_{k+1}) + \\ f_N(\alpha_k)] + [f_N(\alpha_{k+2}) - 2f_N(\alpha_{k+1}) + f_N(\alpha_k)]| \geq \varepsilon\}| \\ &\leq \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f(\alpha_{k+2}) - f_N(\alpha_{k+2})| \geq \frac{\varepsilon}{6}\}| + \\ &+ \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |-2f(\alpha_{k+1}) + 2f_N(\alpha_{k+1})| \geq \frac{\varepsilon}{6}\}| + \\ &+ \lim_{r \rightarrow \infty} |\{k \in I_r : |f(\alpha_k) - f_N(\alpha_k)| \geq \frac{\varepsilon}{6}\}| + \\ &+ \lim_{r \rightarrow \infty} |\{k \in I_r : |f_N(\alpha_{k+2}) - 2f_N(\alpha_{k+1}) + f_N(\alpha_k)| \geq \frac{\varepsilon}{6}\}| = 0 + 0 + 0 + 0 = 0. \end{aligned}$$

Thus f maps quasi Cauchy sequences to lacunary statistically δ quasi Cauchy sequences. This completes the proof of the theorem. \square

Corollary 3. *The set of functions on a subset A of \mathbb{R} which map quasi Cauchy sequences to lacunary statistically δ quasi Cauchy sequences is a complete subset of the set of continuous functions on A .*

3. CONCLUSION

In this paper, the concept of a lacunary statistically δ -quasi-Cauchy sequence is investigated. In this investigation, we proved interesting theorems related to lacunary statistically δ -ward continuity, and some other kinds of continuities. One may expect this investigation to be a useful tool in the field of analysis in modeling various problems occurring in many areas of science, dynamical systems, computer science, information theory, and biological science. For a further study, we suggest to investigate lacunary statistically δ -quasi Cauchy sequences of fuzzy points (see [15] for the definitions and related concepts in fuzzy setting), and is to investigate lacunary statistically δ -quasi Cauchy double sequences of points (see [18] for the related concepts in the double case). Another suggestion for another further study is to introduce and give investigations of lacunary statistically δ -quasi-Cauchy sequences in topological vector space valued cone metric spaces (see [19] for basic concepts in topological vector space valued cone metric spaces). However due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work.

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