Article electronically published on January 12, 2017

Commun.Fac.Sci.Univ.Ank.Series A1
Volume 66, Number 2, Pages 80–90 (2017)
DOI: 10.1501/Commua1_0000000803
ISSN 1303–5991

http://communications.science.ankara.edu.tr/index.php?series=A1

COMMUNICATIONS SERIES A1

SOME CESÀRO-TYPE SUMMABILITY SPACES DEFINED BY A MODULUS FUNCTION OF ORDER (α, β)

HACER ŞENGÜL

Abstract. In this article, we introduce strong $w[\theta, f, p]$ −summability of order (α, β) for sequences of complex (or real) numbers and give some inclusion relations between the sets of lacunary statistical convergence of order (α, β), strong $w_{\alpha}^{\beta}[\theta, f, p]$ −summability and strong $w_{\alpha}^{\beta}(p)$ −summability.

1. Introduction

In 1951, Steinhaus [15] and Fast [9] introduced the concept of statistical convergence and later in 1959, Schoenberg [13] reintroduced independently. Caserta et al. [2], Çakallı [3], Connor [8], Çolak [7], Et [4], Fridy [10], Gadjiev and Orhan [5], Kolk [6], Salat [14] and many others investigated some arguments related to this notion.

Çolak [7] studied statistical convergence order α by giving the definition as follows:

We say that the sequence $x = (x_k)$ is statistically convergent of order α to ℓ if there is a complex number ℓ such that

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{k \le n : |x_k - \ell| \ge \varepsilon\}| = 0.$$

Let $0 < \alpha \le \beta \le 1$. We define the (α, β)−density of the subset E of $\mathbb{N}$ by

$$\delta_{\alpha}^{\beta}(E) = \lim_{n} \frac{1}{n^{\alpha}} |\{k \le n : k \in E\}|^{\beta}$$

provided the limit exists (finite or infinite), where $|\{k \le n : k \in E\}|^{\beta}$ denotes the βth power of number of elements of E not exceeding n.

If a sequence $x = (x_k)$ satisfies property $P(k)$ for all k except a set of (α, β)−density zero, then we say that x_k satisfies $P(k)$ for "almost all k according to β" and we abbreviate this by "$a.a.k\ (\alpha, \beta)$".

Received by the editors: August 04, 2016; Accepted: November 01, 2016.
2010 *Mathematics Subject Classification.* Primary 40A05, 40C05; Secondary 46A45.
Key words and phrases. Lacunary sequence, modulus function, statistical convergence.

Throughout this paper w indicate the space of sequences of real number.

Let $0 < \beta \leq 1, 0 < \alpha \leq 1$, $\alpha \leq \beta$ and $x = (x_k) \in w$. The sequence $x = (x_k)$ is said to be statistically convergent of order (α, β) if there is a complex number L such that

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right|^{\beta} = 0$$

i.e. for $a.a.k(\alpha, \beta)$ $|x_k - L| < \varepsilon$ for every $\varepsilon > 0$, in that case a sequence x is said to be statistically convergent of order (α, β), to L. This convergence is indicated by $S_{\alpha}^{\beta} - \lim x_k = L$ ([16]).

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$ and $\alpha \in (0, 1]$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r. Lacunary sequence spaces were studied in ([11], [12], [17], [18]).

First of all, the notion of a modulus was given by Nakano [20]. Maddox [25] and Ruckle [28] used a modulus function to construct some sequence spaces. Afterwards different sequence spaces defined by modulus have been studied by Altın [1], Et ([26], [27]) , Gaur and Mursaleen [21], Işık [23], Nuray and Savaş [22], Pehlivan and Fisher [29] and everybody else.

We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

i) $f(x) = 0$ if and only if $x = 0$,

ii) $f(x + y) \leq f(x) + f(y)$ for $x, y \geq 0$,

iii) f is increasing,

iv) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$.

The following inequality will be used frequently throughout the paper:

$$|a_k + b_k|^{p_k} \leq D \left(|a_k|^{p_k} + |b_k|^{p_k}\right) \tag{1}$$

where $a_k, b_k \in \mathbb{C}$, $0 < p_k \leq \sup p_k = H$, $D = \max\left(1, 2^{H-1}\right)$ ([24]).

2. Main Results

In this part we will describe the sets of strongly $w_{\alpha}^{\beta}(p)$ −summable sequences and strongly $w_{\alpha}^{\beta}[\theta, f, p]$ −summable sequences with respect to the modulus function f. We will examine these spaces and we give some inclusion relations between the $S_{\alpha}^{\beta}(\theta)$− statistical convergent, strong $w_{\alpha}^{\beta}[\theta, f, p]$ −summability and strong $w_{\alpha}^{\beta}(p)$ −summability.

Definition 1. *Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq \beta \leq 1$ be given. We say that the sequence $x = (x_k)$ is $S_{\alpha}^{\beta}(\theta)$ −statistically convergent (or lacunary statistically convergent sequences of order (α, β)) if there is a real number L such* *that*

$$\lim_{r\to\infty}\frac{1}{h_r^{\alpha}}\left|\left\{k\in I_r:\left|x_k-L\right|\geq\varepsilon\right\}\right|^{\beta}=0,$$

where $I_r=(k_{r-1},k_r]$ *and* h_r^{α} *denotes the* α*th power* $(h_r)^{\alpha}$ *of* h_r, *that is* $h^{\alpha}=(h_r^{\alpha})=(h_1^{\alpha},h_2^{\alpha},...,h_r^{\alpha},...)$ *and* $\left|\{k\leq n:k\in E\}\right|^{\beta}$ *denotes the* β*th power of number of elements of* E *not exceeding* n. *In the present case this convergence is indicated by* $S_{\alpha}^{\beta}(\theta)-\lim x_k=L$. $S_{\alpha}^{\beta}(\theta)$ *will indicate the set of all* $S_{\alpha}^{\beta}(\theta)-$*statistically convergent sequences. If* $\theta=(2^r)$, *then we will write* S_{α}^{β} *in the place of* $S_{\alpha}^{\beta}(\theta)$. *If* $\alpha=\beta=1$ *and* $\theta=(2^r)$, *then we will write* S *in the place of* $S_{\alpha}^{\beta}(\theta)$.

Definition 2. *Let* $\theta=(k_r)$ *be a lacunary sequence,* $0<\alpha\leq\beta\leq1$ *and* p *be a positive real number. We say that the sequence* $x=(x_k)$ *is strongly* $N_{\alpha}^{\beta}(\theta,p)-$*summable (or strongly* $N(\theta,p)-$*summable of order* (α,β)*) if there is a real number* L *such that*

$$\lim_{r\to\infty}\frac{1}{h_r^{\alpha}}\left(\sum_{k\in I_r}\left|x_k-L\right|^{p}\right)^{\beta}=0.$$

In the present case we denote $N_{\alpha}^{\beta}(\theta,p)-\lim x_k=L$. $N_{\alpha}^{\beta}(\theta,p)$ *will denote the set of all strongly* $N(\theta,p)-$*summable of order* (α,β). *If* $\alpha=\beta=1$, *then we will write* $N(\theta,p)$ *in the place of* $N_{\alpha}^{\beta}(\theta,p)$. *If* $\theta=(2^r)$, *then we will write* $w_{\alpha}^{\beta}(p)$ *in the place of* $N_{\alpha}^{\beta}(\theta,p)$. *If* $L=0$, *then we will write* $w_{\alpha,0}^{\beta}(p)$ *in the place of* $w_{\alpha}^{\beta}(p)$. $N_{\alpha,0}^{\beta}(\theta,p)$ *will denote the set of all strongly* $N_{\theta}(p)-$*summable of order* (α,β) *to* 0.

Definition 3. *Let* f *be a modulus function,* $p=(p_k)$ *be a sequence of strictly positive real numbers and* $0<\alpha\leq\beta\leq1$ *be real numbers. We say that the sequence* $x=(x_k)$ *is strongly* $w_{\alpha}^{\beta}[\theta,f,p]-$*summable to* L *(a real number) such that*

$$w_{\alpha}^{\beta}[\theta,f,p]=\left\{x=(x_k):\lim_{r\to\infty}\frac{1}{h_r^{\alpha}}\left(\sum_{k\in I_r}\left[f\left(\left|x_k-L\right|\right)\right]^{p_k}\right)^{\beta}=0,\ \text{for some } L\right\}.$$

In the present case, we denote $w_{\alpha}^{\beta}[\theta,f,p]-\lim x_k=L$. *In the special case* $p_k=1$, *for all* $k\in\mathbb{N}$ *and* $f(x)=x$ *we will denote* $N_{\alpha}^{\beta}(\theta,p)$ *in the place of* $w_{\alpha}^{\beta}[\theta,f,p]$. $w_{\alpha,0}^{\beta}[\theta,f,p]$ *will denote the set of all strongly* $w[\theta,f,p]-$*summable of order* (α,β) *to* 0.

In the following theorems we shall assume that the sequence $p=(p_k)$ is bounded and $0<h=\inf_k p_k\leq p_k\leq\sup_k p_k=H<\infty$.

Theorem 1. *The class of sequences* $w_{\alpha,0}^{\beta}[\theta,f,p]$ *is linear space.*

Proof. Omitted. □

Theorem 2. *The space* $w_{\alpha,0}^{\beta}[\theta,f,p]$ *is paranormed by*

$$g(x)=\sup_r\left\{\frac{1}{h_r^\alpha}\left(\sum_{k\in I_r}[f(|x_k|)]^{p_k}\right)^\beta\right\}^{\frac{1}{M}}$$

where $0<\alpha\le\beta\le1$ *and* $M=\max(1,H)$.

Proof. Clearly $g(0)=0$ and $g(x)=g(-x)$. Take any $x,y\in w_{\alpha,0}^\beta[\theta,f,p]$. Since $\frac{p_k}{\frac{M}{\beta}}\le1$ and $\frac{M}{\beta}\ge1$, using the Minkowski's inequality and definition of f, we can write

$$\begin{aligned}\left\{\frac{1}{h_r^\alpha}\left(\sum_{k\in I_r}[f(|x_k+y_k|)]^{p_k}\right)^\beta\right\}^{\frac{1}{M}} &\le \left\{\frac{1}{h_r^\alpha}\left(\sum_{k\in I_r}[f(|x_k|)+f(|y_k|)]^{p_k}\right)^\beta\right\}^{\frac{1}{M}}\\ &= \frac{1}{h_r^{\frac{\alpha}{M}}}\left(\sum_{k\in I_r}[f(|x_k|)+f(|y_k|)]^{p_k}\right)^{\frac{1}{\frac{M}{\beta}}}\\ &\le \frac{1}{h_r^{\frac{\alpha}{M}}}\left\{\left(\sum_{k\in I_r}[f(|x_k|)]^{p_k}\right)^\beta\right\}^{\frac{1}{M}}\\ &\quad+\frac{1}{h_r^{\frac{\alpha}{M}}}\left\{\left(\sum_{k\in I_r}[f(|y_k|)]^{p_k}\right)^\beta\right\}^{\frac{1}{M}}.\end{aligned}$$

Therefore $g(x+y)\le g(x)+g(y)$ for $x,y\in w_{\alpha,0}^\beta[\theta,f,p]$. Let λ be complex number. By definition of f we have

$$g(\lambda x)=\sup_r\left\{\frac{1}{h_r^\alpha}\left(\sum_{k\in I_r}[f(|\lambda x_k|)]^{p_k}\right)^\beta\right\}^{\frac{1}{M}}\le K^{\frac{H}{\frac{M}{\beta}}}g(x)$$

where $[\lambda]$ denotes the integer part of λ, and $K=1+[|\lambda|]$. Now, let $\lambda\to0$ for any fixed x with $g(x)\ne0$. By definition of f, for $|\lambda|<1$ and $0<\alpha\le\beta\le1$, we have

$$\frac{1}{h_r^\alpha}\left(\sum_{k\in I_r}[f(|\lambda x_k|)]^{p_k}\right)^\beta<\varepsilon \text{ for } n>N(\varepsilon). \tag{2}$$

Also, for $1\le n\le N$, taking λ small enough, since f is continuous we have

$$\frac{1}{h_r^\alpha}\left(\sum_{k\in I_r}[f(|\lambda x_k|)]^{p_k}\right)^\beta<\varepsilon \tag{3}$$

(2) and (3) together imply that $g(\lambda x)\to0$ as $\lambda\to0$. □

Proposition 1. ([19]) *Let f be a modulus and $0<\delta<1$. Then for each $\|u\|\geq\delta$, we have $f(\|u\|)\leq 2f(1)\delta^{-1}\|u\|$.*

Theorem 3. *If $0<\alpha=\beta\leq 1$, $p>1$ and $\liminf_{u\to\infty}\frac{f(u)}{u}>0$, then $w_{\alpha}^{\beta}[\theta,f,p]=w_{\alpha}^{\beta}(p)$.*

Proof. Let $p_k=p$ be a positive real number. If $\liminf_{u\to\infty}\frac{f(u)}{u}>0$ then there exists a number $c>0$ such that $f(u)>cu$ for $u>0$. We have $x\in w_{\alpha}^{\beta}[\theta,f,p]$. Clearly

$$\frac{1}{h_r^{\alpha}}\left(\sum_{k\in I_r}[f(|x_k-L|)]^p\right)^{\beta}\geq\frac{1}{h_r^{\alpha}}\left(\sum_{k\in I_r}[c|x_k-L|]^p\right)^{\beta}=\frac{c^{p\beta}}{h_r^{\alpha}}\left(\sum_{k\in I_r}|x_k-L|^p\right)^{\beta},$$

therefore $w_{\alpha}^{\beta}[\theta,f,p]\subseteq w_{\alpha}^{\beta}(p)$.

Let $x\in w_{\alpha}^{\beta}(p)$. Then we have

$$\frac{1}{h_r^{\alpha}}\left(\sum_{k\in I_r}|x_k-L|^p\right)^{\beta}\to 0 \text{ as } r\to\infty.$$

Let $\varepsilon>0$, $\alpha=\beta$ and choose δ with $0<\delta<1$ such that $cu<f(u)<\varepsilon$ for every u with $0\leq u\leq\delta$. We can write

$$\begin{aligned}\frac{1}{h_r^{\alpha}}\left(\sum_{k\in I_r}[f(|x_k-L|)]^p\right)^{\beta} &= \frac{1}{h_r^{\alpha}}\left(\sum_{\substack{k\in I_r\\|x_k-L|\leq\delta}}[f(|x_k-L|)]^p\right)^{\beta}\\ &\quad+\frac{1}{h_r^{\alpha}}\left(\sum_{\substack{k\in I_r\\|x_k-L|>\delta}}[f(|x_k-L|)]^p\right)^{\beta}\\ &\leq \frac{1}{h_r^{\alpha}}\varepsilon^{p\beta}h_r^{\beta}+\frac{1}{h_r^{\alpha}}\left(\sum_{\substack{k\in I_r\\|x_k-L|>\delta}}\left[2f(1)\delta^{-1}|x_k-L|\right]^p\right)^{\beta}\\ &\leq \frac{1}{h_r^{\alpha}}\varepsilon^{p\beta}h_r^{\beta}+\frac{2^{p\beta}f(1)^{p\beta}}{h_r^{\alpha}\delta^{p\beta}}\left(\sum_{k\in I_r}|x_k-L|^p\right)^{\beta}\end{aligned}$$

by Proposition 1. Therefore $x\in w_{\alpha}^{\beta}[\theta,f,p]$. □

Example 1. *We now give an example to show that $w_{\alpha}^{\beta}[\theta,f,p]\neq w_{\alpha}^{\beta}(p)$ in this case when $\liminf_{u\to\infty}\frac{f(u)}{u}=0$. Consider the sequence $f(x)=\sqrt{x}$ of modulus function.*

Define $x = (x_k)$ *by*

$$x_k = \begin{cases} h_r, & if \quad k = k_r \\ 0, & if \quad otherwise. \end{cases}$$

We have, for $L = 0$, $p = \frac{3}{2}$ *and* $\alpha = \beta$

$$\frac{1}{h_r^{\alpha}} \left(\sum_{k \in I_r} [f(|x_k|)]^p \right)^{\beta} = \frac{1}{h_r^{\alpha}} \left(\sqrt{h_r} \right)^{\frac{3}{2}\beta} \to 0 \ as \ r \to \infty$$

and so $x \in w_{\alpha}^{\beta}[\theta, f, p]$. *But*

$$\frac{1}{h_r^{\alpha}} \left(\sum_{k \in I_r} |x_k|^p \right)^{\beta} = \frac{(h_r)^{\frac{3}{2}\beta}}{h_r^{\alpha}} \to \infty \ as \ r \to \infty$$

and so $x \notin w_{\alpha}^{\beta}(p)$.

Theorem 4. *Let* $0 < \alpha \leq \beta \leq 1$ *and* $\liminf p_k > 0$. *Then* $x_k \to L$ *implies* $w_{\alpha}^{\beta}[\theta, f, p] - \lim x_k = L$.

Proof. Let $x_k \to L$. By definition of f we have $f(|x_k - L|) \to 0$. Since $\liminf p_k > 0$, we have $[f(|x_k - L|)]^{p_k} \to 0$. Therefore $w_{\alpha}^{\beta}[\theta, f, p] - \lim x_k = L$. □

Theorem 5. *Let* $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0, 1]$ *be real numbers such that* $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$, f *be a modulus function and let* $\theta = (k_r)$ *be a lacunary sequence, then* $w_{\alpha_1}^{\beta_2}[\theta, f, p] \subset S_{\alpha_2}^{\beta_1}(\theta)$.

Proof. Let $x \in w_{\alpha_1}^{\beta_2}[\theta, f, p]$ and let $\varepsilon > 0$ be given and $\sum_1$ and $\sum_2$ denote the sums over $k \in I_r$, $|x_k - L| \geq \varepsilon$ and $k \in I_r$, $|x_k - L| < \varepsilon$ respectively. Since $h_r^{\alpha_1} \leq h_r^{\alpha_2}$ for each r we may write

$$\begin{aligned} &\frac{1}{h_r^{\alpha_1}} \left(\sum_{k \in I_r} [f(|x_k - L|)]^{p_k} \right)^{\beta_2} \\ &\quad = \frac{1}{h_r^{\alpha_1}} \left[\sum\nolimits_1 [f(|x_k - L|)]^{p_k} + \sum\nolimits_2 [f(|x_k - L|)]^{p_k} \right]^{\beta_2} \\ &\quad \geq \frac{1}{h_r^{\alpha_2}} \left[\sum\nolimits_1 [f(|x_k - L|)]^{p_k} + \sum\nolimits_2 [f(|x_k - L|)]^{p_k} \right]^{\beta_2} \\ &\quad \geq \frac{1}{h_r^{\alpha_2}} \left[\sum\nolimits_1 [f(\varepsilon)]^{p_k} \right]^{\beta_2} \\ &\quad \geq \frac{1}{h_r^{\alpha_2}} \left[\sum\nolimits_1 \min([f(\varepsilon)]^h, [f(\varepsilon)]^H) \right]^{\beta_2} \\ &\quad \geq \frac{1}{h_r^{\alpha_2}} |\{k \in I_r : |x_k - L| \geq \varepsilon\}|^{\beta_1} \left[\min([f(\varepsilon)]^h, [f(\varepsilon)]^H) \right]^{\beta_1}. \end{aligned}$$

Hence $x \in S_{\alpha_2}^{\beta_1}(\theta)$. □

Theorem 6. *If the modulus f is bounded and $\lim_{r\to\infty}\frac{h_r^{\beta_2}}{h_r^{\alpha_1}}=1$ then $S_{\alpha_1}^{\beta_2}(\theta)\subset w_{\alpha_2}^{\beta_1}[\theta,f,p]$.*

Proof. Let $x\in S_{\alpha_1}^{\beta_2}(\theta)$. Assume that f is bounded. Therefore $f(x)\leq K$, for a positive integer K and all $x\geq 0$. Then for each $r\in\mathbb{N}$ and $\varepsilon>0$ we can write

$$\begin{aligned}\frac{1}{h_r^{\alpha_2}}\left(\sum_{k\in I_r}[f(|x_k-L|)]^{p_k}\right)^{\beta_1} &\leq \frac{1}{h_r^{\alpha_1}}\left(\sum_{k\in I_r}[f(|x_k-L|)]^{p_k}\right)^{\beta_1}\\ &= \frac{1}{h_r^{\alpha_1}}\left(\sum\nolimits_1[f(|x_k-L|)]^{p_k}+\sum\nolimits_2[f(|x_k-L|)]^{p_k}\right)^{\beta_1}\\ &\leq \frac{1}{h_r^{\alpha_1}}\left(\sum\nolimits_1\max\left(K^h,K^H\right)+\sum\nolimits_2[f(\varepsilon)]^{p_k}\right)^{\beta_1}\\ &\leq \left(\max\left(K^h,K^H\right)\right)^{\beta_2}\frac{1}{h_r^{\alpha_1}}\left|\{k\in I_r: f(|x_k-L|)\geq\varepsilon\}\right|^{\beta_2}\\ &\quad+\frac{h_r^{\beta_2}}{h_r^{\alpha_1}}\left(\max\left(f(\varepsilon)^h,\ f(\varepsilon)^H\right)\right)^{\beta_2}.\end{aligned}$$

Hence $x\in w_{\alpha_2}^{\beta_1}[\theta,f,p]$. □

Theorem 7. *Let f be a modulus function. If $\lim p_k>0$, then $w_\alpha^\beta[\theta,f,p]-\lim x_k=L$ uniquely.*

Proof. Let $\lim p_k=s>0$. Assume that $w_\alpha^\beta[\theta,f,p]-\lim x_k=L_1$ and $w_\alpha^\beta[\theta,f,p]-\lim x_k=L_2$. Then

$$\lim_r\frac{1}{h_r^\alpha}\left(\sum_{k\in I_r}[f(|x_k-L_1|)]^{p_k}\right)^\beta=0,$$

and

$$\lim_r\frac{1}{h_r^\alpha}\left(\sum_{k\in I_r}[f(|x_k-L_2|)]^{p_k}\right)^\beta=0.$$

By definition of f and using (1), we have

$$\begin{aligned}&\frac{1}{h_r^\alpha}\left(\sum_{k\in I_r}[f(|L_1-L_2|)]^{p_k}\right)^\beta\\ &\quad\leq\frac{D}{h_r^\alpha}\left(\sum_{k\in I_r}[f(|x_k-L_1|)]^{p_k}+\sum_{k\in I_r}[f(|x_k-L_2|)]^{p_k}\right)^\beta\\ &\quad\leq\frac{D}{h_r^\alpha}\left(\sum_{k\in I_r}[f(|x_k-L_1|)]^{p_k}\right)^\beta+\frac{D}{h_r^\alpha}\left(\sum_{k\in I_r}[f(|x_k-L_2|)]^{p_k}\right)^\beta\end{aligned}$$

where $\sup_k p_k = H$, $0 < \alpha \leq \beta \leq 1$ and $D = \max\left(1, 2^{H-1}\right)$. Hence

$$\lim_r \frac{1}{h_r^{\alpha}} \left(\sum_{k \in I_r} \left[f\left(|L_1 - L_2| \right) \right]^{p_k} \right)^{\beta} = 0.$$

Since $\lim_{k\to\infty} p_k = s$ we have $L_1 - L_2 = 0$. Thus the limit is unique. □

Theorem 8. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let $\alpha_1, \alpha_2, \beta_1$ and β_2 be such that $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$,*
(i) If

$$\liminf_{r\to\infty} \frac{h_r^{\alpha_1}}{\ell_r^{\alpha_2}} > 0 \tag{4}$$

then $w_{\alpha_2}^{\beta_2}\left[\theta', f, p\right] \subset w_{\alpha_1}^{\beta_1}\left[\theta, f, p\right]$,
(ii) If the modulus f is bounded and

$$\lim_{r\to\infty} \frac{\ell_r}{h_r^{\alpha_2}} = 1 \tag{5}$$

then $w_{\alpha_1}^{\beta_2}\left[\theta, f, p\right] \subset w_{\alpha_2}^{\beta_1}\left[\theta', f, p\right]$.

Proof. (i) Let $x \in w_{\alpha_2}^{\beta_2}\left[\theta', f, p\right]$. We can write

$$\begin{aligned} \frac{1}{\ell_r^{\alpha_2}} \left(\sum_{k \in J_r} \left[f\left(|x_k - L| \right) \right]^{p_k} \right)^{\beta_2} &= \frac{1}{\ell_r^{\alpha_2}} \left(\sum_{k \in J_r - I_r} \left[f\left(|x_k - L| \right) \right]^{p_k} \right)^{\beta_2} \\ &\quad + \frac{1}{\ell_r^{\alpha_2}} \left(\sum_{k \in I_r} \left[f\left(|x_k - L| \right) \right]^{p_k} \right)^{\beta_2} \\ &\geq \frac{1}{\ell_r^{\alpha_2}} \left(\sum_{k \in I_r} \left[f\left(|x_k - L| \right) \right]^{p_k} \right)^{\beta_2} \\ &\geq \frac{h_r^{\alpha_1}}{\ell_r^{\alpha_2}} \frac{1}{h_r^{\alpha_1}} \left(\sum_{k \in I_r} \left[f\left(|x_k - L| \right) \right]^{p_k} \right)^{\beta_1}. \end{aligned}$$

Thus if $x \in w_{\alpha_2}^{\beta_2}\left[\theta', f, p\right]$, then $x \in w_{\alpha_1}^{\beta_1}\left[\theta, f, p\right]$.

(ii) Let $x = (x_k) \in w_{\alpha_1}^{\beta_2}\left[\theta, f, p\right]$ and (2) holds. Assume that f is bounded. Therefore $f(x) \leq K$, for a positive integer K and all $x \geq 0$. Now, since $I_r \subseteq J_r$

and $h_r \leq \ell_r$ for all $r \in \mathbb{N}$, we can write

$$\begin{aligned} &\frac{1}{\ell_r^{\alpha_2}}\left(\sum_{k\in J_r}\left[f\left(|x_k - L|\right)\right]^{p_k}\right)^{\beta_1} \\ &= \frac{1}{\ell_r^{\alpha_2}}\left(\sum_{k\in J_r - I_r}\left[f\left(|x_k - L|\right)\right]^{p_k}\right)^{\beta_1} + \frac{1}{\ell_r^{\alpha_2}}\left(\sum_{k\in I_r}\left[f\left(|x_k - L|\right)\right]^{p_k}\right)^{\beta_1} \\ &\leq \left(\frac{\ell_r - h_r}{\ell_r^{\alpha_2}}\right)^{\beta_1} K^{p_k\beta_1} + \frac{1}{\ell_r^{\alpha_2}}\left(\sum_{k\in I_r}\left[f\left(|x_k - L|\right)\right]^{p_k}\right)^{\beta_1} \\ &\leq \left(\frac{\ell_r - h_r^{\alpha_2}}{h_r^{\alpha_2}}\right) K^{H\beta_1} + \frac{1}{h_r^{\alpha_2}}\left(\sum_{k\in I_r}\left[f\left(|x_k - L|\right)\right]^{p_k}\right)^{\beta_2} \\ &\leq \left(\frac{\ell_r}{h_r^{\alpha_2}} - 1\right) K^{H\beta_1} + \frac{1}{h_r^{\alpha_1}}\left(\sum_{k\in I_r}\left[f\left(|x_k - L|\right)\right]^{p_k}\right)^{\beta_2} \end{aligned}$$

for every $r \in \mathbb{N}$. Therefore $w_{\alpha_1}^{\beta_2}[\theta, f, p] \subset w_{\alpha_2}^{\beta_1}\left[\theta', f, p\right]$. □

Now as a result of Theorem 8 we have the following Corollary 1.

Corollary 1. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$.*

If (4) holds then, for $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$

(i) If $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq 1$ and $\beta_2 = 1$, then $w_{\alpha_2}\left[\theta', f, p\right] \subset w_{\alpha_1}^{\beta_1}[\theta, f, p]$,

(ii) If $0 < \alpha_1 \leq \alpha_2 \leq 1$ and $\beta_1 = \beta_2 = 1$, then $w_{\alpha_2}\left[\theta', f, p\right] \subset w_{\alpha_1}[\theta, f, p]$,

(iii) If $0 < \alpha_1 \leq 1$ and $\alpha_2 = \beta_1 = \beta_2 = 1$, then $w\left[\theta', f, p\right] \subset w_{\alpha_1}[\theta, f, p]$,

(iv) If $0 < \alpha_1 \leq \alpha_2 \leq 1$ and $\beta_1 = \beta_2 = \beta$, then $w_{\alpha_2}^{\beta}\left[\theta', f, p\right] \subset w_{\alpha_1}^{\beta}[\theta, f, p]$,

(v) If $\alpha_1 = \alpha_2 = \alpha$ and $0 < \beta_1 \leq \beta_2 \leq 1$, then $w_{\alpha}^{\beta_2}\left[\theta', f, p\right] \subset w_{\alpha}^{\beta_1}[\theta, f, p]$,

(vi) If $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 1$, then $w\left[\theta', f, p\right] \subset w[\theta, f, p]$.

If (5) holds then, for $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$

(i) If $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq 1$ and $\beta_2 = 1$, then $w_{\alpha_1}[\theta, f, p] \subset w_{\alpha_2}^{\beta_1}\left[\theta', f, p\right]$,

(ii) If $0 < \alpha_1 \leq \alpha_2 \leq 1$ and $\beta_1 = \beta_2 = 1$, then $w_{\alpha_1}[\theta, f, p] \subset w_{\alpha_2}\left[\theta', f, p\right]$,

(iii) If $0 < \alpha_1 \leq 1$ and $\alpha_2 = \beta_1 = \beta_2 = 1$, then $w_{\alpha_1}[\theta, f, p] \subset w\left[\theta', f, p\right]$,

(iv) If $0 < \alpha_1 \leq \alpha_2 \leq 1$ and $\beta_1 = \beta_2 = \beta$, then $w_{\alpha_1}^{\beta}[\theta, f, p] \subset w_{\alpha_2}^{\beta}\left[\theta', f, p\right]$,

(v) If $\alpha_1 = \alpha_2 = \alpha$ and $0 < \beta_1 \leq \beta_2 \leq 1$, then $w_{\alpha}^{\beta_2}[\theta, f, p] \subset w_{\alpha}^{\beta_1}\left[\theta', f, p\right]$,

(vi) If $\alpha_1 = \alpha_2 = 1$ *and* $\beta_1 = \beta_2 = 1$, *then* $w[\theta, f, p] \subset w\left[\theta', f, p\right]$.

References

[1] Altın, Y., Properties of some sets of sequences defined by a modulus function, *Acta Math. Sci. Ser. B Engl. Ed.* 29(2) (2009), 427–434.

[2] Caserta, A., Giuseppe, Di M. and Kočinac, L. D. R., Statistical convergence in function spaces, *Abstr. Appl. Anal.* 2011, Art. ID 420419, 11 pp.

[3] Çakallı, H., A study on statistical convergence, *Funct. Anal. Approx. Comput.* 1(2) (2009), 19–24.

[4] Et, M., Generalized Cesàro difference sequence spaces of non-absolute type involving lacunary sequences, *Appl. Math. Comput.* 219(17) (2013), 9372–9376.

[5] Gadjiev, A. D., and Orhan, C., Some approximation theorems via statistical convergence, *Rocky Mountain J. Math.* 32(1) (2002), 129-138.

[6] Kolk, E., The statistical convergence in Banach spaces, *Acta Comment. Univ. Tartu*, 928 (1991), 41-52.

[7] Çolak, R., Statistical convergence of order α, *Modern Methods in Analysis and Its Applications, New Delhi, India: Anamaya Pub*, 2010: 121–129.

[8] Connor, J. S., The Statistical and Strong p-Cesaro Convergence of Sequences, *Analysis*,8, pp. (1988), 47-63.

[9] Fast, H., Sur La Convergence Statistique, *Colloq. Math.*, 2, pp. (1951), 241–244.

[10] Fridy, J., On Statistical Convergence, *Analysis*, 5, pp. (1985), 301-313.

[11] Fridy, J., and Orhan, C., Lacunary Statistical Convergence, *Pacific J. Math.*, 160, pp. (1993), 43–51.

[12] Fridy, J., and Orhan, C., Lacunary Statistical Summability, *J. Math. Anal. Appl.*, 173, pp. (1993), no. 2, 497–504.

[13] Schoenberg, I. J., The Integrability of Certain Functions and Related Summability Methods, *Amer. Math. Monthly*, 66, pp. (1959), 361–375.

[14] Salat, T., On Statistically Convergent Sequences of Real Numbers, *Math. Slovaca.*, 30, pp. (1980), 139-150.

[15] Steinhaus, H., Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.* 2 (1951) 73-74.

[16] Şengül, H., On Statistical Convergence of Order (α, β). (In rewiev)

[17] Et, M., and Şengül, H., Some Cesaro-type summability spaces of order α and lacunary statistical convergence of order α, *Filomat* 28 (2014), no. 8, 1593–1602.

[18] Şengül, H., and Et, M., On lacunary statistical convergence of order α, *Acta Math. Sci. Ser. B Engl. Ed.* 34 (2014), no. 2, 473–482.

[19] Pehlivan, S., and Fisher, B., Some sequence spaces defined by a modulus, *Mathematica Slovaca* vol. 45, no. 3, pp. 275–280,1995.

[20] Nakano, H., Modulared sequence spaces, *Proc. Japan Acad.* 27 (1951), 508–512.

[21] Gaur, A. K., and Mursaleen, M., Difference sequence spaces defined by a sequence of moduli, *Demonstratio Math.* 31(2) (1998), 275–278.

[22] Nuray, F., and Savaş, E., Some new sequence spaces defined by a modulus function, *Indian J. Pure Appl. Math.* 24(11) (1993), 657–663.

[23] Işık, M., Strongly almost $(w, \lambda, q)-$summable sequences, *Math. Slovaca* 61(5) (2011), 779–788.

[24] Maddox, I. J., Elements of Functional Analysis, Cambridge University Press, 1970.

[25] Maddox, I. J., Sequence spaces defined by a modulus, *Math. Proc. Camb. Philos. Soc,* 1986, 100:161-166.

[26] Et, M., Strongly almost summable difference sequences of order m defined by a modulus, *Studia Sci. Math. Hungar.* 40(4) (2003), 463–476.

[27] Et, M., Spaces of Cesàro difference sequences of order r defined by a modulus function in a locally convex space, *Taiwanese J. Math.* 10(4) (2006), 865-879.

[28] Ruckle, W. H., FK spaces in which the sequence of coordinate vectors is bounded, *Canad. J. Math.* 25 (1973), 973–978.

[29] Pehlivan, S. and Fisher, B., Lacunary strong convergence with respect to a sequence of modulus functions, *Comment. Math. Univ. Carolin.* 36 (1995), no. 1, 69-76.

Current address: Hacer Şengül: Department of Mathematics ; Siirt University 56100; Siirt; TURKEY.

E-mail address: hacer.sengul@hotmail.com