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ON VECTOR-VALUED CLASSICAL AND VARIABLE EXPONENT AMALGAM SPACES

ISMAIL AYDIN

Abstract. Let $1 \leq p, q, s \leq \infty$ and $1 \leq r(.) \leq \infty$, where $r(.)$ is a variable exponent. In this paper, we introduce firstly vector-valued variable exponent amalgam spaces $\left(L^{r(\cdot)}(\mathbb{R}, E), \ell^s\right)$. Secondly, we investigate some basic properties of $\left(L^{r(\cdot)}(\mathbb{R}, E), \ell^s\right)$ spaces. Finally, we recall vector-valued classical amalgam spaces $\left(L^p(G, A), \ell^q\right)$, and inquire the space of multipliers from $\left(L^{p_1}(G, A), \ell^{q_1}\right)$ to $\left(L^{p_2'}(G, A^*), \ell^{q_2'}\right)$.

1. Introduction

The amalgam of L^p and l^q on the real line is the space $(L^p, l^q)(\mathbb{R})$ (or shortly (L^p, l^q)) consisting of functions which are locally in L^p and have l^q behavior at infinity. Several special cases of amalgam spaces, such as $\left(L^1, l^2\right)$, $\left(L^2, l^\infty\right)$, $\left(L^\infty, l^1\right)$ and $\left(L^1, l^\infty\right)$ were studied by N. Wiener [30]. Comprehensive information about amalgam spaces can be found in some papers, such as [16], [29], [15], [10] and [11]. Recently, there have been many interesting and important papers appeared in variable exponent amalgam spaces $\left(L^{r(\cdot)}, \ell^s\right)$, such as Aydın and Gürkanlı [3], Aydın [5], Gürkanli and Aydın [14], Kokilashvili, Meskhi and Zaighum [17], Meskhi and Zaighum[23], Gürkanli [13], Kulak and Gürkanli [20]. Vector-valued classical amalgam spaces $\left(L^p(\mathbb{R}, E), \ell^q\right)$ on the real line were defined by Lakshmi and Ray [21] in 2009. They described and discussed some fundamental properties of these spaces, such as embeddings and separability. In their following paper [22], they investigated convolution product and obtained a similar result to Young's convolution theorem on $\left(L^p(\mathbb{R}, E), \ell^q\right)$. They also showed classical result on Fourier transform of convolution product for $\left(L^p(\mathbb{R}, E), \ell^q\right)$. Vector-valued variable exponent Bochner-Lebesgue spaces $L^{r(\cdot)}(\mathbb{R}, E)$ defined by Cheng and Xu [7] in 2013. They proved dual space, the reflexivity, uniformly convexity and uniformly smoothness of

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$L^{r(\cdot)}(\mathbb{R}, E)$. Furthermore, they gave some properties of the Banach valued Bochner-Sobolev spaces with variable exponent. In this paper, we give some information about $\left(L^{r(\cdot)}(\mathbb{R}, E), \ell^{s}\right)$, and obtain the generalization of some results in Sağır [27] and similar consequences in Avcı and Gürkanli [1] and Öztop and Gürkanli [24]. Finally, our original aim is to prove that the space of multipliers from $\left(L^{p_1}(G, A), \ell^{q_1}\right)$ to $\left(L^{p_2'}(G, A^*), \ell^{q_2'}\right)$ is isometrically isomorphic to $\left(\mathbf{A}_{p_1,p_2}^{q_1,q_2}(G, A)\right)^*$.

2. DEFINITION AND PRELIMINARY RESULTS

In this section, we give several definitions and theorems for vector-valued variable exponent Lebesgue spaces $L^{r(\cdot)}(\mathbb{R}, E)$.

Definition 1. *For a measurable function $r : \mathbb{R} \to [1, \infty)$ (called a variable exponent on $\mathbb{R}$), we put*

$$r^- = \underset{x\in\mathbb{R}}{essinf} r(x), \qquad r^+ = \underset{x\in\mathbb{R}}{esssup} r(x).$$

The variable exponent Lebesgue spaces $L^{r(\cdot)}(\mathbb{R})$ consist of all measurable functions f such that $\varrho_{r(.)}(\lambda f) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|f\|_{r(.)} = \inf\left\{\lambda > 0 : \varrho_{r(.)}(\frac{f}{\lambda}) \leq 1\right\},$$

where

$$\varrho_{r(.)}(f) = \int_{\mathbb{R}} |f(x)|^{r(x)}\, dx.$$

If $r^+ < \infty$, then $f \in L^{r(\cdot)}(\mathbb{R})$ iff $\varrho_{r(.)}(f) < \infty$. The space $\left(L^{r(\cdot)}(\mathbb{R}), \|.\|_{r(.)}\right)$ is a Banach space. If $r(x) = r$ is a constant function, then the norm $\|.\|_{r(.)}$ coincides with the usual Lebesgue norm $\|.\|_r$ [18], [2], [4]. In this paper we assume that $r^+ < \infty$.

Definition 2. *We denote by $L_{loc}^{r(\cdot)}(\mathbb{R})$ the space of (equivalence classes of) functions on $\mathbb{R}$ such that f restricted to any compact subset K of $\mathbb{R}$ belongs to $L^{r(\cdot)}(\mathbb{R})$.*

Let $1 \leq r(.), s < \infty$ and $J_k = [k, k+1)$, $k \in \mathbb{Z}$. The variable exponent amalgam spaces $\left(L^{r(\cdot)}, \ell^s\right)$ are the normed spaces

$$\left(L^{r(\cdot)}, \ell^s\right) = \left\{f \in L_{loc}^{r(\cdot)}(\mathbb{R}) : \|f\|_{\left(L^{r(\cdot)}, \ell^s\right)} < \infty\right\},$$

where

$$\|f\|_{\left(L^{r(\cdot)}, \ell^s\right)} = \left(\sum_{k\in\mathbb{Z}} \left\|f\chi_{J_k}\right\|_{r(.)}^{s}\right)^{1/s}.$$

It is well known that $\left(L^{r(\cdot)}, \ell^s\right)$ is a Banach space and does not depend on the particular choice of J_k, that is, J_k can be equal to $[k, k+1)$, $[k, k+1]$ or $(k, k+1)$. Thus, we have same spaces $\left(L^{r(\cdot)}, \ell^s\right)$ [15]. Furthermore, it can be seen in references

[3], [5] and [14] to obtain some basic properties for $\left(L^{r(.)}, \ell^s\right)$ spaces. It is well known that $L^{r(.)}(\mathbb{R})$ is not translation invariant. So, the convolution operator and multipliers are useless in this space. By using Theorem 3.3 in [13] we also obtain $\left(L^{r(.)}, \ell^s\right)$ is not translation invariant.

Let $(E, \|.\|_E)$ be a Banach space and E^* its dual space and (Ω, Σ, μ) be a measure space.

Definition 3. *A function $f : \Omega \to E$ is Bochner (or strongly) μ -measurable if there exists a sequence $\{f_n\}$ of simple functions $f_n : \Omega \to E$ such that $f_n(x) \xrightarrow{E} f(x)$ as $n \to \infty$ for almost all $x \in \Omega$ [9].*

Definition 4. *A μ-measurable function $f : \Omega \to E$ is called Bochner integrable if there exists a sequence of simple functions $\{f_n\}$ such that*

$$\lim_{n\to\infty} \int_\Omega \|f_n - f\|_E \, d\mu = 0$$

for almost all $x \in \Omega$ [9].

Theorem 1. *A μ-measurable function $f : \Omega \to E$ is Bochner integrable if and only if $\int_\Omega \|f\|_E \, d\mu < \infty$ [9].*

Definition 5. *A function $F : \Sigma \to E$ is called a vector measure, if for all sequences (A_n) of pairwise disjoint members of Σ such that $\bigcup_{n=1}^{\infty} A_n \in \Sigma$ and $F\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} F(A_n)$, where the series converges in the norm topology of E.*

Let $F : \Sigma \to E$ be a vector measure. The variation of F is the function $\|F\| : \Sigma \to [0, \infty]$ defined by

$$\|F\|(A) = \sup_{\pi} \sum_{B\in\pi}^{\infty} \|F(B)\|_E \, ,$$

where the supremum is taken over all finite disjoint partitions π of A. If $\|F\|(\Omega) < \infty$, then F is called a measure of bounded variation [7],[9].

Definition 6. *A Banach space E has the Radon-Nikodym property (RNP) with respect to (Ω, Σ, μ) if for each vector measure $F : \Sigma \to E$ of bounded variation, which is absolutely continuous with respect to μ, there exists a function $g \in L^1(\Omega, E)$ such that*

$$F(A) = \int_A g d\mu$$

for all $A \in \Sigma$ [7],[9].

Definition 7. *The variable exponent Bochner- Lebesgue space $L^{r(\cdot)}(\mathbb{R},E)$ stands for all (equivalence classes of) E-valued Bochner integrable functions f on $\mathbb{R}$ such that*

$$L^{r(\cdot)}(\mathbb{R},E)=\left\{f:\|f\|_{r(.),E}<\infty\right\},$$

where

$$\|f\|_{r(.),E}=\inf\left\{\lambda>0:\varrho_{r(.),E}(\frac{f}{\lambda})\leq 1\right\}$$

and

$$\varrho_{r(.),E}(f)=\int\limits_{\mathbb{R}}\|f(x)\|_E^{r(.)}\,dx.$$

The following properties proved by Cheng and Xu [7];

(i) $f\in L^{r(\cdot)}(\mathbb{R},E)\Leftrightarrow\|f(x)\|_E^{r(.)}\in L^1(\mathbb{R})\Leftrightarrow\|f(x)\|_E\in L^{r(\cdot)}(\mathbb{R})$

(ii) $L^{r(\cdot)}(\mathbb{R},E)$ *is a Banach space with respect to* $\|.\|_{r(.),E}$ *.*

(iii) $L^{r(\cdot)}(\mathbb{R},E)$ *is a generalization of the* $L^r(\mathbb{R},E)$ *spaces.*

(iv) *If* $E=\mathbb{R}$ *or* $\mathbb{C}$*, then* $L^{r(\cdot)}(\mathbb{R},E)=L^{r(\cdot)}(\mathbb{R})$ *.*

(v) *If* E *is reflexive and* $1<r^-\leq r^+<\infty$*, then* $L^{r(\cdot)}(\mathbb{R},E)$ *is reflexive.*

Theorem 2. *If E^* has the Radon-Nikodym Property (RNP), then the mapping $g\mapsto\varphi_g$, $\frac{1}{r(.)}+\frac{1}{q(.)}=1$, $L^{q(\cdot)}(\mathbb{R},E^*)\to\left(L^{r(\cdot)}(\mathbb{R},E)\right)^*$ which is defined by*

$$<\varphi_g,f>=\int\limits_{\mathbb{R}}<g,f>dx$$

for any $f\in L^{r(\cdot)}(\mathbb{R},E)$ is a linear isomorphism and

$$\|g\|_{q(.),E^*}\leq\left\|\varphi_g\right\|_{\left(L^{r(\cdot)}(\mathbb{R},E)\right)^*}\leq 2\,\|g\|_{q(.),E^*}\,.$$

Hence, the dual space $\left(L^{r(\cdot)}(\mathbb{R},E)\right)^*$ is isometrically isomorphic to $L^{q(\cdot)}(\mathbb{R},E^*)$, where E^* has RNP. In addition, for $f\in L^{r(\cdot)}(\mathbb{R},E)$ and $g\in L^{q(\cdot)}(\mathbb{R},E^*)$ (g defines a continuous linear functional), the dual pair $<f(.),g(.)>\in L^1(\mathbb{R})$ and Hölder inequality implies

$$\begin{aligned}\int\limits_{\mathbb{R}}|<f(.),g(.)>|\,dx &\leq \int\limits_{\mathbb{R}}\|f\|_E\,\|g\|_{E^*}\,dx\\ &\leq C\,\|f\|_{r(.),E}\,\|g\|_{q(.),E^*}\end{aligned}$$

for some $C>0$ [7].

3. VECTOR-VALUED VARIABLE EXPONENT AMALGAM SPACES

In this section, we define vector-valued variable exponent amalgam spaces $\left(L^{r(.)}(\mathbb{R},E),\ell^{s}\right)$. We also discuss some basic and significant properties of $\left(L^{r(.)}(\mathbb{R},E),\ell^{s}\right)$.

Definition 8. *Let $1\leq r(.)<\infty$, $1\leq s\leq\infty$ and $J_k=[k,k+1)$, $k\in\mathbb{Z}$. The vector-valued variable exponent amalgam spaces $\left(L^{r(.)}(\mathbb{R},E),\ell^{s}\right)$ are the normed space*

$$\left(L^{r(.)}(\mathbb{R},E),\ell^{s}\right)=\left\{f\in L_{loc}^{r(.)}(\mathbb{R},E):\|f\|_{\left(L^{r(.)}(\mathbb{R},E),\ell^{s}\right)}<\infty\right\},$$

where

$$\|f\|_{\left(L^{r(.)}(\mathbb{R},E),\ell^{s}\right)}=\left(\sum_{k\in\mathbb{Z}}\left\|f\chi_{J_k}\right\|_{r(.),E}^{s}\right)^{1/s},1\leq s<\infty$$

and

$$\|f\|_{\left(L^{r(.)}(\mathbb{R},E),\ell^{\infty}\right)}=\sup_{k}\left\|f\chi_{J_k}\right\|_{r(.),E}\ ,s=\infty.$$

It can be proved that $\left(L^{r(.)}(\mathbb{R},E),\ell^{s}\right)$ is a Banach space with respect to the norm $\|.\|_{\left(L^{r(.)}(\mathbb{R},E),\ell^{s}\right)}$ [21]. Moreover, $\left(L^{r(.)}(\mathbb{R},E),\ell^{s}\right)$ has some inclusions and embeddings similar to [3].

The proof of the following Theorem is proved by using techniques in Theorem 2.6 in [11], [p. 32, 29] and [p.359,19].

Theorem 3. *Let E^{*} has RNP and $1<r^{-}\leq r^{+}<\infty$ and $1<s<\infty$. Then the dual space of $\left(L^{r(.)}(\mathbb{R},E),\ell^{s}\right)$ is isometrically isomorphic to $\left(L^{q(.)}(\mathbb{R},E^{*}),\ell^{t}\right)$ for $\frac{1}{r(.)}+\frac{1}{q(.)}=1$ and $\frac{1}{s}+\frac{1}{t}=1$.*

Proof. Let $\{A_k\}_{k\in\mathbb{Z}}$ be a family of Banach spaces. We define

$$\ell^{s}(A_k)=\{x=(x_k):x_k\in A_k,\|x\|<\infty\}$$

where $\|x\|=\left(\sum_{k\in\mathbb{Z}}\|x_k\|_{A_k}^{s}\right)^{\frac{1}{s}}$. It can be seen that $\ell^{s}(A_k)$ is a Banach space under the norm $\|.\|$. It is also well known that the dual of $\ell^{s}(A_k)$ is $\ell^{t}(A_k^{*})$. Moreover, $\left(L^{r(.)}(\mathbb{R},E),\ell^{s}\right)$ is particular case of $\ell^{s}(A_k)$. Indeed, if we take $A_k=L^{r(.)}(J_k,E)$ and $J_k=[k,k+1)$, then the map $f\mapsto(f_k)$, $f_k=f\chi_{J_k}$ is an isometric isomorphism from $\left(L^{r(.)}(\mathbb{R},E),\ell^{s}\right)$ to $\ell^{s}\left(L^{r(.)}(J_k,E)\right)$. Hence, we have $\left(L^{r(.)}(\mathbb{R},E),\ell^{s}\right)^{*}\cong\left(L^{q(.)}(\mathbb{R},E^{*}),\ell^{t}\right)$ by Theorem 2. □

Corollary 1. *Let $1<r^{-}\leq r^{+}<\infty$ and $1<s<\infty$. If E is reflexive, then $\left(L^{r(.)}(\mathbb{R},E),\ell^{s}\right)$ is reflexive.*

Theorem 4. (***Generalized Hölder Inequality***) *Let E^* has RNP and $m^+ < \infty$, $1 \leq s \leq \infty$. If $\frac{1}{r(.)} + \frac{1}{q(.)} = \frac{1}{m(.)}$ and $\frac{1}{s} + \frac{1}{t} = \frac{1}{n}$, then there exists a $C > 0$ such that*

$$\|< f(.), g(.) >\|_{\left(L^{m(.)}(\mathbb{R}),\ell^n\right)} \leq C \|f\|_{\left(L^{r(.)}(\mathbb{R},E),\ell^s\right)} \|g\|_{\left(L^{q(.)}(\mathbb{R},E^*),\ell^t\right)}$$

and $< f(.), g(.) >\in \left(L^{m(.)}(\mathbb{R}), \ell^n\right)$ *for* $f \in \left(L^{r(.)}(\mathbb{R}, E), \ell^s\right)$, $g \in \left(L^{q(.)}(\mathbb{R}, E^*), \ell^t\right)$.

Proof. Let $\widetilde{f}(x) = \|f(x)\|_E$ and $\widetilde{g}(x) = \|g(x)\|_{E^*}$ be given for any $x \in \mathbb{R}$. If $f \in \left(L^{r(.)}(\mathbb{R}, E), \ell^s\right)$ and $g \in \left(L^{q(.)}(\mathbb{R}, E^*), \ell^t\right)$, then we have $\widetilde{f} \in \left(L^{r(.)}, \ell^s\right)$, $\widetilde{g} \in \left(L^{q(.)}, \ell^t\right)$ and $\left\|\widetilde{f}\right\|_{\left(L^{r(.)},\ell^s\right)} = \|f\|_{\left(L^{r(.)}(\mathbb{R},E),\ell^s\right)}$, $\|\widetilde{g}\|_{\left(L^{q(.)},\ell^t\right)} = \|g\|_{\left(L^{q(.)}(\mathbb{R},E^*),\ell^t\right)}$. Therefore, by using Hölder inequality for $L^{m(.)}$ [18], we can write the following inequality

$$\begin{aligned} \|< f(.), g(.) >\|_{m(.),J_k} &\leq \left\|\|f(.)\|_E \|g(.)\|_{E^*}\right\|_{m(.),J_k} \\ &\leq C \left\|\widetilde{f}\right\|_{r(.),J_k} \|\widetilde{g}\|_{q(.),J_k} \\ &= C \left\|f\chi_{J_k}\right\|_{r(.),E} \left\|g\chi_{J_k}\right\|_{q(.),E^*}. \end{aligned}$$

By Corollary 2.4 in [3] and Jensen's inequality for ℓ^sspaces, we obtain

$$\|< f(.), g(.) >\|_{\left(L^{m(.)}(\mathbb{R}),\ell^n\right)} \leq C \|f\|_{\left(L^{r(.)}(\mathbb{R},E),\ell^s\right)} \|g\|_{\left(L^{q(.)}(\mathbb{R},E^*),\ell^t\right)}.$$

This completes the proof. □

Definition 9. *We define $c_0(\mathbb{Z}) \subset l^\infty$ to be the linear space of $(a_k)_{k\in\mathbb{Z}}$ such that $\lim_k a_k = 0$, that is, given $\varepsilon > 0$ there exists a compact subset K of $\mathbb{R}$ such that $|a_k| < \varepsilon$ for all $k \notin K$.*

The vector-valued type variable exponent amalgam spaces $\left(L^{r(.)}(\mathbb{R}, E), c_0\right)$ are the normed spaces

$$\left(L^{r(.)}(\mathbb{R}, E), c_0\right) = \left\{f \in \left(L^{r(.)}(\mathbb{R}, E), \ell^\infty\right) : \left\{\left\|f\chi_{J_k}\right\|_{r(.),E}\right\}_{k\in\mathbb{Z}} \in c_0\right\},$$

where

$$\|f\|_{\left(L^{r(.)}(\mathbb{R},E),\ell^\infty\right)} = \sup_k \left\|f\chi_{J_k}\right\|_{r(.),E}$$

for $f \in \left(L^{r(.)}(\mathbb{R}, E), c_0\right)$ [29].

Proposition 1. *Let E^* has RNP and $m^+ < \infty$, $1 \leq s \leq \infty$. If $f \in \left(L^{r(.)}(\mathbb{R}, E), c_0\right)$ and $g \in \left(L^{q(.)}(\mathbb{R}, E^*), c_0\right)$, then there exists a $C > 0$ such that*

$$\|< f(.), g(.) >\|_{\left(L^{m(.)}(\mathbb{R}),\ell^\infty\right)} \leq C \|f\|_{\left(L^{r(.)}(\mathbb{R},E),\ell^\infty\right)} \|g\|_{\left(L^{q(.)}(\mathbb{R},E^*),\ell^\infty\right)}$$

and $< f(.), g(.) >\in \left(L^{m(.)}(\mathbb{R}), c_0\right)$ *for* $\frac{1}{r(.)} + \frac{1}{q(.)} = \frac{1}{m(.)}$.

Proof. If $f \in \left(L^{r(\cdot)}(\mathbb{R},E), c_0\right)$ and $g \in \left(L^{q(\cdot)}(\mathbb{R},E^*), c_0\right)$, then by Theorem 4 we can write $< f(.), g(.) > \in \left(L^{m(\cdot)}(\mathbb{R}), \ell^\infty\right)$ and

$$\| < f(.), g(.) > \|_{m(.),J_k} \leq C \left\| f\chi_{J_k} \right\|_{r(.),E} \left\| g\chi_{J_k} \right\|_{q(.),E^*},$$

where C does not depend on for any $k \in \mathbb{Z}$. This implies that

$$\lim_k \| < f(.), g(.) > \|_{m(.),J_k} \leq \lim_k C \left\| f\chi_{J_k} \right\|_{r(.),E} \lim_k \left\| g\chi_{J_k} \right\|_{q(.),E^*} = 0$$

and $< f(.), g(.) > \in \left(L^{m(\cdot)}(\mathbb{R}), c_0\right)$. If we use the definition of the norm $\|.\|_{\left(L^{r(\cdot)}(\mathbb{R},E),\ell^\infty\right)}$, then we get

$$\begin{aligned} \| < f(.), g(.) > \|_{\left(L^{r(\cdot)},\ell^\infty\right)} &= \sup_k \left\| f\chi_{J_k} \right\|_{r(.)} \\ &\leq C \sup_k \left\| f\chi_{J_k} \right\|_{r(.),E} \left\| g\chi_{J_k} \right\|_{q(.),E^*} \\ &\leq C \sup_k \left\| f\chi_{J_k} \right\|_{r(.),E} \sup_k \left\| g\chi_{J_k} \right\|_{r(.),E} \\ &= C \|f\|_{\left(L^{r(\cdot)}(\mathbb{R},E),\ell^\infty\right)} \|g\|_{\left(L^{q(\cdot)}(\mathbb{R},E^*),\ell^\infty\right)} \end{aligned}$$

□

Definition 10. *$L_c^{r(\cdot)}(\mathbb{R},E)$ denotes the functions f in $L^{r(\cdot)}(\mathbb{R},E)$ such that supp$f \subset \mathbb{R}$ is compact,that is,*

$$L_c^{r(\cdot)}(\mathbb{R},E) = \left\{ f \in L^{r(\cdot)}(\mathbb{R},E) : suppf \ \ compact \right\}.$$

Let $K \subset \mathbb{R}$ be given. The cardinality of the set

$$S(K) = \{J_k : J_k \cap K \neq \varnothing\}$$

is denoted by $|S(K)|$, where $\{J_k\}_{k\in\mathbb{Z}}$ is a collection of intervals.

Proposition 2. *If g belongs to $L_c^{r(\cdot)}(\mathbb{R},E)$, then*

(i) *$\|g\|_{\left(L^{r(\cdot)}(\mathbb{R},E),\ell^s\right)} \leq |S(K)|^{\frac{1}{s}} \|g\|_{r(.),E}$ for $1 \leq s < \infty$,*

(ii) *$\|g\|_{\left(L^{r(\cdot)}(\mathbb{R},E),\ell^\infty\right)} \leq |S(K)| \|g\|_{r(.),E}$ for $s = \infty$,*

(iii) *$L_c^{r(\cdot)}(\mathbb{R},E) \subset \left(L^{r(\cdot)}(\mathbb{R},E), \ell^s\right)$ for $1 \leq s \leq \infty$,*

where K is the compact support of g.

Proof. **(i)** Since K is compact, then $K \subset \bigcup_{i=1}^{|S(K)|} J_{k_i}$ and

$$\begin{aligned} \|g\|_{\left(L^{r(\cdot)}(\mathbb{R},E),\ell^s\right)} &= \left(\sum_{k\in\mathbb{Z}} \left\| g\chi_{J_k} \right\|_{r(.),E}^s \right)^{1/s} \\ &= \left(\sum_{J_{k_i}\in S(K)} \left\| g\chi_{J_k} \right\|_{r(.),E}^s \right)^{1/s} \\ &\leq |S(K)|^{\frac{1}{s}} \|g\|_{r(.),E} \end{aligned}$$

for $1 \leq s < \infty$, where the number of J_{k_i} is finite.

(ii) Let $s = \infty$.Then

$$\begin{aligned} \|g\|_{\left(L^{r(\cdot)}(\mathbb{R},E),\ell^s\right)} &= \sup_{k\in\mathbb{Z}} \left\|g\chi_{J_k}\right\|_{r(.),E} \\ &= \sup_{i=1,2,..|S(K)|} \left\|g\chi_{J_{k_i}}\right\|_{r(.),E} \\ &\leq |S(K)| \, \|g\|_{r(.),E} \, . \end{aligned}$$

□

Theorem 5. ***(i)*** *$L_c^{r(.)}(\mathbb{R},E)$ is subspace of $\left(L^{r(\cdot)}(\mathbb{R},E), c_0\right)$ for $1 \leq s < \infty$.*

(ii) *$C_0(\mathbb{R},E)$ is subspace of $\left(L^{r(\cdot)}(\mathbb{R},E), c_0\right)$.*

Proof. **(i)** Firstly, we show that $L_c^{r(.)}(\mathbb{R},E) \subset \left(L^{r(\cdot)}(\mathbb{R},E), c_0\right)$. Let $f \in L_c^{r(.)}(\mathbb{R},E)$ be given. Since f has compact support, then $\left\|f\chi_{J_k}\right\|_{r(.),E}$ is zero for all, but finitely many J_k. By definition of c_0, we get $\left\{\left\|f\chi_{J_k}\right\|_{r(.),E}\right\}_{k\in\mathbb{Z}} \in c_0$. Hence, $L_c^{r(.)}(\mathbb{R},E) \subset \left(L^{r(\cdot)}(\mathbb{R},E), c_0\right)$.

(ii) If $f \in C_0(\mathbb{R},E)$ and given $0 < \varepsilon < 1$, then there exists a compact set $K \subset \mathbb{R}$ such that $\|f(x)\|_E < \varepsilon$ for all $x \notin K$. Since K is compact, then $K \subset \cup_{i=1}^{n} J_{k_i}$ (n is finite) and $\left\|f(.)\chi_{J_k}\right\|_{r(.),E} < \varepsilon$ for all $k \neq k_i$, $i = 1, 2, ..n$. Indeed, by using $\varrho_{r(.),E}(f) \to 0 \Leftrightarrow \|f(.)\|_{r(.),E} \to 0$ $(r^+ < \infty)$, and $|J_k| = 1$ (measure of J_k) it is written that

$$\begin{aligned} \varrho_{r(.),E}(f) &= \int_{J_k} \|f(x)\|_E^{r(.)} \, dx \\ &\leq \varepsilon^{r^-} |J_k| \to 0. \end{aligned}$$

Therefore, we obtain $f \in \left(L^{r(\cdot)}(\mathbb{R},E), c_0\right)$ due to definition of norm of $\left(L^{r(\cdot)}(\mathbb{R},E), c_0\right)$. □

4. VECTOR-VALUED CLASSICAL AMALGAM SPACES

In this section, we consider that G is a locally compact Abelian group, and A is a commutative Banach algebra with Haar measure μ. By the Structure Theorem, $G = \mathbb{R}^a \times G_1$, where a is a nonnegative integer and G_1 is a locally compact abelian group which contains an open compact subgroup H. Let $I = [0,1)^a \times H$ and $J = \mathbb{Z}^a \times T$, where T is a transversal of H in G_1, i.e. $G_1 = \bigcup_{t\in T}(t+H)$ is a coset decomposition of G_1. For $\alpha \in J$ we define $I_\alpha = \alpha + I$, and therefore G is equal to the disjoint union of relatively compact sets I_α. We normalize μ so that $\mu(I) = \mu(I_\alpha) = 1$ for all α [11], [29].

Definition 11. *Let $1 \leq p, q < \infty$. The vector-valued classical amalgam spaces $(L^p(G,A), \ell^q)$ are the normed space*

$$(L^p(G,A), \ell^q) = \left\{ f \in L^p_{loc}(G,A) : \|f\|_{(L^p(G,A),\ell^q)} < \infty \right\},$$

where

$$\|f\|_{(L^p(G,A),\ell^q)} = \left(\sum_{\alpha \in J} \left\| f\chi_{I_\alpha} \right\|^q_{p,A} \right)^{1/q}, 1 \leq p, q < \infty.$$

Now we give Young's inequality for vector-valued amalgam spaces.

Theorem 6. ([22])*Let $1 \leq p_1, q_1, p_2, q_2 < \infty$. If $f \in (L^{p_1}(G,A), \ell^{q_1})$ and $g \in (L^{p_2}(G,A), \ell^{q_2})$, then $f * g \in (L^{r_1}(G,A), \ell^{r_2})$, where $\frac{1}{p_1} + \frac{1}{p_2} \geq 1$, $\frac{1}{q_1} + \frac{1}{q_2} \geq 1$, $\frac{1}{r_1} = \frac{1}{p_1} + \frac{1}{p_2} - 1$ and $\frac{1}{r_2} = \frac{1}{q_1} + \frac{1}{q_2} - 1$. Moreover, there exists a $C > 0$ such that*

$$\|f * g\|_{(L^{r_1}(G,A),\ell^{r_2})} \leq C \|f\|_{(L^{p_1}(G,A),\ell^{q_1})} \|g\|_{(L^{p_2}(G,A),\ell^{q_2})}.$$

Definition 12. *Let A be a Banach algebra. A Banach space B is said to be a Banach $A-$module if there exists a bilinear operation $\cdot : A \times B \to B$ such that*

(i) $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ for all $f, g \in A$, $h \in B$.

(ii) For some constant $C \geq 1$, $\|f \cdot h\|_B \leq C \|f\|_A \|h\|_B$ for all $f \in A$, $h \in B$.

By Theorem 6, we have the following inequality

$$\|f * g\|_{(L^p(G,A),\ell^q)} \leq C \|f\|_{(L^p(G,A),\ell^q)} \|g\|_{(L^1(G,A),\ell^1)} = C \|f\|_{(L^p(G,A),\ell^q)} \|g\|_{L^1(G,A)}$$

for all $f \in (L^p(G,A), \ell^q)$ and $g \in L^1(G,A)$, where $C \geq 1$, i.e. the amalgam space $(L^p(G,A), \ell^q)$ is a Banach $L^1(G,A)-$module with respect to convolution. Moreover, it is easy to see that the amalgam space $\left(L^p(G,A), \ell^1\right)$ is a Banach algebra under convolution $p \geq 1$, if we define the norm $|||f|||_{(L^p(G,A),\ell^1)} = C \|f\|_{(L^p(G,A),\ell^1)}$ for $\left(L^p(G,A), \ell^1\right)$. Recall that $\left(L^p(G,A), \ell^1\right) \subset L^1(G,A)$.

Definition 13. *Let V and W be two Banach modules over a Banach algebra A. Then a multiplier from V into W is a bounded linear operator T from V into W, which commutes with module multiplication, i.e. $T(av) = aT(v)$ for $a \in A$ and $v \in V$. We denote by $Hom_A(V,W)$ the space of all multipliers from V into W.*

Let V and W be left and right Banach $A-$modules, respectively, and $V \otimes_\gamma W$ be the projective tensor product of V and W [6], [28]. If K is the closed linear subspace of $V \otimes_\gamma W$, which is spanned by all elements of the form $av \otimes w - v \otimes aw$, $a \in A$, $v \in V, w \in W$, then the $A-$module tensor product $V \otimes_A W$ is defined to be the quotient Banach space $(V \otimes_\gamma W)/K$. Every element t of $(V \otimes_\gamma W)/K$ can be written

$$t = \sum_{i=1}^{\infty} v_i \otimes w_i, \; v_i \in V, w_i \in W,$$

where $\sum_{i=1}^{\infty} \|v_i\| \|w_i\| < \infty$. Moreover, $(V \otimes_\gamma W)/K$ is a normed space according to

$$\|t\| = \inf \left\{ \sum_{i=1}^{\infty} \|v_i\| \|w_i\| < \infty \right\},$$

where the infimum is taken over all possible representations for t [26]. It is known that

$$Hom_A(V, W^*) \cong (V \otimes_A W)^*,$$

where W^* *is dual of* W *(Corollary 2.13, [25]). The linear functional* T *on* $Hom_A(V, W^*)$*, which corresponds to* $t \in V \otimes_A W$ *gets the value*

$$< t, T >= \sum_{i=1}^{\infty} < w_i, Tv_i > .$$

Moreover, it is well known that the ultraweak-operator topology on* $Hom_A(V, W^*)$ *corresponds to the weak*-topology on* $(V \otimes_A W)^*$ *[26].*

5. THE MULTIPLIERS OF THE SPACE $\mathbf{A}_{p_1,p_2}^{q_1,q_2}(G, A)$

By Theorem 6, a linear operator $b : (L^{p_1}(G, A), \ell^{q_1}) \times (L^{p_2}(G, A), \ell^{q_2}) \to (L^{r_1}(G, A), \ell^{r_2})$ can be defined by

$$b(f, g) = \widetilde{f} * g, \ f \in (L^{p_1}(G, A), \ell^{q_1}), g \in (L^{p_2}(G, A), \ell^{q_2}),$$

where $\widetilde{f}(x) = f(-x)$ and $\left\|\widetilde{f}\right\|_{(L^{p_1}(G,A),\ell^{q_1})} = \|f\|_{(L^{p_1}(G,A),\ell^{q_1})}$. It is easy to see that $\|b\| \leq C$. Furthermore, there exists a bounded linear operator B from $(L^{p_1}(G, A), \ell^{q_1}) \otimes_\gamma (L^{p_2}(G, A), \ell^{q_2})$ into $(L^{r_1}(G, A), \ell^{r_2})$ such that $B(f \otimes g) = b(f, g)$, where $f \in (L^{p_1}(G, A), \ell^{q_1}), g \in (L^{p_2}(G, A), \ell^{q_2})$ and $\|B\| \leq C$ by Theorem 6 in [6].

Definition 14. $A_{p_1,p_2}^{q_1,q_2}(G, A)$ *denotes the range of* B *with the quotient norm. Hence, we write*

$$A_{p_1,p_2}^{q_1,q_2}(G, A) = \left\{ h = \sum_{i=1}^{\infty} \widetilde{f}_i * g_i : \sum_{i=1}^{\infty} \|f_i\|_{(L^{p_1}(G,A),\ell^{q_1})} \|g_i\|_{(L^{p_2}(G,A),\ell^{q_2})} < \infty \right\}$$

for $f_i \in (L^{p_1}(G, A), \ell^{q_1}), g_i \in (L^{p_2}(G, A), \ell^{q_2})$ *and*

$$|||h||| = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\|_{(L^{p_1}(G,A),\ell^{q_1})} \|g_i\|_{(L^{p_2}(G,A),\ell^{q_2})} : h = \sum_{i=1}^{\infty} \widetilde{f}_i * g_i \right\}.$$

It is clear that $A_{p_1,p_2}^{q_1,q_2}(G, A) \subset (L^{r_1}(G, A), \ell^{r_2})$ *and* $\|h\|_{(L^{r_1}(G,A),\ell^{r_2})} \leq C\,|||h|||$. *Moreover, by using the technique given in Theorem 2.4 by [12],* $A_{p_1,p_2}^{q_1,q_2}(G, A)$ *can be showed a Banach space with respect to* $|||.|||$.

Let K *be the closed linear subspace of* $(L^{p_1}(G, A), \ell^{q_1}) \otimes_\gamma (L^{p_2}(G, A), \ell^{q_2})$*, which is spanned by all elements of the form* $(\varphi * f) \otimes g - f \otimes (\widetilde{\varphi} * g)$*, where* $f \in (L^{p_1}(G, A), \ell^{q_1})$*,* $g \in (L^{p_2}(G, A), \ell^{q_2})$ *and* $\varphi \in L^{p_1}(G, A)$*. Then the* $L^1(G, A)$*–module tensor product* $(L^{p_1}(G, A), \ell^{q_1}) \otimes_{L^1(G,A)} (L^{p_2}(G, A), \ell^{q_2})$ *is defined to be the quotient Banach space* $((L^{p_1}(G, A), \ell^{q_1}) \otimes_\gamma (L^{p_2}(G, A), \ell^{q_2}))/K$*. We define the norm*

$$\|h\| = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\|_{(L^{p_1}(G,A),\ell^{q_1})} \|g_i\|_{(L^{p_2}(G,A),\ell^{q_2})} : h = \sum_{i=1}^{\infty} f_i \otimes g_i \right\}.$$

for $h\in\left(L^{p_1}(G,A),\ell^{q_1}\right)\otimes_{L^1(G,A)}\left(L^{p_2}(G,A),\ell^{q_2}\right)$ *Also, it is well known that this space is a Banach space [26].*

In the following Lemma, we use numbers p *and* q *given in Lemma 3.2 in [1].*

Lemma 1. *Let* $1\le p_1,q_1,p_2,q_2<\infty$, $\frac{1}{p_1}+\frac{1}{p_2}\ge 1$, $\frac{1}{q_1}+\frac{1}{q_2}\ge 1$, $\frac{1}{r_1}=\frac{1}{p_1}+\frac{1}{p_2}-1$ *and* $\frac{1}{r_2}=\frac{1}{q_1}+\frac{1}{q_2}-1$, $p=\frac{p_1p_2'}{p_1p_2'+p_1-p_2'}$ *and* $q=\frac{q_1q_2'}{q_1q_2'+q_1-q_2'}$, *where* $\frac{1}{p_2}+\frac{1}{p_2'}=1$ *and* $\frac{1}{q_2}+\frac{1}{q_2'}=1$. *If we define* $T_\varphi f=f*\varphi$ *for* $f\in\left(L^{p_1}(G,A),\ell^{q_1}\right)$ *and* $\varphi\in C_C(G,A)$, *then we have* $T_\varphi\in Hom_{L^1(G,A)}\left(\left(L^{p_1}(G,A),\ell^{q_1}\right),\left(L^{p_2'}(G,A^*),\ell^{q_2'}\right)\right)$ *and the inequality*

$$\|T_\varphi\|\le C\,\|\varphi\|_{(L^p(G,A),\ell^q)}$$

for some $C>0$.

Proof. Let $f\in\left(L^{p_1}(G,A),\ell^{q_1}\right)$ and $\varphi\in C_C(G,A)\subset\left(L^p(G,A),\ell^q\right)$ be given. Due to $p=\frac{p_1p_2'}{p_1p_2'+p_1-p_2'}$ and $q=\frac{q_1q_2'}{q_1q_2'+q_1-q_2'}$, $\frac{1}{p_1}+\frac{1}{p}=1+\frac{1}{p_2'}>1$ and $\frac{1}{r_1}=\frac{1}{p_1}+\frac{1}{p}-1$, $\frac{1}{q_1}+\frac{1}{q}=1+\frac{1}{q_2'}>1$ and $\frac{1}{r_2}=\frac{1}{q_1}+\frac{1}{q}-1$, we have $r_1=p_2'$ and $r_2=q_2'$. Therefore, by Theorem 6 we obtain $f*\varphi\in\left(L^{r_1}(G,A),\ell^{r_2}\right)=\left(L^{p_2'}(G,A),\ell^{q_2'}\right)$ and

$$\|T_\varphi f\|_{\left(L^{p_2'}(G,A),\ell^{q_2'}\right)}\le C\,\|f\|_{(L^{p_1}(G,A),\ell^{q_1})}\,\|\varphi\|_{(L^p(G,A),\ell^q)},$$

i.e. $T_\varphi f$ is continuous. Also, we can write the inequality

$$\|T_\varphi\|\le C\,\|\varphi\|_{(L^p(G,A),\ell^q)}$$

for some $C>0$.

Definition 15. *A locally compact Abelian group* G *is said to satisfy the property* $P_{(p_1,p_2)}^{(q_1,q_2)}$ *if every element of* $Hom_{L^1(G,A)}\left(\left(L^{p_1}(G,A),\ell^{q_1}\right),\left(L^{p_2'}(G,A^*),\ell^{q_2'}\right)\right)$ *can be approximated in the ultraweak*-operator topology by operators* $T_\varphi,\varphi\in C_C(G,A)$.

□

Theorem 7. *Let* G *be a locally compact Abelian group. If* $\frac{1}{p_1}+\frac{1}{p_2}\ge 1$, $\frac{1}{q_1}+\frac{1}{q_2}\ge 1$, $\frac{1}{r_1}=\frac{1}{p_1}+\frac{1}{p_2}-1$ *and* $\frac{1}{r_2}=\frac{1}{q_1}+\frac{1}{q_2}-1$, *then the following statements are equivalent:*

(i) *G satisfies the property* $P_{(p_1,p_2)}^{(q_1,q_2)}$.

(ii) *The kernel of* B *is* K *such that*

$$\left(L^{p_1}(G,A),\ell^{q_1}\right)\otimes_{L^1(G,A)}\left(L^{p_2}(G,A),\ell^{q_2}\right)\cong\mathbf{A}_{p_1,p_2}^{q_1,q_2}(G,A).$$

Proof. Since $B\left((\varphi*f)\otimes g-f\otimes(\widetilde{\varphi}*g)\right)=(\varphi*f)^{\sim}*g-\widetilde{f}*(\widetilde{\varphi}*g)=0$, then $K\subset KerB$. Assume that G satisfies the property $P_{(p_1,p_2)}^{(q_1,q_2)}$. To display $KerB\subset K$ it is enough to prove that $K^\perp\subset(KerB)^\perp$ due to the fact that K is the closed linear subspace of $\left(L^{p_1}(G,A),\ell^{q_1}\right)\otimes_\gamma\left(L^{p_2}(G,A),\ell^{q_2}\right)$. It is well known that

$$K^\perp\cong\left(\left(\left(L^{p_1}(G,A),\ell^{q_1}\right)\otimes_\gamma\left(L^{p_2}(G,A),\ell^{q_2}\right)\right)/K\right)^*$$

[8]. Also, we write

$$K^{\perp} \cong \left(\left(L^{p_1}(G,A), \ell^{q_1} \right) \otimes_{L^1(G,A)} \left(L^{p_2}(G,A), \ell^{q_2} \right) \right)^* .$$

Moreover, by using Corollary 2.13 in [25], we obtain

$$K^{\perp} \cong Hom_{L^1(G,A)} \left(\left(L^{p_1}(G,A), \ell^{q_1} \right), \left(L^{p_2'}(G,A^*), \ell^{q_2'} \right) \right).$$

Thus, there is a multiplier $T \in Hom_{L^1(G,A)} \left(\left(L^{p_1}(G,A), \ell^{q_1} \right), \left(L^{p_2'}(G,A^*), \ell^{q_2'} \right) \right)$ which corresponds to $F \in K^{\perp}$ such that

$$< t, F >= \sum_{i=1}^{\infty} < g_i, Tf_i >,$$

where $t \in KerB$, $t = \sum_{i=1}^{\infty} f_i \otimes g_i$, $\sum_{i=1}^{\infty} \|f_i\|_{(L^{p_1}(G,A),\ell^{q_1})} \|g_i\|_{(L^{p_2}(G,A),\ell^{q_2})} < \infty$ and $f_i \in \left(L^{p_1}(G,A), \ell^{q_1} \right), g_i \in \left(L^{p_2}(G,A), \ell^{q_2} \right)$. Also, due to $t \in KerB$ we get

$$B\left(\sum_{i=1}^{\infty} f_i \otimes g_i \right) = \sum_{i=1}^{\infty} \widetilde{f_i} * g_i = 0. \tag{5.1}$$

Now, we show that

$$< t, F >= \sum_{i=1}^{\infty} < g_i, Tf_i >= 0. \tag{5.2}$$

Furthermore, since G satisfies the property $P_{(p_1,p_2)}^{(q_1,q_2)}$, then there exists a net $\left\{ \left(\varphi_j \right) : j \in I \right\} \subset C_C(G,A)$ such that the operators T_{φ_j} defined in Lemma 1 converges to T in the ultraweak*-operator topology, that is,

$$\lim_{j \in I} \sum_{i=1}^{\infty} < g_i, T_{\varphi_j} f_i >= \lim_{j \in I} \sum_{i=1}^{\infty} < g_i, f_i * \varphi_j >= \sum_{i=1}^{\infty} < g_i, Tf_i > .$$

So, to obtain (5.2), it is enough to show that

$$\sum_{i=1}^{\infty} < g_i, f_i * \varphi_j >= 0$$

for all $j \in I$. It can be seen easily

$$\sum_{i=1}^{\infty} < g_i, f_i * \varphi_j >= \sum_{i=1}^{\infty} < \widetilde{f_i} * g_i, \varphi_j > . \tag{5.3}$$

If we use the equalities (5.1), (5.3) and Hölder inequality for amalgam spaces, then we have

$$\left| \sum_{i=1}^{\infty} < g_i, f_i * \varphi_j > \right| \leq \left\| \sum_{i=1}^{\infty} \widetilde{f_i} * g_i \right\|_{(L^{r_1}(G,A),\ell^{r_2})} \left\| \varphi_j \right\|_{\left(L^{r_1'}(G,A),\ell^{r_2'} \right)} = 0,$$

where $\sum_{i=1}^{\infty} \widetilde{f_i} * g_i \in \left(L^{r_1}(G,A), \ell^{r_2} \right)$ and $\varphi_j \in C_C(G,A) \subset \left(L^{r_1'}(G,A), \ell^{r_2'} \right)$. Therefore, $< t, F >= 0$ for all $t \in KerB$. That means $F \in (KerB)^{\perp}$. Hence $KerB = K$.

By definition of $\mathbf{A}_{p_1,p_2}^{q_1,q_2}(G,A)$ and the First Isomorphism Theorem, we get

$$\left(L^{p_1}(G,A), \ell^{q_1}\right) \otimes_\gamma \left(L^{p_2}(G,A), \ell^{q_2}\right)/K \cong \mathbf{A}_{p_1,p_2}^{q_1,q_2}(G,A)$$

and

$$\left(L^{p_1}(G,A), \ell^{q_1}\right) \otimes_{L^1(G,A)} \left(L^{p_2}(G,A), \ell^{q_2}\right) = \left(L^{p_1}(G,A), \ell^{q_1}\right) \otimes_\gamma \left(L^{p_2}(G,A), \ell^{q_2}\right)/K.$$

This proves

$$\left(L^{p_1}(G,A), \ell^{q_1}\right) \otimes_{L^1(G,A)} \left(L^{p_2}(G,A), \ell^{q_2}\right) \cong \mathbf{A}_{p_1,p_2}^{q_1,q_2}(G,A).$$

Suppose conversely that $KerB = K$. If we show that the operators of the form T_φ for $\varphi \in C_C(G,A)$ are dense in $Hom_{L^1(G,A)}\left(\left(L^{p_1}(G,A), \ell^{q_1}\right), \left(L^{p_2'}(G,A^*), \ell^{q_2'}\right)\right)$ in the ultraweak*-operator topology, then we finish the proof. Hence, it is sufficient to prove that the corresponding functionals are dense in

$$\left(\left(L^{p_1}(G,A), \ell^{q_1}\right) \otimes_{L^1(G,A)} \left(L^{p_2}(G,A), \ell^{q_2}\right)\right)^*$$

in the weak* topology by Theorem 1.4 in [26]. Let M be set of the linear functionals corresponding to the operators T_φ. Let $t \in KerB$ and $F \in M$. Since

$$\left(\left(L^{p_1}(G,A), \ell^{q_1}\right) \otimes_{L^1(G,A)} \left(L^{p_2}(G,A), \ell^{q_2}\right)\right)^* \cong K^\perp,$$

then $F \in M \subset K^\perp$ and $< t, F >= 0$. Hence we find $KerB \subset M^\perp$. Conversely, let $t \in M^\perp$. Since $M^\perp \subset \left(L^{p_1}(G,A), \ell^{q_1}\right) \otimes_{L^1(G,A)} \left(L^{p_2}(G,A), \ell^{q_2}\right)$, then there exist $f_i \in \left(L^{p_1}(G,A), \ell^{q_1}\right)$, $g_i \in \left(L^{p_2}(G,A), \ell^{q_2}\right)$ such that

$$t = \sum_{i=1}^{\infty} f_i \otimes g_i, \quad \sum_{i=1}^{\infty} \|f_i\|_{\left(L^{p_1}(G,A),\ell^{q_1}\right)} \|g_i\|_{\left(L^{p_2}(G,A),\ell^{q_2}\right)} < \infty$$

and $< t, F >= 0$ for all $F \in M$. Also, there is an operator

$$T_\varphi \in Hom_{L^1(G,A)}\left(\left(L^{p_1}(G,A), \ell^{q_1}\right), \left(L^{p_2'}(G,A^*), \ell^{q_2'}\right)\right)$$

corresponding to F such that

$$< t, F >=< t, T_\varphi >= \sum_{i=1}^{\infty} < g_i, T_\varphi f_i >= \sum_{i=1}^{\infty} < \widetilde{f_i} * g_i, \varphi_j >$$

$$=< \sum_{i=1}^{\infty} \widetilde{f_i} * g_i, \varphi_j >= 0$$

by using (5.2). Therefore, we obtain

$$B(t) = \sum_{i=1}^{\infty} \widetilde{f_i} * g_i = 0$$

and $M^\perp \subset KerB$. Consequently, $KerB = M^\perp$. This completes the theorem.

Corollary 2. *Let G be a locally compact Abelian group. If $\frac{1}{p_1} + \frac{1}{p_2} \geq 1$, $\frac{1}{q_1} + \frac{1}{q_2} \geq 1$, $\frac{1}{r_1} = \frac{1}{p_1} + \frac{1}{p_2} - 1$ and $\frac{1}{r_2} = \frac{1}{q_1} + \frac{1}{q_2} - 1$ and G satisfies the property $P_{(p_1,p_2)}^{(q_1,q_2)}$, then*

$$Hom_{L^1(G,A)}\left(\left(L^{p_1}(G,A), \ell^{q_1}\right), \left(L^{p_2'}(G,A^*), \ell^{q_2'}\right)\right) \cong \left(\mathbf{A}_{p_1,p_2}^{q_1,q_2}(G,A)\right)^*.$$

□

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Current address, Ismail AYDIN: Sinop University, Faculty of Sciences and Letters Department of Mathematics, Sinop, Turkey.
E-mail address: iaydin@sinop.edu.tr iaydinmath@gmail.com