Article electronically published on February 17, 2017

Commun.Fac.Sci.Univ.Ank.Series A1
Volume 66, Number 2, Pages 141–161 (2017)
DOI: 10.1501/Commua1_0000000808
ISSN 1303–5991

http://communications.science.ankara.edu.tr/index.php?series=A1

ROBUST PARAMETER ESTIMATION FOR THE MARSHALL-OLKIN EXTENDED BURR XII DISTRIBUTION

YEŞIM GÜNEY AND OLCAY ARSLAN

Abstract. In this paper, we consider the parameter estimation of the Marshall–Olkin extended Burr XII (MOEBXII) distribution, which is a generalization of the Burr XII distribution. For the estimation of the parameters in the MOEBXII, maximum likelihood (ML) is available. However, this is not robust estimator. In this paper we proposed a robust estimator based on M estimation method to estimate the parameters of the MOEBXII distribution. We perform a small simulation study to illustrate the performance of proposed method. We also reanalyze two data sets to asses the capability of the robust estimators over the ML and LS estimators.

1. Introduction

The Burr XII distribution [3] appears very often in practice when modelling unimodal data. Several researchers have investigated various inference problems using Burr distribution as it has been found useful in the study of actuarial science [11], economics [15], life testing and reliability, [1], [16], [17], failure time modeling [8] among others. The relationship between the Burr distribution and the various other distributions, is summarized by [18] and [21].

In the literature, various methods have been used to generalize Burr XII distribution. In addition Marshall and Olkin [14] introduced a method of adding a new parameter into a family of distributions. The resulting distribution is known as Marshall Olkin extended distribution. It is obtained as follows:

Let $\overline{F}(x) = 1 - F(x)$ denote the survival function of a continuous random variable X. Then, the corresponding Marshall-Olkin (MO) extended distribution has survival function defined by

$$\overline{F}(x) = \frac{\alpha \overline{F}(x)}{1 - \overline{\alpha}\overline{F}(x)} \tag{1.1}$$

Received by the editors: December 10, 2016, Accepted: February 10, 2017.

2010 *Mathematics Subject Classification.* Primary 05C38, 15A15; Secondary 05A15, 15A18.

Key words and phrases. Marshall-Olkin extended Burr XII (MOEBXII) distribution, robust estimator,maximum likelihood estimator, least squares estimator.

©2017 Ankara University

where $\alpha > 0$ and $\overline{\alpha} = 1 - \alpha$. The new family contains the initial family as a particular case, obtained when $\alpha = 1$.

There have been various studies reported in the literature dealing with the parameter estimation methods for Burr XII distribution. ML estimation of parameters for fitting Burr distribution to life test data has been studied by [22] and [23]. ML and maximum product of spacings (MPS) methods were compared by [19]. In addition, estimation of parameters in the presence of outliers for a Burr XII distribution with LS, ML and MPS methods were compared by [9]. The minimum variance linear unbiased estimators (MVLUE), the best linear invariant estimators (BLIE) and the maximum likelihood estimators (MLE) based on n-selected generalized order statistics are presented for the parameters of the Burr XII distribution by [13]. An alternative robust estimation methods based on M estimators and optimal B-robust estimation method for the parameters of Burr XII distribution have been proposed by [4] and [5], respectively. However, there is not much work for MOEBXII distribution. The parameters of MOEBXII distribution was estimated by using ML estimation method by [2]. However, it is well established that in the presence of outliers in the data, the traditional methods do not provide reliable estimations. Therefore robust estimation methods can be used for the parameters of the MOEBXII distribution if the data contains outliers.

In this paper, we propose a robust estimation procedures based on M-estimation method to estimate the parameters of the MOEBXII distribution. This is done by changing LS objective function with robust objective function and minimizing it. We compare the performance of the method with the ML and LS estimation methods by a simulation study and real data examples.

The rest of the paper is organized as follows; In section 2, we described the MOEBXII distribution. The maximum likelihood estimator and least square estimator are provided in sections 3. In section 4, simulation results are presented. In Section 5, we reanalyze two data sets for illustrative purpose. Finally, conclusions are given in section 6.

2. Marshall-Olkin Extended Burr XII Distribution

A random variable X is said to have a Burr XII distribution with shape parameters $c > 0$ and $k > 0$ if its probability density function (pdf) is given by

$$f(x; c, k) = ck \frac{x^{(c-1)}}{(1 + x^c)^{k+1}}, \ x \geq 0. \tag{2.1}$$

The cumulative density function (cdf) of X is given by

$$F(x; c, k) = 1 - \frac{1}{(1 + x^c)^k}, x \geq 0. \tag{2.2}$$

Substituting (2.2) in (1.1) we obtain a Marshall-Olkin Extended Burr XII distribution denoted by $MOEBXII(\alpha, c, k)$ with the following pdf and cdf

$$f(x;\alpha,c,k) = \alpha ck \frac{x^{(c-1)}(1+x^c)^{-(k+1)}}{\left[1-(1-\alpha)\,(1+x^c)^{-k}\right]^2},\ x \geq 0. \tag{2.3}$$

$$F(x;\alpha,c,k) = \frac{1-(1+x^c)^{-k}}{1-(1-\alpha)\,(1+x^c)^{-k}}, x \geq 0. \tag{2.4}$$

where α,c and $k > 0$ [2]. Note that the MOEBXII distribution is an extended model to analyze more complex data and it generalizes some of the distributions. In particular for $\alpha = 1$ MOEBXII becomes Burr XII distribution with two parameters c and k. And also, when $c = 1$, the MOEBXII becomes the Marshall-Olkin extended Lomax distribution. Clearly, MOEBXII distribution is more flexible than the Burr XII distribution, because of the presence of the shape parameter. Figure 1 shows the plots of pdf for MOEBXII distribution for some values of the parameters. For more details, see [2].

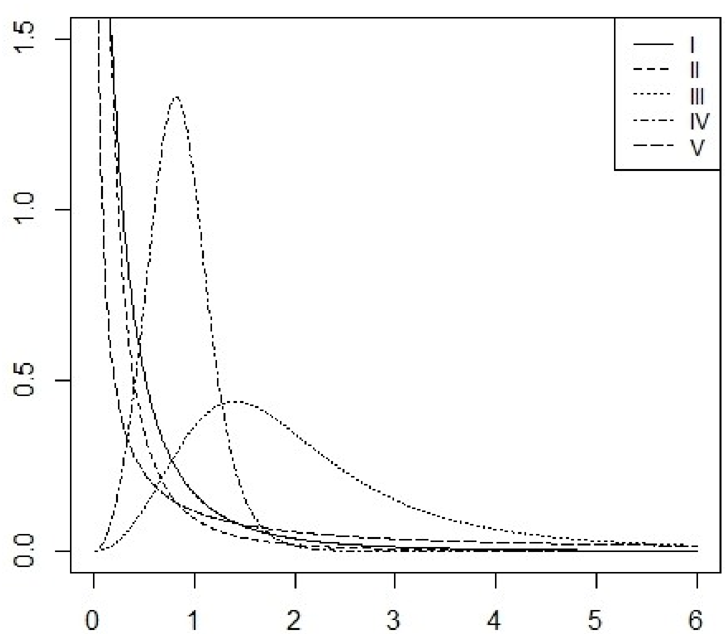

FIGURE 1. Plot of pdf of the MOEBXII distribution. (I)$\alpha = 1, c = 0.9, k = 3$, (II)$\alpha = 0.5, c = 0.9, k = 3$, (III)$\alpha = 5, c = 3, k = 0.9$, (IV)$\alpha = 3, c = 3, k = 3$, (V)$\alpha = 0.8, c = 0.7, k = 0.8$

3. Estimation of the Parameters of MOEBXII Distribution

In this study to estimate α, c and k, we consider ML, LS and robust estimation methods.

3.1. Maximum Likelihood Estimation of MOEBXII Distribution. Let $\underline{x} = (x_1, x_2, ..., x_n)$ be a random sample from $MOEBXII(\alpha, c, k)$. In order to estimate the parameters of the MOEBXII distribution, the log-likelihood of the sample is maximized with respect to the parameters. Log-likelihood function can be written as

$$\begin{aligned} l(\alpha, c, k) &= n \log(\alpha c k) + (c-1) \sum_{i=1}^{n} \log x_i - (k+1) \sum_{i=1}^{n} \log(1 + x_i^c) \\ &\quad -2 \sum_{i=1}^{n} \log\left(1 - (1-\alpha)(1 + x_i^c)^{-k}\right) \end{aligned} \tag{3.1}$$

The associated nonlinear loglikelihood system for MLE's is

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - 2 \sum_{i=1}^{n} \frac{(1 + x_i^c)^{-k}}{1 - (1-\alpha)(1 + x_i^c)^{-k}} = 0 \tag{3.2}$$

$$\begin{aligned} \frac{\partial l}{\partial c} &= \frac{n}{c} + \sum_{i=1}^{n} \log x_i - (k+1) \sum_{i=1}^{n} \frac{x_i^c \log(x_i)}{(1 + x_i^c)} \\ &\quad -2k(1-\alpha) \sum_{i=1}^{n} \frac{x_i^c (1 + x_i^c)^{-(k+1)} \log(x_i)}{1 - (1-\alpha)(1 + x_i^c)^{-k}} = 0 \end{aligned} \tag{3.3}$$

$$\begin{aligned} \frac{\partial l}{\partial k} &= \frac{n}{k} - \sum_{i=1}^{n} \log(1 + x_i^c) - (k+1) \sum_{i=1}^{n} \frac{x_i^c \log(x_i)}{(1 + x_i^c)} \\ &\quad -2(1-\alpha) \sum_{i=1}^{n} \frac{(1 + x_i^c)^{-k} \log(1 + x_i^c)}{1 - (1-\alpha)(1 + x_i^c)^{-k}} = 0. \end{aligned} \tag{3.4}$$

Notice that there are no explicit solutions to (3.2),(3.3) and (3.4). Hence, numerical methods are applied to solve the required equations.

3.2. Least Squares Estimation Method of MOEBXII Distribution. In this section we will discuss the least squares method for estimating α, c and k. As for the Burr XII distribution [4], LS estimation method can be used as an alternative to the ML estimation method to estimate the parameters of the MOEBXII distribution.

The LS method is a combination of parametric (F) and non-parametric $\left(\widehat{F}\right)$ distribution functions. The procedure attempts to minimize the following function with respect to α, c and k

$$S(\alpha,c,k) = \sum_{i=1}^{n}\left(\widehat{F}\left(x_i\right) - F\left(x_i\right)\right)^2 \tag{3.5}$$

$$= \sum_{i=1}^{n}\left(\widehat{F}\left(x_i\right) - \frac{1-(1+x_i^c)^{-k}}{1-(1-\alpha)\left(1+x_i^c\right)^{-k}}\right)^2. \tag{3.6}$$

Since the cdf of MOEBXII does not have a linear form according to the parameters, it will be difficult to minimize the equation (3.6). For this reason, we get the linear form of $F(x)$. To obtain the linear form of $F(x)$, we use the following transformation

$$\log\log\left(\frac{1}{1-F(x)}\right) = \log\log(\alpha-1) - \log\log(\alpha) + \log(k) + \log\log(1+x_i^c).$$

Instead of minimizing the squares of the difference between the $\widehat{F}(\cdot)$ and $F(\cdot)$, we minimize the squares of the difference between the linear form of $F(\cdot)$ and the same transformation of $\widehat{F}(\cdot)$.

On the other hand, $\widehat{F}(x)$ is unknown, we use $\widehat{F}_{X_{(i)}}(x)$ as follows

$$\widehat{F}_{X_{(i)}}(x) = \frac{i-0.5}{n}, \quad i = 1,2,...,n \tag{3.7}$$

where $X_{(i)}$ is the i. order statistics of the random sample of the size n from MOEBXII distribution. Hence for the MOEBXII distribution, to obtain thee LS estimates $\widehat{\alpha}_{LS}$, $\widehat{c}_{LS}$,and $\widehat{k}_{LS}$ of the parameters α, c and k we can define the following objective function:

$$S(\alpha,c,k) = \sum_{i=1}^{n}\left(y_{(i)} - \log\log(\alpha-1) + \log\log(\alpha) - \log(k) - \log\log(1+x_{(i)}^c)\right)^2 \tag{3.8}$$

where $y_{(i)} = \log\log\left(\frac{1}{1-\widehat{F}_{X_{(i)}}(x)}\right)$. The goal is to find α, c and k that minimize the objective function. This requires us to find the values of α, c and k such that

$$\sum_{i=1}^{n}\left(y_{(i)} - \log\log(\alpha-1) + \log\log(\alpha) - \log(k) - \log\log(1+x_{(i)}^c)\right) = 0, \tag{3.9}$$

$$\sum_{i=1}^{n}\left(\begin{array}{c}\left(y_{(i)} - \log\log(\alpha-1) + \log\log(\alpha) - \log(k) - \log\log(1+x_{(i)}^c)\right) \\ \times\frac{x_{(i)}^c\log\left(x_{(i)}\right)}{(1+x_{(i)}^c)\log(1+x_{(i)}^c)}\end{array}\right) = 0, \tag{3.10}$$

$$\overline{y} - \log\log(\alpha-1) + \log\log(\alpha) - \log k - \frac{1}{n}\sum_{i=1}^{n}\log\log(1+x_{(i)}^c) = 0. \tag{3.11}$$

3.3. **Robust Estimation for the MOEBXII Distribution.** We observe that, as in the Burr XII distribution case, the score functions for c and k are unbounded function of x. This implies that the ML estimators for c and k may be affected from outliers in data. It is also happens for the LS estimators as well since the objective function for the LS method and the corresponding score functions are unbounded function of x. This can be easily checked from the equation (3.3)-(3.4) and (3.10)-(3.11). Therefore in the presence of outliers, instead of using ML or LS estimation methods robust methods should be used to get estimators that are not sensitive to the outliers.

In this paper we proposed a robust estimation method based on M estimation method proposed by Huber [10]. In this method we will use a objective function, say ρ as in used in robustness theory which is less decreasing than square function or bounded to reduced the effect of outliers on the estimators. The method will be carried out as follows. We will be minimize following objective function with respect to the parameters of interest instead of minimizing of the objective function given in equation (3.8) or maximizing the loglikelihood function given in equation (3.1).

$$Q(\alpha, c, k) = \sum_{i=1}^{n} \rho\left(y_i - \log\log(\alpha - 1) + \log\log(\alpha) - \log(k) - \log\log(1 + x_i^c)\right). \tag{3.12}$$

By taking the derivatives of the objective function Q with respect to the parameters we obtain following equations

$$\frac{\partial Q(\alpha, c, k)}{\partial \alpha} = \sum_{i=1}^{n} \left(\begin{array}{c} \rho'\left(y_i - \log\log(\alpha - 1) + \log\log(\alpha) - \log(k) - \log\log(1 + x_i^c)\right) \\ \times \left(\frac{\frac{1}{\alpha-1}}{\log(\alpha-1)} - \frac{\frac{1}{\alpha}}{\log(\alpha)} \right) \end{array} \right) = 0, \tag{3.13}$$

$$\frac{\partial Q(\alpha, c, k)}{\partial c} = \sum_{i=1}^{n} \left(\begin{array}{c} \rho'\left(y_i - \log\log(\alpha - 1) + \log\log(\alpha) - \log(k) - \log\log(1 + x_i^c)\right) \\ \times \frac{x_i^c \log(x_i)}{(1+x_i^c)\log(1+x_i^c)} \end{array} \right) = 0, \tag{3.14}$$

$$\frac{\partial Q(\alpha, c, k)}{\partial k} = -\sum_{i=1}^{n} \left(\frac{\rho'\left(y_i - \log\log(\alpha - 1) + \log\log(\alpha) - \log(k) - \log\log(1 + x_i^c)\right)}{k} \right) = 0. \tag{3.15}$$

There many ρ functions used in robust statistical analysis. However, since Huber's and Tukey's ρ functions are widely used in literature we will use Huber's and Tukey's ρ functions. This functions are

$$\rho(x) = \begin{cases} x^2 & , \quad |x| \leq b_1 \\ 2b_1 |x| - b_1^2 & , \quad |x| > b_1 \end{cases} \tag{3.16}$$

$$\rho(x)=\left\{\begin{array}{ccc} 1-\left(1-(x/b_2)^2\right)^2 & , & |x|\le b_2 \\ 1 & , & |x|>b_2 \end{array}\right. , \tag{3.17}$$

respectively. Here b_1 and b_2 are called the robustness tuning constants After rearranging the equations (3.13), (3.14) and (3.15) these estimates can also be obtained by solving the nonlinear equations:

$$\log\widehat{k}=\frac{\sum_{i=1}^{n}\omega_i y_i}{\sum_{i=1}^{n}\omega_i}-\frac{\sum_{i=1}^{n}\omega_i\log\log(1+x_i^c)}{\sum_{i=1}^{n}\omega_i}-\log\log(\alpha-1)+\log\log(\alpha), \tag{3.18}$$

$$\sum_{i=1}^{n}\left(\begin{array}{c}\omega_i\left(y_i-\log\log(\alpha-1)+\log\log(\alpha)-\log(k)-\log\log(1+x_i^c)\right)\\ \times\frac{x_i^c\log(x_i)}{(1+x_i^c)\log(1+x_i^c)}\end{array}\right)=0, \tag{3.19}$$

$$\log\left(\log(\alpha)\log(\alpha-1)\right)=\frac{\sum_{i=1}^{n}\omega_i y_i}{\sum_{i=1}^{n}\omega_i}-\frac{\sum_{i=1}^{n}\omega_i\log\log(1+x_i^c)}{\sum_{i=1}^{n}\omega_i}-\log(k) \tag{3.20}$$

where ω_i's are the weights. When the Huber's ρ function is used, the weights will be

$$\omega_i=\min\left\{1,\frac{b_1}{\left|\left(y_i-\log\log(\alpha-1)+\log\log(\alpha)-\log(k)-\log\log(1+x_i^c)\right)\right|}\right\}. \tag{3.21}$$

If the Tukey's ρ function is used, the weights will be

$$\omega_i=\left(\begin{array}{c}\left(1-\left(\frac{(y_i-\log\log(\alpha-1)+\log\log(\alpha)-\log(k)-\log\log(1+x_i^c))}{b_2}\right)^2\right)^2\\ \times I\left(\left|\left(y_i-\log\log(\alpha-1)+\log\log(\alpha)-\log(k)-\log\log(1+x_i^c)\right)\right|\le b_2\right)\end{array}\right). \tag{3.22}$$

4. Simulation Study

Generating data from the MOEBXII distribution. For the Burr distribution the data generating procedures are available in literature (e.g., random('burr',c,k) code in Matlab). However for the MOEBXII distribution, we can not be able to see any available data generating procedures. Therefore, before the simulation study we will give a brief outline of the data generating scheme use in our simulation study for the MOEBXII distribution.

We generate the data from the MOEBXII distribution by using Inverse Transform Method. First we generate random numbers $u_1, u_2, ..., u_n$ from the uniform distribution on the interval $(0,1)$. Then we find the inverse of cdf of the MOEBXII distribution $F^{-1}(\cdot)$ for any given α, c and k values and calculate the value $F^{-1}(u_i)$ for $i = 1, 2, ..., n$.

Next, we provide a small simulation study to compare the performance of the estimation methods given in section 3. We generate N=100 samples of size n ($n = 20$, $n = 40$, and $n = 100$) from the MOEBXII distribution. We have taken parameter values $(\alpha, c, k) = (3,1,1)$, $(3,1,2)$, $(3,2,1)$, $(3,2,2)$, $(3,3,3)$, $(5,1,1)$, $(5,1,2)$, $(5,2,1)$ and $(5,2,2)$.

To assess the performance of the methods, we calculated the bias and the RMSE for the estimates of $\theta = \alpha, c$ and k obtained from the simulated data sets

$$Bias\left(\widehat{\theta}\right) = \frac{1}{N}\sum_{i=1}^{n}\left(\widehat{\theta}_i - \theta\right), \tag{4.1}$$

$$RMSE = \left(\widehat{\theta}\right) = \sqrt{\frac{1}{N}\sum_{i=1}^{n}\left(\widehat{\theta}_i - \theta\right)^2}. \tag{4.2}$$

Robust estimations of the parameters are obtained with tuning constant $b_1 = 3.5$ for Huber's ρ function and $b_2 = 1.345$ for Tukey's ρ function.

The results of our simulation study are presented in the Tables 1–6. In the tables, we present the bias and RMSE for the estimators obtain from the methods described in Section 3.

The results of our simulation study for the data sets without outliers are presented in the Tables 1–3. This results show that all estimation methods considered in this paper perform well in estimating the parameters of the MOEBXII distribution when the data sets do not contain any outliers. However the robust estimator based on Tukey's ρ function generally outperforms others in terms of the bias and RMSE. In addition, the average bias and RMSE of all the estimators of the parameters c and k generally decrease as n increases.

Table 1 The Bias and RMSE (Parenthesis) for $n = 20$

Parameter α				
	ML	LS	Huber	Tukey
$(3,1,1)$	0.2703 (0.2907)	0.3599 (0.4274)	0.3411 (0.4149)	0.2356 (0.3806)
$(3,1,2)$	0.1239 (0.1820)	0.1928 (0.5397)	0.1267 (0.2185)	0.0078 (0.1872)
$(3,2,1)$	0.5183 (0.5499)	0.2208 (0.6144)	0.5446 (0.5664)	0.5362 (0.5612)
$(3,2,2)$	0.1034 (0.2470)	0.2287 (0.2971)	0.0957 (0.2843)	0.0892 (0.2538)
$(3,3,3)$	0.3755 (1.1090)	0.1285 (0.3726)	0.1462 (0.1916)	0.0967 (0.1970)
$(5,1,1)$	0.0965 (0.3055)	0.3622 (0.3786)	0.0727 (0.2655)	0.0437 (0.2282)
$(5,1,2)$	0.3204 (0.6993)	0.1776 (0.1887)	0.1955 (0.1976)	0.1962 (0.1981)
$(5,2,1)$	0.4591 (0.6376)	0.2003 (0.4086)	0.2413 (0.4166)	0.2160 (0.4063)
$(5,2,2)$	0.1620 (0.1857)	0.4056 (0.8534)	0.1660 (0.1854)	0.1358 (0.2245)

Parameter c.

	ML	LS	Huber	Tukey
$(3, 1, 1)$	0.5789 (0.5822)	0.5851 (0.5870)	0.5824 (0.5857)	0.2436 (0.3245)
$(3, 1, 2)$	0.5885 (0.5924)	0.8937 (0.9443)	0.5913 (0.5934)	0.5821 (0.5891)
$(3, 2, 1)$	0.1333 (0.3235)	0.0531 (0.3903)	0.1097 (0.3149)	0.1511 (0.3153)
$(3, 2, 2)$	0.2093 (0.3370)	0.2346 (0.3390)	0.2071 (0.3297)	0.0310 (0.4851)
$(3, 3, 3)$	0.1599 (0.1974)	0.1651 (0.3172)	0.1559 (0.1954)	0.1506 (0.1953)
$(5, 1, 1)$	0.6275 (0.6698)	0.5779 (0.5891)	0.5948 (0.5958)	0.5833 (0.5909)
$(5, 1, 2)$	0.6766 (0.7201)	0.5990 (0.5990)	0.5975 (0.5976)	0.5924 (0.5945)
$(5, 2, 1)$	0.3222 (0.4300)	0.1843 (0.3417)	0.2328 (0.3456)	0.2003 (0.4086)
$(5, 2, 2)$	0.3137 (0.4210)	0.2769 (0.3694)	0.2755 (0.3883)	0.2744 (0.3837)

Parameter k.

	ML	LS	Huber	Tukey
$(3, 1, 1)$	0.5785 (0.5801)	0.4808 (0.5291)	0.5768 (0.5780)	0.3869 (0.4556)
$(3, 1, 2)$	0.1279 (0.1502)	0.3546 (0.5322)	0.1682 (0.1813)	0.0695 (0.1335)
$(3, 2, 1)$	0.5843 (0.5863)	0.5950 (0.5951)	0.5937 (0.5941)	0.3517 (0.4567)
$(3, 2, 2)$	0.1526 (0.3134)	0.3803 (1.0688)	0.1392 (0.3096)	0.1222 (0.2978)
$(3, 3, 3)$	0.5488 (0.8331)	0.1878 (0.3971)	0.1105 (0.2105)	0.0388 (0.2250)
$(5, 1, 1)$	0.3524 (0.3930)	0.4922 (1.0249)	0.3981 (0.4235)	0.1626 (0.3129)
$(5, 1, 2)$	0.2053 (0.5287)	0.1955 (0.1986)	0.1933 (0.1998)	0.1908 (0.1949)
$(5, 2, 1)$	0.4993 (0.5164)	0.1628 (0.2043)	0.5051 (0.5200)	0.4935 (0.5164)
$(5, 2, 2)$	0.5043 (0.6367)	0.2200 (0.5748)	0.1571 (0.2087)	0.1406 (0.2272)

Table 2 The Bias and RMSE (Parenthesis) for $n = 40$

Parameter α				
	ML	LS	Huber	Tukey
$(3,1,1)$	0.3787 (0.4215)	0.4084 (0.4373)	0.3134 (0.4223)	0.0038 (0.3636)
$(3,1,2)$	0.1859 (0.2659)	0.1996 (0.2629)	0.1547 (0.4467)	0.0543 (0.7690)
$(3,2,1)$	0.4862 (0.4918)	0.2783 (0.2840)	0.4365 (0.6060)	0.3921 (0.5403)
$(3,2,2)$	0.1259 (0.3692)	0.0921 (0.2560)	0.7029 (0.5622)	0.0751 (0.2333)
$(3,3,3)$	0.1303 (1.3301)	0.2127 (0.3781)	0.2281 (0.4004)	0.0077 (0.2743)
$(5,1,1)$	0.4041 (0.5928)	0.2336 (0.4158)	0.3505 (0.3610)	0.2873 (0.4083)
$(5,1,2)$	0.2633 (0.5182)	0.3413 (0.3626)	0.2618 (0.4991)	0.4320 (0.4531)
$(5,2,1)$	0.4017 (0.4121)	0.1647 (0.4358)	0.4194 (0.4307)	0.2258 (0.4345)
$(5,2,2)$	0.2049 (0.3830)	0.1414 (0.3478)	0.4065 (0.8821)	0.0230 (0.2731)
Parameter c.				
	ML	LS	Huber	Tukey
$(3,1,1)$	0.2311 (0.3102)	0.4149 (0.5571)	0.4917 (0.4938)	0.4959 (0.4963)
$(3,1,2)$	0.2269 (0.2810)	0.4959 (0.4966)	0.3990 (0.3361)	0.2941 (0.4973)
$(3,2,1)$	0.0258 (0.3802)	0.0706 (0.3337)	0.0706 (0.3244)	0.0201 (0.2980)
$(3,2,2)$	0.1390 (0.3383)	0.1886 (0.3392)	0.0661 (0.448)	0.0259 (0.4815)
$(3,3,3)$	1.0137 (1.0836)	0.4633 (0.4712)	0.9990 (1.0880)	0.4329 (0.4646)
$(5,1,1)$	0.6905 (0.7355)	0.4969 (0.4973)	0.6500 (0.6931)	0.4017 (0.4121)
$(5,1,2)$	0.7259 (0.7683)	0.4041 (0.5928)	0.6964 (0.7622)	0.1573 (0.3372)
$(5,2,1)$	0.3261 (0.4219)	0.1493 (0.3335)	0.3581 (0.4144)	0.0964 (0.3111)
$(5,2,2)$	0.3466 (0.4326)	0.3076 (0.4397)	0.2061 (0.3551)	0.1573 (0.3372)
Parameter k.				
	ML	LS	Huber	Tukey
$(3,1,1)$	0.3658 (0.5287)	0.4905 (0.4908)	0.4883 (0.4891)	0.0275 (0.2457)
$(3,1,2)$	0.3934 (0.4905)	0.0275 (0.2457)	0.3731 (0.5183)	0.0037 (0.2295)
$(3,2,1)$	0.4956 (0.4970)	0.4894 (0.4913)	0.4981 (0.4983)	0.2336 (0.4158)
$(3,2,2)$	0.3340 (0.7965)	0.1092 (0.2957)	0.4097 (0.9214)	0.1069 (0.2704)
$(3,3,3)$	0.6937 (0.9322)	0.2630 (0.3787)	0.7339 (0.9491)	0.2061 (0.3796)
$(5,1,1)$	0.3255 (0.3471)	0.3274 (0.3567)	0.3466 (0.4326)	0.1647 (0.4358)
$(5,1,2)$	0.6905 (0.7355)	0.4834 (0.4858)	0.4969 (0.4973)	0.2463 (0.7562)
$(5,2,1)$	0.6937 (0.9322)	0.4312 (0.4451)	0.4480 (0.4588)	0.2630 (0.3787)
$(5,2,2)$	0.8649 (1.0009)	0.3702 (0.4095)	0.3261 (0.4219)	0.1069 (0.2704)

Table 3 The Bias and RMSE (Parenthesis) for $n = 100$

Parameter α				
	ML	LS	Huber	Tukey
$(3, 1, 1)$	0.0597 (0.1429)	0.0835 (0.1021)	0.0721 (0.1037)	0.0115 (0.1037)
$(3, 1, 2)$	0.1491 (0.2890)	0.2981 (0.3841)	0.0785 (0.3016)	0.0332 (0.1245)
$(3, 2, 1)$	0.0180 (0.1225)	0.1465 (0.5922)	0.1798 (0.1927)	0.1968 (0.1981)
$(3, 2, 2)$	0.0517 (0.2840)	0.0887 (0.2847)	0.0376 (0.1207)	0.0257 (0.3115)
$(3, 3, 3)$	0.0926 (0.4129)	0.1635 (0.3879)	0.0513 (0.1556)	0.0056 (0.4082)
$(5, 1, 1)$	0.0222 (0.1335)	0.0316 (0.1361)	0.0941 (0.1611)	0.0142 (0.1323)
$(5, 1, 2)$	0.0459 (0.1172)	0.0364 (0.1514)	0.0507 (0.1162)	0.0178 (0.1341)
$(5, 2, 1)$	0.1982 (0.1983)	0.1615 (0.4114)	0.2118 (0.3724)	0.2113 (0.3707)
$(5, 2, 2)$	0.1486 (0.3654)	0.2067 (0.3864)	0.1867 (0.3723)	0.0929 (0.1024)
Parameter c				
	ML	LS	Huber	Tukey
$(3, 1, 1)$	0.0385 (0.1419)	0.0842 (0.5175)	0.1946 (0.1970)	0.0094 (0.1324)
$(3, 1, 2)$	0.4973 (0.4977)	0.4986 (0.4988)	0.4974 (0.4978)	0.1995 (0.1996)
$(3, 2, 1)$	0.1797 (0.3181)	0.1242 (0.2960)	0.0992 (0.2954)	0.0005 (0.3589)
$(3, 2, 2)$	0.1962 (0.3680)	0.2169 (0.3439)	0.1979 (0.3686)	0.0262 (0.1352)
$(3, 3, 3)$	0.3345 (0.4441)	0.3354 (0.4510)	0.3430 (0.4567)	0.0949 (0.1011)
$(5, 1, 1)$	0.0932 (0.1025)	0.0957 (0.1010)	0.0978 (0.1625)	0.0143 (0.1275)
$(5, 1, 2)$	0.1264 (0.1750)	0.1594 (0.1857)	0.1570 (0.1861)	0.1075 (0.0459)
$(5, 2, 1)$	0.2646 (0.3608)	0.1864 (0.3436)	0.2230 (0.3498)	0.1197 (0.3876)
$(5, 2, 2)$	0.1363 (0.3576)	0.1601 (0.3279)	0.1597 (0.3565)	0.0270 (0.1173)
Parameter k				
	ML	LS	Huber	Tukey
$(3, 1, 1)$	0.0864 (0.1024)	0.0831 (0.1013)	0.0913 (0.1539)	0.0562 (0.0975)
$(3, 1, 2)$	0.1844 (0.3232)	0.1977 (0.1982)	0.1170 (0.3341)	0.0257 (0.2820)
$(3, 2, 1)$	0.0501 (0.1433)	0.2524 (0.7771)	0.0529 (0.1413)	0.0509 (0.1416)
$(3, 2, 2)$	0.0859 (0.3318)	0.1380 (0.1779)	0.0259 (0.3263)	0.0814 (0.3295)
$(3, 3, 3)$	0.1643 (0.3750)	0.2457 (0.3703)	0.1148 (0.3868)	0.0753 (0.1087)
$(5, 1, 1)$	0.1767 (0.1866)	0.1791 (0.1873)	0.1597 (0.1814)	0.1088 (0.0423)
$(5, 1, 2)$	0.3285 (0.4698)	0.1405 (0.1637)	0.1767 (0.1866)	0.1597 (0.1814)
$(5, 2, 1)$	0.4746 (0.9614)	0.4583 (0.4710)	0.3553 (0.4097)	0.1172 (0.1600)
$(5, 2, 2)$	0.3160 (0.4152)	0.3311 (0.4163)	0.3117 (0.4193)	0.1878 (0.1942)

Table 4–6 list the bias and RMSE for the data sets with outliers. For the sample size $n = 20$, there is one outlier, for the sample sizes $n = 40$ and $n = 100$, there are two and four outliers, respectively. The outliers are taken 100×largest observation. From Tables 4-6, we observe that outliers induce a large influence on the bias and RMSE of the ML and LS estimators. In particular the ML and LS estimators

compared to the robust estimators are drastically effected from the outliers when the number of outliers is four.

The simulation results in Tables 4-6 clearly indicates that the robust estimator based on Tukey's ρ function has the smallest bias in all cases and smallest RMSE in most of the cases with outliers. In addition, the robust estimator based on Huber's ρ function outperforms LS and ML estimator in terms of bias and RMSE when the data set contains outlier. For example it can be seen from Tables 4-6 that the largest difference of bias for parameter c arises in Table 4 for the cases (3,2,2) and (5,2,2) and for the parameter k the same happens for the cases (5,1,2) and (5,2,1). Similarly we can observe superiority of the robust estimators in terms of bias and RMSE in Tables 5-6.

Finally in Table 6 we observe that when the number of outliers increases, the LS and the ML estimators dramatically worsen compared to the robust estimators. It can be seen that bias and the RMSE values are very large for the LS and ML estimators.

Table 4 The Bias and RMSE (Parenthesis) for $n = 20$ with one outlier

Parameter α				
	ML	LS	Huber	Tukey
$(3,1,1)$	-0.8720 (0.8857)	0.1889 (0.1939)	-0.0467 (0.4116)	0.0399 (0.1566)
$(3,1,2)$	-0.8637 (0.9271)	-0.8936 (0.9683)	-0.1737 (0.1887)	0.0103 (0.4528)
$(3,2,1)$	-0.8621 (0.9096)	0.1848 (0.1911)	0.1845 (0.2419)	0.1663 (0.1856)
$(3,2,2)$	-0.7024 (1.7847)	-0.7262 (0.7852)	-0.0939 (0.1828)	0.0376 (0.2696)
$(3,3,3)$	0.4533 (0.9336)	-0.3358 (0.6387)	-0.1379 (0.1944)	-0.1262 (0.1953)
$(5,1,1)$	-0.9716 (0.9736)	-0.5881 (0.7478)	-0.0861 (0.1790)	0.0148 (0.1702)
$(5,1,2)$	0.8130 (0.8402)	-0.7250 (0.7646)	0.0324 (0.5353)	-0.0025 (0.1813)
$(5,2,1)$	0.9896 (1.0093)	-0.1615 (0.3716)	0.1293 (0.1579)	0.0768 (0.1479)
$(5,2,2)$	-0.9334 (0.9401)	-0.4343 (0.7769)	-0.1725 (0.1992)	0.0688 (0.5022)
Parameter c				
	ML	LS	Huber	Tukey
$(3,1,1)$	0.4250 (0.5100)	0.8450 (0.8713)	0.1939 (0.1961)	0.1930 (0.1963)
$(3,1,2)$	0.4187 (0.5125)	0.8329 (0.8595)	0.1986 (0.1988)	0.1976 (0.1981)
$(3,2,1)$	-0.6027 (0.6895)	0.1951 (0.1966)	-0.0545 (0.0575)	0.0190 (0.0200)
$(3,2,2)$	-0.6474 (0.7040)	-1.2856 (1.7346)	0.0821 (0.2429)	-0.0689 (0.0713)
$(3,3,3)$	-0.9999 (0.9999)	-0.7101 (0.8230)	-0.1903 (0.1966)	-0.0928 (0.1877)
$(5,1,1)$	0.6388 (0.6456)	0.8683 (0.8889)	0.0414 (0.2640)	-0.0080 (0.0714)
$(5,1,2)$	0.5336 (0.6383)	-0.6467 (0.7541)	0.1995 (0.1996)	0.1617 (0.1958)
$(5,2,1)$	-0.8357 (0.8441)	0.2568 (0.2670)	0.1131 (0.5104)	0.0395 (0.4005)
$(5,2,2)$	1.6584 (1.6625)	-0.5162 (0.5476)	-0.4642 (0.4703)	-0.1641 (0.2074)

Parameter k				
	ML	LS	Huber	Tukey
$(3,1,1)$	0.4511 (0.5099)	0.7417 (0.7680)	0.1979 (0.1981)	0.0301 (0.1697)
$(3,1,2)$	-0.5508 (0.6172)	0.2754 (0.2825)	-0.4002 (0.4035)	-0.1672 (0.5026)
$(3,2,1)$	0.4795 (0.5475)	0.5920 (0.6455)	0.1653 (0.1911)	-0.0312 (0.4778)
$(3,2,2)$	-0.3657 (0.4728)	-0.3149 (0.4629)	0.1476 (0.2398)	-0.0444 (0.0898)
$(3,3,3)$	-0.9983 (0.9983)	-0.1501 (0.6614)	-0.1480 (0.1900)	-0.1294 (0.1969)
$(5,1,1)$	0.4275 (0.4415)	0.8678 (0.8734)	0.3440 (0.4601)	0.0070 (0.0097)
$(5,1,2)$	1.1828 (1.3903)	0.2941 (0.3833)	0.1354 (0.1590)	-0.0880 (0.1827)
$(5,2,1)$	-0.7448 (0.7760)	1.1564 (1.1823)	-0.5800 (0.5275)	0.0758 (0.4103)
$(5,2,2)$	-0.8840 (0.9824)	0.3214 (0.4134)	0.1406 (0.1705)	0.1053 (0.1702)

Table 5 The Bias and RMSE (Parenthesis) for $n = 40$ with two outliers

Parameter α				
	ML	LS	Huber	Tukey
$(3,1,1)$	-0.8720 (0.8857)	0.8450 (0.8713)	0.1889 (0.1939)	0.0399 (0.1566)
$(3,1,2)$	-0.8637 (0.9271)	-0.7101 (0.8230)	-0.1936 (0.1968)	-0.1737 (0.1887)
$(3,2,1)$	-0.8621 (0.9096)	0.2845 (0.4419)	0.1848 (0.1911)	0.1663 (0.1856)
$(3,2,2)$	1.1558 (1.3132)	0.8130 (0.8402)	-0.0939 (0.1828)	-0.1617 (0.1938)
$(3,3,3)$	0.4533 (0.9336)	-0.3358 (0.6387)	-0.1379 (0.1944)	-0.1262 (0.1953)
$(5,1,1)$	1.2561 (1.2758)	1.1057 (1.1161)	0.0688 (0.5022)	-0.0861 (0.1790)
$(5,1,2)$	-0.9467 (0.9541)	1.2686 (1.2866)	0.1995 (0.1996)	0.1617 (0.1958)
$(5,2,1)$	-1.4959 (1.4992)	1.2676 (1.2897)	0.1293 (0.1579)	0.0768 (0.1479)
$(5,2,2)$	1.1825 (1.2136)	-2.7170 (2.7186)	0.0539 (0.1066)	0.1976 (0.3438)
Parameter c				
	ML	LS	Huber	Tukey
$(3,1,1)$	0.4250 (0.5100)	0.7417 (0.7680)	0.1939 (0.1961)	0.1930 (0.1963)
$(3,1,2)$	0.4187 (0.5125)	0.8683 (0.8889)	0.1986 (0.1988)	0.1976 (0.1981)
$(3,2,1)$	-0.6027 (0.6895)	0.5920 (0.6455)	0.0376 (0.2696)	0.0324 (0.5353)
$(3,2,2)$	-0.6474 (0.7040)	-0.7250 (0.7646)	0.0450 (0.4822)	0.0148 (0.1702)
$(3,3,3)$	-0.9999 (0.9999)	-0.7448 (0.7760)	-0.0928 (0.1877)	-0.1903 (0.1966)
$(5,1,1)$	-0.5716 (0.5736)	0.5824 (0.6063)	0.1999 (0.1999)	0.0395 (0.4005)
$(5,1,2)$	-0.4312 (0.4346)	-0.4775 (0.4997)	-0.1725 (0.1992)	-0.0025 (0.1813)
$(5,2,1)$	-0.8357 (0.8441)	0.8007 (0.8093)	-0.1306 (0.6490)	0.0218 (0.0226)
$(5,2,2)$	-0.9334 (0.9401)	0.2990 (0.3033)	0.1131 (0.5104)	0.0326 (0.1214)

Parameter k

	ML	LS	Huber	Tukey
$(3,1,1)$	0.4511 (0.5099)	0.8329 (0.8595)	0.1979 (0.1981)	0.0301 (0.1697)
$(3,1,2)$	-0.5508 (0.6172)	-0.1672 (0.5026)	-0.0312 (0.4778)	0.0103 (0.4528)
$(3,2,1)$	0.4795 (0.5475)	-0.5881 (0.7478)	0.1951 (0.1966)	0.1653 (0.1911)
$(3,2,2)$	-0.3657 (0.4728)	-0.3149 (0.4629)	0.2941 (0.3833)	-0.1501 (0.6614)
$(3,3,3)$	-0.9983 (0.9983)	0.8211 (0.8301)	-0.1480 (0.1900)	-0.1294 (0.1969)
$(5,1,1)$	0.9388 (0.9456)	-0.7595 (0.7975)	0.1354 (0.1590)	-0.0880 (0.1827)
$(5,1,2)$	0.9336 (0.9383)	0.3214 (0.4134)	0.1406 (0.1705)	0.1053 (0.1702)
$(5,2,1)$	0.2870 (0.3752)	-0.3589 (0.3638)	-0.1189 (0.6799)	-0.0262 (0.1368)
$(5,2,2)$	0.3440 (0.4601)	0.7412 (0.7497)	-0.0560 (0.0581)	0.0363 (0.0383)

Table 6 The Bias and RMSE (Parenthesis) for $n = 100$ with four outliers

Parameter α				
	ML	LS	Huber	Tukey
$(3,1,1)$	0.7900 (0.8791)	0.2435 (0.2490)	0.1382 (0.1505)	0.0358 (0.2485)
$(3,1,2)$	0.8968 (0.9036)	0.8227 (0.8334)	-0.3528 (0.3592)	0.2308 (0.2401)
$(3,2,1)$	0.8221 (0.8448)	0.3423 (0.5849)	0.2498 (0.2549)	0.1626 (0.1685)
$(3,2,2)$	0.4911 (0.4965)	0.8581 (0.8714)	0.2080 (0.2207)	0.1124 (0.2244)
$(3,3,3)$	0.3057 (0.3102)	0.9621 (0.9791)	0.2184 (0.3691)	0.0638 (0.1133)
$(5,1,1)$	1.1084 (1.1178)	1.3824 (1.4463)	0.0219 (0.0227)	0.0066 (0.0128)
$(5,1,2)$	0.7998 (0.9205)	0.5392 (0.6319)	-0.4114 (0.4148)	0.0862 (0.0865)
$(5,2,1)$	1.1389 (1.8608)	1.2747 (1.2935)	0.2454 (1.1181)	0.1201 (0.1637)
$(5,2,2)$	-1.3619 (1.7776)	1.1672 (1.2235)	0.0658 (0.0677)	0.0930 (0.0938)
Parameter c.				
	ML	LS	Huber	Tukey
$(3,1,1)$	0.7809 (0.7833)	0.8679 (0.8724)	0.0607 (0.0894)	-0.0138 (0.1207)
$(3,1,2)$	0.7901 (0.7911)	0.5867 (0.6052)	0.2806 (0.3403)	0.1063 (0.2110)
$(3,2,1)$	0.5419 (0.7389)	0.3041 (0.3601)	0.0929 (0.1971)	0.0678 (0.2110)
$(3,2,2)$	0.4285 (0.4522)	0.5794 (0.6902)	0.1652 (0.2198)	0.0790 (0.1963)
$(3,3,3)$	0.9075 (0.9379)	0.9224 (0.9363)	0.1497 (0.2743)	0.0042 (0.0077)
$(5,1,1)$	0.4965 (0.4977)	0.2558 (0.2861)	0.2028 (0.2714)	0.0881 (0.0902)
$(5,1,2)$	0.8489 (1.2422)	0.7832 (0.7851)	0.3018 (0.3221)	0.0703 (0.1476)
$(5,2,1)$	0.7857 (0.9828)	0.4994 (0.4994)	0.4203 (0.4351)	0.1879 (0.4997)
$(5,2,2)$	0.4969 (0.4972)	0.9050 (0.9874)	0.1310 (0.2423)	0.0905 (0.2533)
Parameter k				
	ML	LS	Huber	Tukey
$(3,1,1)$	-0.6921 (0.7195)	0.7463 (0.7487)	0.6352 (0.6455)	0.0581 (0.1203)
$(3,1,2)$	-0.5644 (0.5910)	0.3400 (0.3718)	0.2710 (0.2796)	0.1336 (0.1693)
$(3,2,1)$	0.7565 (0.7653)	0.7244 (0.7316)	0.6246 (0.6361)	0.3896 (0.4025)
$(3,2,2)$	0.6129 (0.6151)	0.4793 (0.6053)	0.4256 (0.4369)	0.3153 (0.3249)
$(3,3,3)$	0.5887 (0.9686)	0.8370 (0.8502)	0.2330 (0.3735)	0.4510 (0.4533)
$(5,1,1)$	0.8774 (0.9798)	0.7986 (0.7986)	0.3030 (0.3253)	0.0976 (0.1495)
$(5,1,2)$	0.6951 (1.3824)	0.9917 (1.0100)	0.6741 (0.6818)	0.0160 (0.0172)
$(5,2,1)$	0.3103 (0.3320)	0.4815 (0.5001)	-0.0624 (0.0657)	0.0412 (0.2582)
$(5,2,2)$	0.5932 (0.6090)	1.1397 (1.3610)	0.0588 (0.3338)	-0.0651 (0.0676)

To sum up when there are potential outliers in the data the robust methods should be used to estimate the parameters of the MOEBXII distribution instead of using ML estimators.

5. Real Data Examples

In this section, we consider two real life data sets to illustrate the proposed methods and verify how our estimators work in practice. The first one is presented in [12] and related to the failure times of 20 mechanical components. The second data set is electrical insulating described in [12] in which the length of time until breakdown.

5.1. **Failure Time Data Set.** Failure time data set has been considered by [4] to illustrate the performance of the proposed robust estimators for the parameters of the Burr XII distribution and by [5] to illustrate the proposed Optimal B-robust estimators for the parameters of the Burr XII distribution. The data set has been also used by [20] to illustrate the potential of the Burr XII power series distributions. The data set contains the failure times of 20 mechanical components. The parameter estimations obtained from the ML, LS and proposed robust estimation methods are given in Table 7.

Table 7 Parameter Estimations for the Failure Time Data Set

	$\widehat{\alpha}$	$\widehat{c}$	$\widehat{k}$
ML	1.3060	1.7359	32.9140
LS	3.0968	2.1149	96.3755
Huber	2.6637	2.0053	108.6202
Tukey	2.4360	2.0651	152.436

The fitted pdfs and histogram of the failure time data set are given in Figure 2.

From Figure 2, we can observe that the data set contains a potential outlier. It has also been observed from Figure 2 that the ML and LS estimators are heavily distorted by this single outlier. However, the robust estimators do not seem very affected by the outliers. In particularly, the robust estimator based on Tukey's ρ function provides better fit than the others in terms of modeling the data. We can observe that the fitted density obtained from Tukey seems summarizing data more accurant than the others. On the other hand, clearly the fitted density obtained from ML is affected by the outliers and it is not provided a good fit to the data. It is not catching the pick of the data. It may be said that ML method is underestimating the parameters. The fitted density obtained from the robust estimator based on Huber's ρ function is also seem better than that of ML and LS fits. In summary, the performance of the robust methods in terms of the fitting density to the data seems quite satisfactory.

5.2. **Electrical Insulating Data Set.** The electrical insulating data set has been already considered by [2] to illustrate the MOEBXII distribution. The data set is consist of electrical insulating described in [12] in which the lenght of the time until breakdown recorded. The data set is analyzed at 34 kilovolts with sample size $n = 19$ in this application. The ML estimation method is used to estimate the parameters of this distribution and the results are found as $(4, 0.9, 1.002)$ in the paper by Al-Saiari et al. [2].

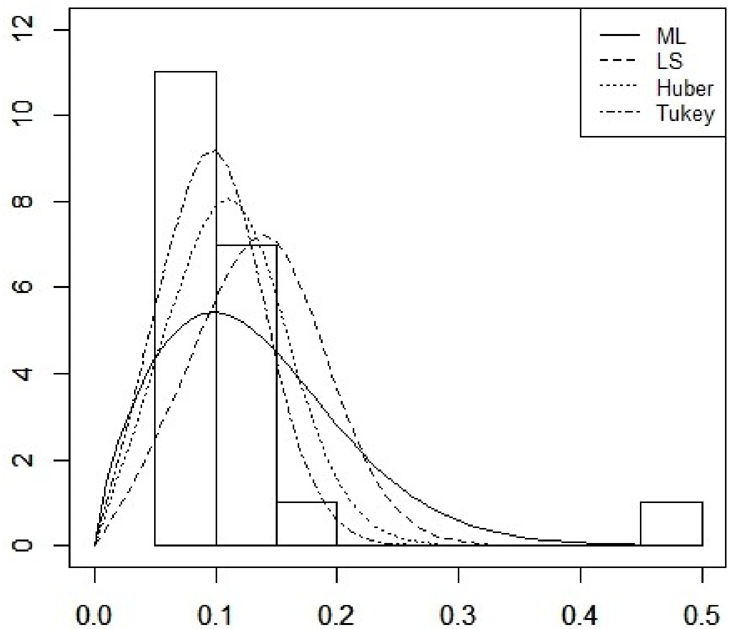

FIGURE 2. Histogram of the Failure Time Data Set and fitted densities

Alternative to the ML estimator we use the LS and the robust estimation methods proposed in this paper. We also recompute the ML estimate. The results are given in Table 8. Note that the ML estimator is very close to the estimate they have given. The other estimates are also close to the ML estimate. Note that to obtain LS and robust estimates we use the ML estimates given by [2] as the initial estimate for the algorithm.

Table 8 Parameter Estimations for the Electrical Insulating Data Set with the initial values $(\alpha_0, c_0, k_0) = (4{,}0.9{,}1)$

	$\widehat{\alpha}$	$\widehat{c}$	$\widehat{k}$
ML	4.003	0.91	1.0032
LS	6.865	0.6434	1.029
Huber	5.217	0.82	1.004
Tukey	5.924	0.932	1.432

Figure 3 shows the histogram of the electrical insulating data set and fitted pdfs obtained form ML, LS, Huber and Tukey according to Table 8.

From Figure 3, we can observe that all estimations of the parameters are closer to each other. However, the LS estimator seems to be more effected by the potential outlier than ML and robust estimators. But the difference of between the estimates

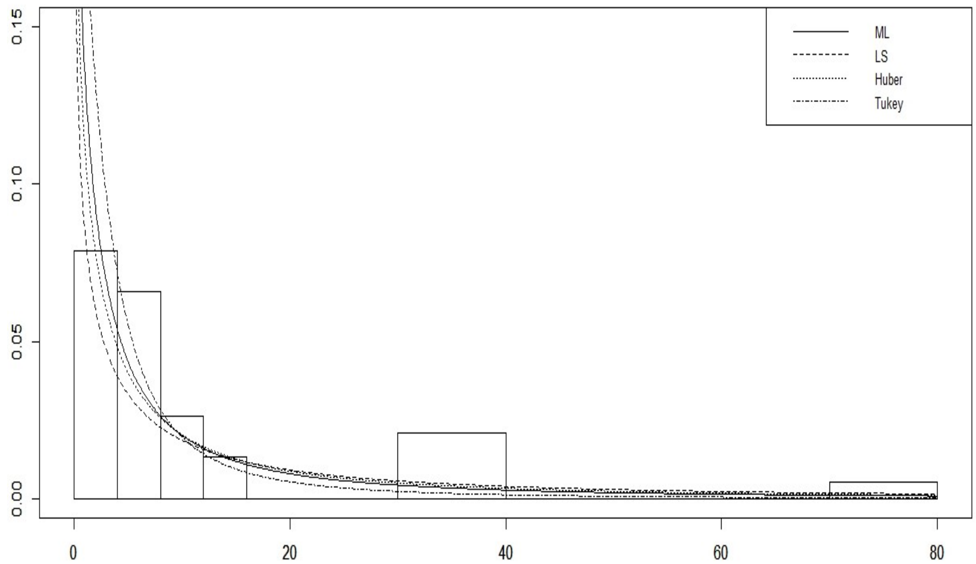

FIGURE 3. Histogram of the Electrical Insulating Data Set and fitted densities ((α_0, c_0, k_0)=(4,1,1))

is not satisfactory enough to claim the superiority of the methods. To gain same more details about the data, we further use the kernel density estimation to see the overall fit to the data. We observe that the fitted densities instead of having L-shaped densities we may have unimodal skewed density. Therefore we taken different initial values for the algorithm such as $(\alpha_0, c_0, k_0) = (20, 1, 1)$ and end it up with the different estimates for the parameters given in Table 9.

Table 9 Parameter Estimations for the Electrical Insulating Data Set with the initial values (α_0, c_0, k_0)=(20,1,1)

	$\widehat{\alpha}$	$\widehat{c}$	$\widehat{k}$
ML	6.002	0.8364	1.4726
LS	51.184	1.19	0.8721
Huber	27.016	1.8721	0.8043
Tukey	23.043	1.8439	0.8727

The histogram of the electrical insulating data set and fitted pdfs obtained form ML, LS, Huber and Tukey according to Table 9 are given in Figure 4.

Now, we can clearly observe that the LS estimator underfit the data. It can be seen that the fitted pdf obtain from ML estimator is still L-shaped. In such cases, it might be expected that the frequencies of the smaller values would be quite large,

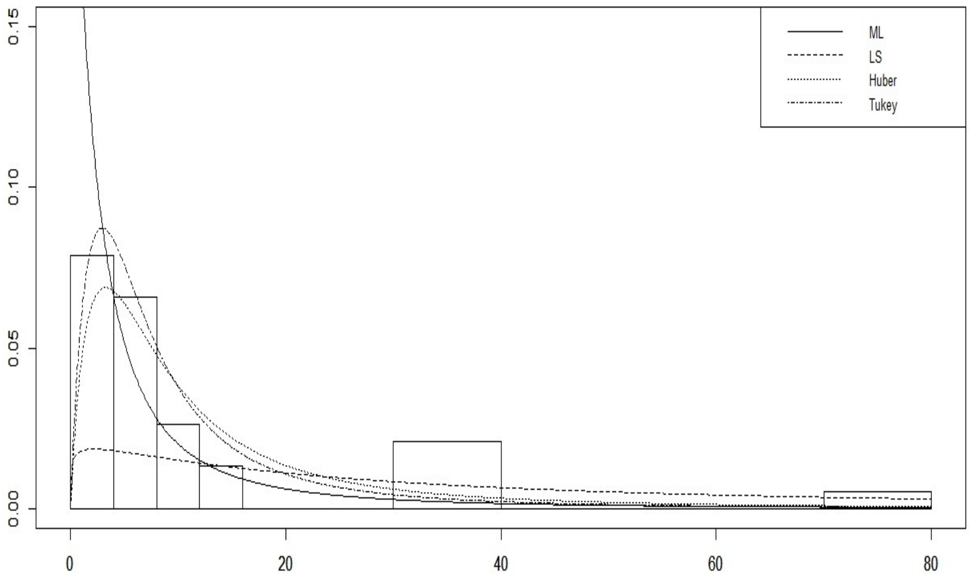

FIGURE 4. Histogram of the Electrical Insulating Data Set and fitted densities $((\alpha_0, c_0, k_0)$=(20,1,1))

and the frequencies of the larger values would be quite small. Therefore, the fitted density obtained from ML is not provided a good fit to the data. The fitted density obtained from the robust estimator based on Huber's function is seem better than that of ML and LS fits. Tukey would be preferable to fit this data set among others as in Failure Time Data Set example since it catch the pick better than the others.

6. Conclusion

In this study, alternative robust estimation methods based on M estimator have been proposed to obtain estimators for the parameters of the MOEBXII distribution. We have compared the performance of these methods through a simulation study and real data examples. It is concluded that all the methods considered show identical performance for estimating the parameters of the MOEBXII distribution unless the data set contain outlier. However, the robust estimation methods perform better for the data sets with outlier than ML and LS estimation methods. From both simulation study and real data examples, the effect of outliers on the LS and ML estimates is fairly obvious. Therefore to eliminate the outliers' effects, robust methods can be preferable. There are other robust estimation methods such as Optimal-B robust estimation that can be used to estimate the parameter of the MOEBXII distribution. This will be future work.

7. ACKNOWLEDGMENT

The authors would like to thank the Editor/referee for the valuable comments on our submitted manuscript.

References

[1] Al-Hussaini, E. K. and Jaheen, Z. F., Bayes Estimation of the Parameters, Reliability and Failure Rate Functions of the Burr Type XII Failure Model, *Journal of Statistical Computation and Simulation* (1992), 41 (1-2), 31-40.

[2] Al-Saiari, A. Y., Baharith, L. A. and Mousa, S. A., Marshall-Olkin Extended Burr Type XII Distribution, *International Journal of Statistics and Probability* (2014); 3 (1), 78-84.

[3] Burr, W., Cumulative Frequency Functions, *The Annals of Mathematical Statistics* (1942), 1 (2), 215-232.

[4] Dogru, F. Z. and Arslan, O., Alternative Robust Estimators for the Shape Parameters of the Burr XII Distribution, *International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering* (2015), 9 (5), 271-276.

[5] Dogru, F. Z. and Arslan, O., Optimal B-robust estimators for the parameters of the Burr XII distribution. *Journal of Statistical Computation and Simulation* (2016), 86 (6), 1133-1149.

[6] Ghitany, M.E., Al-Awadhi, F.A. and Alkhalfan L.A., Marshall–olkin extended lomax distribution and its application to censored data, *Communications in Statistics Theory and Methods* (2007), 36 (10), 1855-1866.

[7] Gui, W., Marshall-Olkin Extended Log-Logistic Distribution and Its Application in Minification Processes. *Applied Mathematical Sciences* (2013), 7 (80), 3947-3961.

[8] Gupta, P.L., Gupta, R.C. and Lvin, S.J., Analysis of failure time data by burr distribution, *Communications in Statistics-Theory and Methods* (1996), 25, 2013-2024

[9] Hossain, A. M. and Nath, S. K., Estimation of parameters in the presence of outliers for a burr XII distribution, *Communications in Statistics - Theory and Methods* (1997), 26 (3), 637-652.

[10] Huber, P. J., Robust estimation of a location parameter, *Ann. Math. Statist.* (1964), 35, 73-101.

[11] Klugman, S. A., Loss Distributions. *Proceedings of Symposia in Applied Mathematics Actuarial Mathematics* (1986), 35, 31-55.

[12] Lawless, J. F., Statistical Models and Methods for Lifetime Data. New York: John Wiley, 2003.

[13] Malinowska, I., Pawlas, P. and Szynal D., Estimation of location and scale parameters for the Burr XII distribution using generalized order statistics, *Linear Algebra and its Applications* (2006), 417 (1), 150-162.

[14] Marshall, A.W. and Olkin, I., A new method for adding a parameter to a family of distributions with application to the exponential and weibull families, *Biometrika* (1997), 84 (3), 641-652.

[15] McDonald, J. B., Some Generalized Function for the Size Distribution of Income, *Econometrica* (1984), 52 (3), 647-663.

[16] Moore, D. and Papadopoulos, A. S., The Burr Type XII Distribution as a Failure Model under Various Loss Functions, *Microelectronics Reliability* (2000), 40 (12), 2117-2122.

[17] Papadopoulos, A. S., The Burr Distribution as a Life Time Model from a Bayesian Approach, *IEEE Transactions on Reliability* (1978), 27 (5), 369-371.

[18] Rodriguez, R. N., A guide to Burr XII distribution, *Biometrika* (1977), 64, 129-134.

[19] Shah, A. and Gokhale, D. V., On maximum product of spagings (mps) estimation for burr xii distributions, *Communications in Statistics- Simulation and Computation* (1993), 22 (3), 615-641.

[20] Silva, R. B. and Cordeiro, G. M., The Burr XII power series distributions: A new compounding family, *Brazilian Journal of Probability and Statistics* (2015), 29 (3), 565-589.
[21] Tadikamalla P. R., A Look at the Burr and Related Distributions, *Int. Stat. Rev.* (1980), 48(3), 337-344.
[22] Wingo, D. R., Maximum likelihood methods for fitting the Burr type XII distribution to life test data, *Biometrical Journal* (1983), 25 (1), 77–84.
[23] Wingo, D. R., Maximum likelihood methods for fitting the Burr type XII distribution to multiply (progressively) censored life test data, *Metrika* (1993), 40, 203–210.

Current address: Yeşim Güney (Corresponding author): Ankara University, Faculty of Sciences, Department of Statistics 06100 Besevler ANKARA TURKEY.
E-mail address: ydone@ankara.edu.tr
Current address: Olcay Arslan: Ankara University, Faculty of Sciences, Department of Statistics 06100 Besevler ANKARA TURKEY.
E-mail address: oarslan@ankara.edu.tr