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# SPECTRAL ANALYSIS OF BOUNDARY VALUE PROBLEMS WITH RETARDED ARGUMENT

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Abstract. In this paper, by modifying some techniques of [S.B. Norkin, Differential equations of the second order with retarded argument, Translations of Mathematical Monographs, Vol. 31, AMS, Providence, RI, 1972] and suggesting own approaches we find asymptotic formulas for the eigenvalues and eigenfunctions of boundary value problems of Sturm-Liouville type for the second order differential equation with retarded argument.

## 1. Formulation of the problem

In this study we shall investigate discontinuous eigenvalue problems which consist of Sturm-Liouville equation

$$a(x)y''(x) + M(x)y(x - \Delta(x)) + \lambda y(x) = 0 \tag{1}$$

on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, with boundary conditions

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \tag{2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \tag{3}$$

and transmission conditions

$$y(\frac{\pi}{2} - 0) - \delta_1 y(\frac{\pi}{2} + 0) = 0, \tag{4}$$

$$y'(\frac{\pi}{2} - 0) - \delta_2 y'(\frac{\pi}{2} + 0) = 0, \tag{5}$$

where $a(x) = a_1^2$ for $x \in \left[0, \frac{\pi}{2}\right)$ and $a(x) = a_2^2$ for $x \in \left(\frac{\pi}{2}, \pi\right]$; the real-valued function $M(x)$ is continuous in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and has a finite limit $M(\frac{\pi}{2} \pm 0) = \lim_{x \to \frac{\pi}{2} \pm 0} M(x)$, the real valued function $\Delta(x) \geq 0$ continuous in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and has a finite limit $\Delta(\frac{\pi}{2} \pm 0) = \lim_{x \to \frac{\pi}{2} \pm 0} \Delta(x)$, $x - \Delta(x) \geq 0$, if $x \in \left[0, \frac{\pi}{2}\right)$; $x - \Delta(x) \geq$

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$\frac{\pi}{2}$, if $x \in \left(\frac{\pi}{2}, \pi\right]$; $\lambda$ is a real spectral parameter; $\delta_i$'s $(i = 1, 2)$ are arbitrary real numbers.

In the book [1] and papers [2-9] the asymptotic formulas for the eigenvalues and eigenfunctions of boundary value problems with retarded argument and a spectral parameter in the differential equation and/or boundary conditions and/or transmission conditions were obtained.

The articles [10-18] are devoted to study of the spectral properties of eigenvalues and eigenfunctions of the classical Sturm-Liouville problems.

If we take $\Delta(x) \equiv 0$ and/or $\delta_1 = \delta_2 = 1$ and/or $a(x) \equiv 1$ then the asymptotic formulas for eigenvalues and eigenfunctions correspond to those for the classical Sturm-Liouville problem [10, 14, 16-18].

Let $\phi_1(x, \lambda)$ be a solution of Eq. (1) on $\left[0, \frac{\pi}{2}\right]$, satisfying the initial conditions

$$\phi_1(0, \lambda) = \sin \alpha, \ \phi_1'(0, \lambda) = -\cos \alpha. \tag{6}$$

The conditions (6) define a unique solution of Eq. (1) on $\left[0, \frac{\pi}{2}\right]$ [1].

After defining the above solution we shall define the solution $\phi_2(x, \lambda)$ of Eq. (1) on $\left[\frac{\pi}{2}, \pi\right]$ by means of the solution $\phi_1(x, \lambda)$ using the initial conditions

$$\phi_2\left(\frac{\pi}{2}, \lambda\right) = \delta_1^{-1}\phi_1\left(\frac{\pi}{2}, \lambda\right), \quad \phi_2'\left(\frac{\pi}{2}, \lambda\right) = \delta_2^{-1}\phi_1'\left(\frac{\pi}{2}, \lambda\right). \tag{7}$$

The conditions (7) are defined as a unique solution of Eq. (1) on $\left[\frac{\pi}{2}, \pi\right]$.

Consequently, the function $\phi(x, \lambda)$ is defined on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ by the equality

$$\phi(x, \lambda) = \begin{cases} \phi_1(x, \lambda), & x \in [0, \frac{\pi}{2}) \\ \phi_2(x, \lambda), & x \in (\frac{\pi}{2}, \pi] \end{cases}$$

is a solution of the Eq. (1) on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$; which satisfies one of the boundary conditions and both transmission conditions.

**Lemma 1.1.** *Let $\phi(x, \lambda)$ be a solution of Eq.(1) and $\lambda > 0$. Then the following integral equations hold:*

$$\phi_1(x, \lambda) = \sin \alpha \cos \frac{s}{a_1} x - \frac{a_1 \cos \alpha}{s} \sin \frac{s}{a_1} x$$
$$- \frac{1}{a_1 s} \int_0^x M(\tau) \sin \frac{s}{a_1}(x - \tau) \phi_1(\tau - \Delta(\tau), \lambda) d\tau \quad \left(s = \sqrt{\lambda}, \lambda > 0\right), \tag{8}$$

$$\phi_2(x, \lambda) = \frac{\phi_1\left(\frac{\pi}{2}, \lambda\right)}{\delta_1} \cos \frac{s}{a_2}\left(x - \frac{\pi}{2}\right) + \frac{a_2 \phi_1'\left(\frac{\pi}{2}, \lambda\right)}{s\delta_2} \sin \frac{s}{a_2}\left(x - \frac{\pi}{2}\right)$$
$$- \frac{1}{a_2 s} \int_{\pi/2}^x M(\tau) \sin \frac{s}{a_2}(x - \tau) \phi_2(\tau - \Delta(\tau), \lambda) d\tau \quad \left(s = \sqrt{\lambda}, \lambda > 0\right). \tag{9}$$

*Proof.* To prove this, it is enough to substitute $-\frac{s^2}{a_1^2}\phi_1(\tau,\lambda)-\frac{\partial^2\phi_1(\tau,\lambda)}{\partial\tau^2}$ and $-\frac{s^2}{a_2^2}\phi_2(\tau,\lambda)-\frac{\partial^2\phi_2(\tau,\lambda)}{\partial\tau^2}$ instead of $-\frac{M(\tau)}{a_1^2}\phi_1(\tau-\Delta(\tau),\lambda)$ and $-\frac{M(\tau)}{a_2^2}\phi_2(\tau-\Delta(\tau),\lambda)$ in the integrals in (8) and (9) respectively and integrate by parts twice. □

**Theorem 1.2.** *The problem* $(1)-(5)$ *can have only simple eigenvalues.*

*Proof.* It is similar to the proof of Theorem 1 in [6]. □

## 2. Existence of solutions of the problem

The function $\phi(x,\lambda)$ defined in introduction is a nontrivial solution of Eq. (1) satisfying conditions (2), (4) and (5). Putting $\phi(x,\lambda)$ into (3), we get the characteristic equation

$$R(\lambda)\equiv\phi(\pi,\lambda)\cos\beta+\phi'(\pi,\lambda)\sin\beta=0. \tag{10}$$

By Theorem 1.2 the set of eigenvalues of boundary-value problem (1)-(5) coincides with the set of real roots of Eq. (10). Let $M_1=a_1^{-1}\int\limits_0^{\pi/2}|M(\tau)|d\tau$ and $M_2=a_2^{-1}\int\limits_{\pi/2}^{\pi}|M(\tau)|\,d\tau$. Also let us assume that $\lambda\geq\max\left\{k^2M_1^2,k^2M_2^2\right\}$, $k>1$ $(k\in\mathbb{R})$. Then for the solution $\phi_1(x,\lambda)$ of Eq. (8), the following inequality holds:

$$|\phi_1(x,\lambda)|\leq\frac{\sqrt{k^2M_1^2\sin^2\alpha+a_1^2\cos^2\alpha}}{(k-1)\,|M_1|},\quad x\in\left[0,\frac{\pi}{2}\right]. \tag{11}$$

Differentiating (8) with respect to $x$, we have

$$\phi_1'(x,\lambda)=-\frac{s}{a_1}\sin\alpha\sin\frac{s}{a_1}x-\cos\alpha\cos\frac{s}{a_1}x-\frac{1}{a_1^2}\int\limits_0^x M(\tau)\cos\frac{s}{a_1}(x-\tau)\,\phi_1(\tau-\Delta(\tau),\lambda)d\tau. \tag{12}$$

Then from (11) and (12) for the solution $\phi_2(x,\lambda)$ of Eq. (9), the following inequality holds:

$$|\phi_2(x,\lambda)|\leq\frac{\sqrt{k^2M_1^2\sin^2\alpha+a_1^2\cos^2\alpha}}{(k-1)^2\,|a_1M_1\delta_1\delta_2|},\quad x\in\left[\frac{\pi}{2},\pi\right]. \tag{13}$$

**Theorem 2.1.** *The problem* $(1)-(5)$ *has an infinite set of positive eigenvalues.*

*Proof.* Differentiating (9) with respect to $x$, we get

$$\phi_2'(x,\lambda)=-\frac{s\phi_1\left(\frac{\pi}{2},\lambda\right)}{a_2\delta_1}\sin\frac{s}{a_2}\left(x-\frac{\pi}{2}\right)+\frac{\phi_1'\left(\frac{\pi}{2},\lambda\right)}{\delta_2}\cos\frac{s}{a_2}\left(x-\frac{\pi}{2}\right)$$
$$-\frac{1}{a_2^2}\int\limits_{\pi/2}^x M(\tau)\cos\frac{s}{a_2}(x-\tau)\,\phi_2(\tau-\Delta(\tau),\lambda)d\tau\quad(s=\sqrt{\lambda},\lambda>0). \tag{14}$$

From (8)-(10), (12) and (14), we get

$$\Bigg[\frac{1}{\delta_1}\Bigg(\sin\alpha\cos\frac{s\pi}{2a_1}-\frac{a_1\cos\alpha}{s}\sin\frac{s\pi}{2a_1}$$
$$-\frac{1}{a_1 s}\int_0^{\frac{\pi}{2}}M(\tau)\sin\frac{s}{a_1}(\frac{\pi}{2}-\tau)\phi_1(\tau-\Delta(\tau),\lambda)d\tau\Bigg)\times\cos\frac{s\pi}{2a_2}$$
$$+\frac{a_2}{s\delta_2}\Bigg(-\frac{s}{a_1}\sin\alpha\sin\frac{s\pi}{2a_1}-\cos\alpha\cos\frac{s\pi}{2a_1}$$
$$-\frac{1}{a_1^2}\int_0^{\frac{\pi}{2}}M(\tau)\cos\frac{s}{a_1}(\frac{\pi}{2}-\tau)\phi_1(\tau-\Delta(\tau),\lambda)d\tau\Bigg)\times\sin\frac{s\pi}{2a_2}$$
$$-\frac{1}{a_2 s}\int_{\pi/2}^{\pi}M(\tau)\sin\frac{s}{a_2}(\pi-\tau)\phi_2(\tau-\Delta(\tau),\lambda)d\tau\Bigg]\cos\beta$$
$$+\Bigg[-\frac{s}{a_2\delta_1}\Bigg(\sin\alpha\cos\frac{s\pi}{2a_1}-\frac{a_1\cos\alpha}{s}\sin\frac{s\pi}{2a_1}$$
$$-\frac{1}{a_1 s}\int_0^{\frac{\pi}{2}}M(\tau)\sin\frac{s}{a_1}(\frac{\pi}{2}-\tau)\phi_1(\tau-\Delta(\tau),\lambda)d\tau\Bigg)\times\sin\frac{s\pi}{2a_2}$$
$$+\frac{1}{\delta_2}\Bigg(-\frac{s}{a_1}\sin\alpha\sin\frac{s\pi}{2a_1}-\cos\alpha\cos\frac{s\pi}{2a_1}-$$
$$\frac{1}{a_1^2}\int_0^{\frac{\pi}{2}}M(\tau)\cos\frac{s}{a_1}(\frac{\pi}{2}-\tau)\phi_1(\tau-\Delta(\tau),\lambda)d\tau\Bigg)\times\cos\frac{s\pi}{2a_2}$$
$$-\frac{1}{a_2^2}\int_{\frac{\pi}{2}}^{\pi}M(\tau)\cos\frac{s}{a_2}(\pi-\tau)\phi_2(\tau-\Delta(\tau),\lambda)d\tau\Bigg]\sin\beta=0 \quad (15)$$

There are four possible cases:
1. $\sin\alpha\neq 0, \sin\beta\neq 0$;
2. $\sin\alpha\neq 0, \sin\beta=0$;
3. $\sin\alpha=0, \sin\beta\neq 0$;
4. $\sin\alpha=0, \sin\beta=0$.

Let $\lambda$ be sufficiently large and $\delta_1 a_2=\delta_2 a_1$.

**Cases 1 and 4.** Then, by (11) and (13), Eq. (15) may be rewritten in the form

$$s\sin\frac{s\pi\,(a_1+a_2)}{2a_1a_2}+O(1)=0. \quad (16)$$

For large $s$ Eq. (16) has an infinite set of roots.

**Cases 2 and 3.** In these cases, Eq. (15) assumes the form

$$s\cos\frac{s\pi\,(a_1+a_2)}{2a_1a_2}+O(1)=0. \quad (17)$$

Obviously, for large $s$ Eq. (17) has, evidently, an infinite set of roots. The proof is complete. □

Thus, by Theorem 1.2 we conclude that the problem (1)-(5) has infinitely many nontrivial solutions.

## 3. Asymptotic formulas for eigenvalues and eigenfunctions

The function $\phi(x,\lambda)$ defined in introduction is a nontrivial solution of Eq. (1) satisfying conditions (2), (4) and (5). Putting $\phi(x,\lambda)$ into (3), we get the characteristic equation

$$R(\lambda) \equiv \phi(\pi,\lambda)\cos\beta + \phi'(\pi,\lambda)\sin\beta = 0. \tag{10}$$

By Theorem 1 the set of eigenvalues of boundary-value problem (1)-(5) coincides with the set of real roots of Eq. (10). Let $M_1 = a_1^{-1}\int\limits_0^{\pi/2} |M(\tau)|d\tau$ and $M_2 = a_2^{-1}\int\limits_{\pi/2}^{\pi} |M(\tau)|\, d\tau$. Also let us assume that $\lambda \geq \max\left\{k^2M_1^2, k^2M_2^2\right\}$, $k>1$ $(k\in\mathbb{R})$. Then for the solution $\phi_1(x,\lambda)$ of Eq. (8), the following inequality holds:

$$|\phi_1(x,\lambda)| \leq \frac{\sqrt{k^2M_1^2\sin^2\alpha + a_1^2\cos^2\alpha}}{(k-1)\,|M_1|}, \quad x\in\left[0,\frac{\pi}{2}\right]. \tag{11}$$

Differentiating (8) with respect to $x$, we have

$$\phi_1'(x,\lambda) = -\frac{s}{a_1}\sin\alpha\sin\frac{s}{a_1}x - \cos\alpha\cos\frac{s}{a_1}x - \frac{1}{a_1^2}\int\limits_0^x M(\tau)\cos\frac{s}{a_1}(x-\tau)\,\phi_1(\tau-\Delta(\tau),\lambda)d\tau. \tag{12}$$

Then from (11) and (12) for the solution $\phi_2(x,\lambda)$ of Eq. (9), the following inequality holds:

$$|\phi_2(x,\lambda)| \leq \frac{\sqrt{k^2M_1^2\sin^2\alpha + a_1^2\cos^2\alpha}}{(k-1)^2\,|a_1M_1\delta_1\delta_2|}, \quad x\in\left[\frac{\pi}{2},\pi\right]. \tag{13}$$

**Theorem 3.1.** *The problem* (1) − (5) *has an infinite set of positive eigenvalues.*

*Proof.* Differentiating (9) with respect to $x$, we get

$$\phi_2'(x,\lambda) = -\frac{s\phi_1\left(\frac{\pi}{2},\lambda\right)}{a_2\delta_1}\sin\frac{s}{a_2}\left(x-\frac{\pi}{2}\right) + \frac{\phi_1'\left(\frac{\pi}{2},\lambda\right)}{\delta_2}\cos\frac{s}{a_2}\left(x-\frac{\pi}{2}\right)$$
$$-\frac{1}{a_2^2}\int\limits_{\pi/2}^x M(\tau)\cos\frac{s}{a_2}(x-\tau)\,\phi_2(\tau-\Delta(\tau),\lambda)d\tau \quad (s=\sqrt{\lambda},\lambda>0). \tag{14}$$

From (8)-(10), (12) and (14), we get

$$\left[\frac{1}{\delta_1}\left(\sin\alpha\cos\frac{s\pi}{2a_1}-\frac{a_1\cos\alpha}{s}\sin\frac{s\pi}{2a_1}\right.\right.$$
$$\left.-\frac{1}{a_1 s}\int_0^{\frac{\pi}{2}}M(\tau)\sin\frac{s}{a_1}(\frac{\pi}{2}-\tau)\phi_1(\tau-\Delta(\tau),\lambda)d\tau\right)\times\cos\frac{s\pi}{2a_2}$$
$$+\frac{a_2}{s\delta_2}\left(-\frac{s}{a_1}\sin\alpha\sin\frac{s\pi}{2a_1}-\cos\alpha\cos\frac{s\pi}{2a_1}\right.$$
$$\left.-\frac{1}{a_1^2}\int_0^{\frac{\pi}{2}}M(\tau)\cos\frac{s}{a_1}(\frac{\pi}{2}-\tau)\phi_1(\tau-\Delta(\tau),\lambda)d\tau\right)$$
$$\left.\times\sin\frac{s\pi}{2a_2}-\frac{1}{a_2 s}\int_{\pi/2}^{\pi}M(\tau)\sin\frac{s}{a_2}(\pi-\tau)\phi_2(\tau-\Delta(\tau),\lambda)d\tau\right]\cos\beta$$
$$+\left[-\frac{s}{a_2\delta_1}\left(\sin\alpha\cos\frac{s\pi}{2a_1}-\frac{a_1\cos\alpha}{s}\sin\frac{s\pi}{2a_1}\right.\right.$$
$$\left.-\frac{1}{a_1 s}\int_0^{\frac{\pi}{2}}M(\tau)\sin\frac{s}{a_1}(\frac{\pi}{2}-\tau)\phi_1(\tau-\Delta(\tau),\lambda)d\tau\right)\times\sin\frac{s\pi}{2a_2}$$
$$+\frac{1}{\delta_2}\left(-\frac{s}{a_1}\sin\alpha\sin\frac{s\pi}{2a_1}-\cos\alpha\cos\frac{s\pi}{2a_1}\right.$$
$$\left.-\frac{1}{a_1^2}\int_0^{\frac{\pi}{2}}M(\tau)\cos\frac{s}{a_1}(\frac{\pi}{2}-\tau)\phi_1(\tau-\Delta(\tau),\lambda)d\tau\right)\times\cos\frac{s\pi}{2a_2}$$
$$\left.-\frac{1}{a_2^2}\int_{\frac{\pi}{2}}^{\pi}M(\tau)\cos\frac{s}{a_2}(\pi-\tau)\phi_2(\tau-\Delta(\tau),\lambda)d\tau\right]\sin\beta=0. \quad (15)$$

There are four possible cases:
1. $\sin\alpha\neq 0,\sin\beta\neq 0$;
2. $\sin\alpha\neq 0,\sin\beta=0$;
3. $\sin\alpha=0,\sin\beta\neq 0$;
4. $\sin\alpha=0,\sin\beta=0$.

Let $\lambda$ be sufficiently large and $\delta_1 a_2=\delta_2 a_1$.

**Cases 1 and 4.** Then, by (11) and (13), Eq. (15) may be rewritten in the form

$$s\sin\frac{s\pi\,(a_1+a_2)}{2a_1a_2}+O(1)=0. \quad (16)$$

For large $s$ Eq. (16) has an infinite set of roots.

**Cases 2 and 3.** In these cases, Eq. (15) assumes the form

$$s\cos\frac{s\pi\,(a_1+a_2)}{2a_1a_2}+O(1)=0. \tag{17}$$

Obviously, for large $s$ Eq. (17) has, evidently, an infinite set of roots. The proof is complete. □

Thus, by Theorem 1 we conclude that the problem (1)-(5) has infinitely many nontrivial solutions. We shall now study the asymptotic properties of eigenvalues and eigenfunctions. In the following we shall assume that $s$ is sufficiently large. From (8) and (11), we get

$$\phi_1(x,\lambda)=O(1)\quad \text{on}\quad [0,\frac{\pi}{2}] \tag{18}$$

and from (9) and (13), we get

$$\phi_2(x,\lambda)=O(1)\quad \text{on}\quad [\frac{\pi}{2},\pi] \tag{19}$$

in the cases 1 and 2. In cases 3 and 4 we have

$$\phi_1(x,\lambda)=O(\frac{1}{s})\quad \text{on}\quad [0,\frac{\pi}{2}] \tag{20}$$

and

$$\phi_2(x,\lambda)=O(\frac{1}{s})\quad \text{on}\quad [\frac{\pi}{2},\pi]. \tag{21}$$

The existence and continuity of the derivatives $\phi'_{1s}(x,\lambda)$ for $0\le x\le\frac{\pi}{2}$, $|\lambda|<\infty$, and $\phi'_{2s}(x,\lambda)$ for $\frac{\pi}{2}\le x\le\pi$, $|\lambda|<\infty$, follows from Theorem 1.4.1 in [1]. Using the same technique in the proof of Lemma 3.1 in [1], we have the following equalities:

In cases 1 and 2

$$\phi'_{1s}(x,\lambda)=O(1),\quad x\in[0,\frac{\pi}{2}], \tag{22}$$

$$\phi'_{2s}(x,\lambda)=O(1),\quad x\in[\frac{\pi}{2},\pi] \tag{23}$$

and in cases 3 and 4

$$\phi'_{1s}(x,\lambda)=O(\frac{1}{s}),\quad x\in[0,\frac{\pi}{2}], \tag{24}$$

$$\phi'_{2s}(x,\lambda)=O(\frac{1}{s}),\quad x\in[\frac{\pi}{2},\pi] \tag{25}$$

hold.

Let $n$ be a natural number. We shall say that the number $\lambda$ is situated near the number $\frac{4n^2a_1^2a_2^2}{(a_1+a_2)^2}$ or $\frac{(2n+1)^2a_1^2a_2^2}{(a_1+a_2)^2}$ if, respectively $\left|\frac{2na_1a_2}{a_1+a_2}-\sqrt{\lambda}\right|<\frac{1}{k^2}$ or $\left|\frac{(2n+1)a_1a_2}{a_1+a_2}-\sqrt{\lambda}\right|<\frac{1}{k^2}$ $(k>1,k\in\mathbb{R})$.

**Theorem 3.2.** *Let $n$ be a natural number. For each sufficiently large $n$, in cases 1 and 4, there is exactly one eigenvalue of the problem* $(1)-(5)$ *near* $\frac{4n^2a_1^2a_2^2}{(a_1+a_2)^2}$ *and in cases 2 and 3, there is exactly one eigenvalue of this problem near* $\frac{(2n+1)^2a_1^2a_2^2}{(a_1+a_2)^2}$.

*Proof.* It is similar to proof of Theorem 3.3.1 in [1]. □

With the helps of Eqs. (16) and (17) we can find asymptotic formulas for eigenvalues of the problem (1)-(5). Let $n$ be sufficiently large. In what follows we shall denote by $\lambda_n = s_n^2$ the eigenvalue of the problem $(1)-(5)$ situated near $\frac{4n^2a_1^2a_2^2}{(a_1+a_2)^2}$ (or near $\frac{(2n+1)^2a_1^2a_2^2}{(a_1+a_2)^2}$).

**Cases 1 and 4.** We set $s_n = \frac{2na_1a_2}{a_1+a_2} + \delta_n$. From Eq. (16) it follows that $\delta_n = O\left(\frac{1}{n}\right)$. Consequently

$$s_n = \frac{2na_1a_2}{a_1+a_2} + O\left(\frac{1}{n}\right). \tag{26}$$

**Cases 2 and 3.** We set $s_n = \frac{(2n+1)a_1a_2}{a_1+a_2} + \delta_n$. From Eq. (17) it follows that $\delta_n = O\left(\frac{1}{n}\right)$. Consequently

$$s_n = \frac{(2n+1)\,a_1a_2}{a_1+a_2} + O\left(\frac{1}{n}\right). \tag{27}$$

Formula (26) and (27) make it possible to obtain asymptotic expressions for eigenfunction of the problem $(1)-(5)$. In cases 1 and 2, from (8), (12) and (18), we get

$$\phi_1(x,\lambda) = \sin\alpha\cos\frac{s}{a_1}x + O\left(\frac{1}{s}\right), \tag{28}$$

From (9), (19) and (28), we get

$$\phi_2(x,\lambda) = \frac{\sin\alpha}{\delta_1}\cos s\left(\frac{\pi\,(a_2-a_1)}{2a_1a_2} + \frac{x}{a_2}\right) + O\left(\frac{1}{s}\right). \tag{29}$$

In cases 3 and 4, from (8), (12) and (20), we obtain

$$\phi_1(x,\lambda) = -\frac{a_1\cos\alpha}{s}\sin\frac{s}{a_1}x + O\left(\frac{1}{s^2}\right), \tag{30}$$

From (9), (21) and (30), we get

$$\phi_2(x,\lambda) = -\frac{a_1\cos\alpha}{s\delta_1}\sin s\left(\frac{\pi\,(a_2-a_1)}{2a_1a_2} + \frac{x}{a_2}\right) + O\left(\frac{1}{s^2}\right). \tag{31}$$

Now we can write the asymptotic representations of eigenfunctions

$$y_n(x) = \begin{cases} \phi_1\left(x,\lambda_n\right) & \text{for} \quad x \in [0,\frac{\pi}{2}), \\ \phi_2\left(x,\lambda_n\right) & \text{for} \quad x \in (\frac{\pi}{2},\pi]. \end{cases}$$

for the problem (1)-(5):

**Case 1.** By substituting (26) into (28) and (29), we find that

$$\phi_1(x,\lambda_n)=\sin\alpha\cos\frac{2na_2x}{a_1+a_2}+O\left(\frac{1}{n}\right),$$

$$\phi_2(x,\lambda_n)=\frac{\sin\alpha}{\delta_1}\cos\left(\frac{n\pi(a_2-a_1)}{a_1+a_2}+\frac{2na_1x}{a_1+a_2}\right)+O\left(\frac{1}{n}\right).$$

**Case 2.** By substituting (27) into (28) and (29), we find that

$$\phi_1(x,\lambda_n)=\sin\alpha\cos\frac{(2n+1)a_2x}{a_1+a_2}+O\left(\frac{1}{n}\right),$$

$$\phi_2(x,\lambda_n)=\frac{\sin\alpha}{\delta_1}\cos\left(\frac{(2n+1)(a_2-a_1)\pi}{2(a_1+a_2)}+\frac{(2n+1)a_1x}{a_1+a_2}\right)+O\left(\frac{1}{n}\right).$$

**Case 3.** By substituting (27) into (30) and (31), we find that

$$\phi_1(x,\lambda_n)=-\frac{(a_1+a_2)\cos\alpha}{(2n+1)a_2}\sin\frac{(2n+1)a_2x}{a_1+a_2}+O\left(\frac{1}{n^2}\right),$$

$$\phi_2(x,\lambda_n)=-\frac{(a_1+a_2)\cos\alpha}{(2n+1)a_2\delta_1}\sin\left(\frac{(2n+1)(a_2-a_1)\pi}{2(a_1+a_2)}+\frac{(2n+1)a_1x}{a_1+a_2}\right)+O\left(\frac{1}{n^2}\right).$$

**Case 4.** By substituting (26) into (30) and (31), we find that

$$\phi_1(x,\lambda_n)=-\frac{(a_1+a_2)\cos\alpha}{2na_2}\sin\frac{2na_2x}{a_1+a_2}+O\left(\frac{1}{n^2}\right),$$

$$\phi_2(x,\lambda_n)=-\frac{(a_1+a_2)\cos\alpha}{2na_2\delta_1}\sin\left(\frac{n\pi(a_2-a_1)}{a_1+a_2}+\frac{2na_1x}{a_1+a_2}\right)+O\left(\frac{1}{n^2}\right).$$

## 4. Sharper asymptotic formulas for eigenvalues and eigenfunctions

Under some additional conditions the more exact asymptotic formulas which depend upon the retardation may be obtained. Let us assume that the following conditions are fulfilled:

**a)** The derivatives $M'(x)$ and $\Delta''(x)$ exist and are bounded in $[0,\frac{\pi}{2})\bigcup(\frac{\pi}{2},\pi]$ and have finite limits $M'(\frac{\pi}{2}\pm 0)=\lim\limits_{x\to\frac{\pi}{2}\pm 0}M'(x)$ and $\Delta''(\frac{\pi}{2}\pm 0)=\lim\limits_{x\to\frac{\pi}{2}\pm 0}\Delta''(x)$, respectively.

**b)** $\Delta'(x)\leq 1$ in $[0,\frac{\pi}{2})\bigcup(\frac{\pi}{2},\pi]$, $\Delta(0)=0$ and $\lim\limits_{x\to\frac{\pi}{2}+0}\Delta(x)=0$.

By using b), we have

$$x-\Delta(x)\geq 0,\ x\in[0,\frac{\pi}{2}), \tag{32}$$

$$x-\Delta(x)\geq\frac{\pi}{2},\ x\in(\frac{\pi}{2},\pi]. \tag{33}$$

From (28), (29), (32) and (33), in the cases 1 and 2 we have

$$\phi_1(\tau-\Delta(\tau),\lambda)=\sin\alpha\cos\frac{s}{a_1}(\tau-\Delta(\tau))+O\left(\frac{1}{s}\right), \tag{34}$$

$$\phi_2\left(\tau-\Delta\left(\tau\right),\lambda\right)=\frac{\sin\alpha}{\delta_1}\cos\frac{s}{a_2}\left(\frac{\pi\left(a_2-a_1\right)}{2a_1}+\tau-\Delta\left(\tau\right)\right)+O\left(\frac{1}{s}\right). \quad (35)$$

From (30), (31), (32) and (33), in the cases 3 and 4 we have

$$\phi_1\left(\tau-\Delta\left(\tau\right),\lambda\right)=-\frac{a_1\cos\alpha}{s}\sin\frac{s}{a_1}\left(\tau-\Delta\left(\tau\right)\right)+O\left(\frac{1}{s^2}\right), \quad (36)$$

$$\phi_2\left(\tau-\Delta\left(\tau\right),\lambda\right)=-\frac{a_1\cos\alpha}{s\delta_1}\sin\frac{s}{a_2}\left(\frac{\pi\left(a_2-a_1\right)}{2a_1}+\tau-\Delta\left(\tau\right)\right)+O\left(\frac{1}{s^2}\right). \quad (37)$$

Let

$$A\left(x,s,\Delta(\tau)\right)=\tfrac{1}{2}\int_0^x M(\tau)\sin\left(\tfrac{s}{a_1}\Delta(\tau)\right)d\tau,$$
$$B(x,s,\Delta(\tau))=\tfrac{1}{2}\int_0^x M(\tau)\cos\left(\tfrac{s}{a_1}\Delta(\tau)\right)d\tau.$$

It is obvious that these functions are bounded for $0\le x\le\frac{\pi}{2}$ and $0<s<\infty$. Let

$$C(x,s,\Delta(\tau))=\tfrac{1}{2}\int_{\frac{\pi}{2}}^x M(\tau)\sin\left(\tfrac{s}{a_2}\Delta(\tau)\right)d\tau,$$
$$D(x,s,\Delta(\tau))=\tfrac{1}{2}\int_{\frac{\pi}{2}}^x M(\tau)\cos\left(\tfrac{s}{a_2}\Delta(\tau)\right)d\tau.$$

It is obvious that these functions are bounded for $\frac{\pi}{2}\le x\le\pi$ and $0<s<\infty$. Under the conditions a) and b) the following formulas

$$\left\{\begin{array}{l}\int_0^x M(\tau)\cos\frac{s}{a_1}(2\tau-\Delta(\tau))d\tau\\ \int_0^x M(\tau)\sin\frac{s}{a_1}(2\tau-\Delta(\tau))d\tau\\ \int_{\frac{\pi}{2}}^x M(\tau)\cos\frac{s}{a_2}(2\tau-\Delta(\tau))d\tau\\ \int_{\frac{\pi}{2}}^x M(\tau)\sin\frac{s}{a_2}(2\tau-\Delta(\tau))d\tau\end{array}\right.=O\left(\frac{1}{s}\right) \quad (38)$$

can be proved by the same technique in Lemma 3.3.3 in [1].

**Case 1.** Putting the expressions (34) and (35) into (15), and using the equalities in (15), after long operations we have

$$\tan\frac{s\pi\left(a_1+a_2\right)}{2a_1a_2}=\frac{1}{s}\left[-\frac{\cos\beta\sin\alpha}{\delta_1}+\frac{\sin\beta\cos\alpha}{\delta_2}\right.$$
$$\left.+\frac{B\left(\frac{\pi}{2},s,\Delta\left(\tau\right)\right)\sin\beta}{a_1a_2\delta_1}+\frac{D\left(\pi,s,\Delta\left(\tau\right)\right)\sin\alpha\sin\beta}{a_2^2\delta_1}\right]+O\left(\frac{1}{s^2}\right).$$

Again if we take $s_n=\frac{2na_1a_2}{a_1+a_2}+\delta_n$, then

$$\tan\pi\left(n+\frac{\delta_n\left(a_1+a_2\right)}{2a_1a_2}\right)=\tan\frac{\delta_n\pi\left(a_1+a_2\right)}{2a_1a_2}=\frac{a_1+a_2}{2na_1a_2}$$

$$\times\left[-\frac{\cos\beta\sin\alpha}{\delta_1}+\frac{\sin\beta\cos\alpha}{\delta_2}+\frac{B\left(\frac{\pi}{2},\frac{2na_1a_2}{a_1+a_2},\Delta(\tau)\right)\sin\beta}{a_1a_2\delta_1}\right.$$
$$\left.+\frac{D\left(\pi,\frac{2na_1a_2}{a_1+a_2},\Delta(\tau)\right)\sin\alpha\sin\beta}{a_2^2\delta_1}\right]+O\left(\frac{1}{n^2}\right).$$

Hence for large $n$,

$$\delta_n=\frac{1}{n\pi}\left[-\frac{\cos\beta\sin\alpha}{\delta_1}+\frac{\sin\beta\cos\alpha}{\delta_2}+\frac{B\left(\frac{\pi}{2},\frac{2na_1a_2}{a_1+a_2},\Delta(\tau)\right)\sin\beta}{a_1a_2\delta_1}\right.$$
$$\left.+\frac{D\left(\pi,\frac{2na_1a_2}{a_1+a_2},\Delta(\tau)\right)\sin\alpha\sin\beta}{a_2^2\delta_1}\right]+O\left(\frac{1}{n^2}\right)$$

and finally

$$s_n=\frac{2na_1a_2}{a_1+a_2}+\frac{1}{n\pi}\left[-\frac{\cos\beta\sin\alpha}{\delta_1}+\frac{\sin\beta\cos\alpha}{\delta_2}+\frac{B\left(\frac{\pi}{2},\frac{2na_1a_2}{a_1+a_2},\Delta(\tau)\right)\sin\beta}{a_1a_2\delta_1}\right.$$
$$\left.+\frac{D\left(\pi,\frac{2na_1a_2}{a_1+a_2},\Delta(\tau)\right)\sin\alpha\sin\beta}{a_2^2\delta_1}\right]+O\left(\frac{1}{n^2}\right). \tag{39}$$

**Case 2.** Putting the expressions (34) and (35) into (15), and using the equalities in (15), after long operations we have

$$\cot\frac{s\pi(a_1+a_2)}{2a_1a_2}=\frac{1}{s}\left[\frac{a_1\cos\alpha}{\delta_1}+\frac{B\left(\frac{\pi}{2},s,\Delta(\tau)\right)}{a_1\delta_1}+\frac{D(\pi,s,\Delta(\tau))}{a_2\delta_1}\right]+O\left(\frac{1}{s^2}\right)$$

Again if we take $s_n=\frac{(2n+1)a_1a_2}{a_1+a_2}+\delta_n$, then

$$\cot\pi\left(\frac{2n+1}{2}+\frac{\delta_n(a_1+a_2)}{2a_1a_2}\right)=-\tan\frac{\delta_n\pi(a_1+a_2)}{2a_1a_2}=\frac{a_1+a_2}{(2n+1)a_1a_2}$$
$$\times\left[\frac{a_1\cos\alpha}{\delta_1}+\frac{B\left(\frac{\pi}{2},\frac{(2n+1)a_1a_2}{a_1+a_2},\Delta(\tau)\right)}{a_1\delta_1}+\frac{D\left(\pi,\frac{(2n+1)a_1a_2}{a_1+a_2},\Delta(\tau)\right)}{a_2\delta_1}\right]+O\left(\frac{1}{n^2}\right).$$

Hence for large $n$,

$$\delta_n=-\frac{2}{(2n+1)\pi}\left[\frac{a_1\cos\alpha}{\delta_1}+\frac{B\left(\frac{\pi}{2},\frac{(2n+1)a_1a_2}{a_1+a_2},\Delta(\tau)\right)}{a_1\delta_1}+\frac{D\left(\pi,\frac{(2n+1)a_1a_2}{a_1+a_2},\Delta(\tau)\right)}{a_2\delta_1}\right]$$
$$+O\left(\frac{1}{n^2}\right)$$

and finally

$$s_n = \frac{(2n+1)\,a_1a_2}{a_1+a_2} - \frac{2}{(2n+1)\,\pi} \times \left[\frac{a_1\cos\alpha}{\delta_1} + \frac{B\left(\frac{\pi}{2}, \frac{(2n+1)a_1a_2}{a_1+a_2}, \Delta(\tau)\right)}{a_1\delta_1} + \frac{D\left(\pi, \frac{(2n+1)a_1a_2}{a_1+a_2}, \Delta(\tau)\right)}{a_2\delta_1}\right] + O\left(\frac{1}{n^2}\right). \tag{40}$$

**Case 3.** Putting the expressions (36) and (37) into (15), and using the equalities in (15), after long operations we have

$$\cot\frac{s\pi(a_1+a_2)}{2a_1a_2} = \frac{1}{s}\left[\frac{a_1\cos\beta}{\delta_1} - \frac{A\left(\frac{\pi}{2}, s, \Delta(\tau)\right)\sin\beta}{a_2\delta_1} - \frac{a_1D(\pi, s, \Delta(\tau))\sin\beta}{a_2^2\delta_1}\right] + O\left(\frac{1}{s^2}\right)$$

Again if we take $s_n = \frac{(2n+1)a_1a_2}{a_1+a_2} + \delta_n$, then

$$\cot\pi\left(\frac{2n+1}{2} + \frac{\delta_n(a_1+a_2)}{2a_1a_2}\right) = -\tan\frac{\delta_n\pi(a_1+a_2)}{2a_1a_2} = \frac{a_1+a_2}{(2n+1)\,a_1a_2} \times \left[\frac{a_1\cos\beta}{\delta_1} - \frac{A\left(\frac{\pi}{2}, \frac{(2n+1)a_1a_2}{a_1+a_2}, \Delta(\tau)\right)\sin\beta}{a_2\delta_1} - \frac{a_1D\left(\pi, \frac{(2n+1)a_1a_2}{a_1+a_2}, \Delta(\tau)\right)\sin\beta}{a_2^2\delta_1}\right] + O\left(\frac{1}{n^2}\right).$$

Hence for large $n$,

$$\delta_n = -\frac{2}{(2n+1)\,\pi}\left[\frac{a_1\cos\beta}{\delta_1} - \frac{A\left(\frac{\pi}{2}, \frac{(2n+1)a_1a_2}{a_1+a_2}, \Delta(\tau)\right)\sin\beta}{a_2\delta_1} - \frac{a_1D\left(\pi, \frac{(2n+1)a_1a_2}{a_1+a_2}, \Delta(\tau)\right)\sin\beta}{a_2^2\delta_1}\right] + O\left(\frac{1}{n^2}\right)$$

and finally

$$s_n = \frac{(2n+1)\,a_1a_2}{a_1+a_2} - \frac{2}{(2n+1)\,\pi} \times \left[\frac{a_1\cos\beta}{\delta_1} - \frac{A\left(\frac{\pi}{2}, \frac{(2n+1)a_1a_2}{a_1+a_2}, \Delta(\tau)\right)\sin\beta}{a_2\delta_1} - \frac{a_1D\left(\pi, \frac{(2n+1)a_1a_2}{a_1+a_2}, \Delta(\tau)\right)\sin\beta}{a_2^2\delta_1}\right] + O\left(\frac{1}{n^2}\right). \tag{41}$$

**Case 4.** Putting the expressions (36) and (37) into (15), and using the equalities in (15), after long operations we have

$$\tan\frac{s\pi\left(a_1+a_2\right)}{2a_1a_2}=\frac{1}{s}\left[\frac{a_1D\left(\pi,s,\Delta\left(\tau\right)\right)}{a_2\delta_1}-\frac{B\left(\frac{\pi}{2},s,\Delta\left(\tau\right)\right)}{\delta_1}+\frac{a_1C\left(\pi,s,\Delta\left(\tau\right)\right)}{a_2\delta_1}\right]+O\left(\frac{1}{s^2}\right).$$

Again if we take $s_n=\frac{2na_1a_2}{a_1+a_2}+\delta_n$, then

$$\tan\pi\left(n+\frac{\delta_n\left(a_1+a_2\right)}{2a_1a_2}\right)=\tan\frac{\delta_n\pi\left(a_1+a_2\right)}{2a_1a_2}=\frac{a_1+a_2}{2na_1a_2}$$

$$\times\left[\frac{a_1D\left(\pi,\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)}{a_2\delta_1}-\frac{B\left(\frac{\pi}{2},\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)}{\delta_1}\right.$$

$$\left.+\frac{a_1C\left(\pi,\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)}{a_2\delta_1}\right]+O\left(\frac{1}{n^2}\right).$$

Hence for large $n$,

$$\delta_n=\frac{1}{n\pi}\left[\frac{a_1D\left(\pi,\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)}{a_2\delta_1}-\frac{B\left(\frac{\pi}{2},\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)}{\delta_1}\right.$$

$$\left.+\frac{a_1C\left(\pi,\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)}{a_2\delta_1}\right]+O\left(\frac{1}{n^2}\right)$$

and finally

$$s_n=\frac{2na_1a_2}{a_1+a_2}+\frac{1}{n\pi}$$

$$\times\left[\frac{a_1D\left(\pi,\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)}{a_2\delta_1}-\frac{B\left(\frac{\pi}{2},\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)}{\delta_1}+\frac{a_1C\left(\pi,\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)}{a_2\delta_1}\right]+O\left(\frac{1}{n^2}\right). \tag{42}$$

Now, we are ready to obtain a sharper asymptotic formula for the eigenfunctions.

**Case 1.** From (8) and (34)

$$\phi_1(x,\lambda)=\sin\alpha\cos\frac{s}{a_1}x-\frac{a_1\cos\alpha}{s}\sin\frac{s}{a_1}x$$

$$-\frac{\sin\alpha}{a_1s}\int_0^x M\left(\tau\right)\sin\frac{s}{a_1}\left(x-\tau\right)\cos\frac{s}{a_1}\left(\tau-\Delta\left(\tau\right)\right)d\tau. \tag{43}$$

Thus, using (38) and making necessary arrangements we have

$$\phi_1(x,\lambda) = \sin\alpha\cos\frac{s}{a_1}x\left[1+\frac{A\left(x,s,\Delta\left(\tau\right)\right)}{a_1 s}\right] - \frac{\sin\frac{s}{a_1}x}{s}\left[a_1\cos\alpha+\frac{\sin\alpha B\left(x,s,\Delta\left(\tau\right)\right)}{a_1}\right]+O\left(\frac{1}{s^2}\right). \tag{44}$$

Now replacing $s$ by $s_n$ and using (39) we get

$$\begin{aligned}\phi_{1n}(x) &= \sin\alpha\cos\frac{2na_2x}{a_1+a_2}\left[1+\frac{\left(a_1+a_2\right)A\left(x,\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)}{2na_1^2a_2}\right]\\ &-\sin\alpha\sin\frac{2na_2x}{a_1+a_2}\left\{\frac{x}{n\pi}\left[-\frac{\cos\beta\sin\alpha}{\delta_1}+\frac{\sin\beta\cos\alpha}{\delta_2}+\frac{B\left(\frac{\pi}{2},\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)\sin\beta}{a_1a_2\delta_1}\right.\right.\\ &\left.\left.+\frac{D\left(\pi,\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)\sin\alpha\sin\beta}{a_2^2\delta_1}\right]\right\}-\frac{a_1+a_2}{2na_1a_2}\sin\frac{2na_2x}{a_1+a_2}\\ &\times\left[a_1\cos\alpha+\frac{\sin\alpha B\left(x,\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)}{a_1}\right]+O\left(\frac{1}{n^2}\right).\end{aligned}$$

From (12), (34) and (38), we have

$$\begin{aligned}\frac{\phi_1'(x,\lambda)}{s} &= -\frac{\sin\alpha\sin\frac{s}{a_1}x}{a_1}\left[1+\frac{A\left(x,s,\Delta\left(\tau\right)\right)}{a_1s}\right]-\\ &\frac{\cos\frac{s}{a_1}x}{s}\left[a_1\cos\alpha+\frac{\sin\alpha B\left(x,s,\Delta\left(\tau\right)\right)}{a_1}\right]+O\left(\frac{1}{s^2}\right).\end{aligned} \tag{45}$$

From (9), (35), (38), (44) and (45)

$$\begin{aligned}\phi_2(x,\lambda) &= \frac{1}{\delta_1}\left\{\sin\alpha\cos\frac{s\pi}{2a_1}\left[1+\frac{A\left(\frac{\pi}{2},s,\Delta\left(\tau\right)\right)}{a_1s}\right]\right.\\ &\left.-\frac{\sin\frac{s\pi}{2a_1}}{s}\left[a_1\cos\alpha+\frac{B\left(\frac{\pi}{2},s,\Delta\left(\tau\right)\right)\sin\alpha}{a_1}\right]+O\left(\frac{1}{s^2}\right)\right\}\cos\frac{s}{a_2}\left(x-\frac{\pi}{2}\right)\\ &+\frac{a_2}{\delta_2}\left\{-\frac{\sin\alpha\sin\frac{s\pi}{2a_1}}{a_1}\left[1+\frac{A\left(\frac{\pi}{2},s,\Delta\left(\tau\right)\right)}{a_1s}\right]\right.\\ &\left.-\frac{\cos\frac{s\pi}{2a_1}}{a_1s}\left[a_1\cos\alpha+\frac{B\left(\frac{\pi}{2},s,\Delta\left(\tau\right)\right)\sin\alpha}{a_1}\right]+O\left(\frac{1}{s^2}\right)\right\}\sin\frac{s}{a_2}\left(x-\frac{\pi}{2}\right)\\ &-\frac{\sin\alpha}{a_2\delta_1s}\int_{\pi/2}^{x}M\left(\tau\right)\sin\frac{s}{a_2}\left(x-\tau\right)\cos\frac{s}{a_2}\left(\frac{\pi\left(a_2-a_1\right)}{2a_1}+\tau-\Delta\left(\tau\right)\right)+O\left(\frac{1}{s^2}\right)\end{aligned}$$

$$= \frac{\sin\alpha}{\delta_1}\cos\frac{s}{a_2}\left(\frac{\pi\left(a_2-a_1\right)}{2a_1}+x\right)\left[1+\frac{A\left(\frac{\pi}{2},s,\Delta\left(\tau\right)\right)}{a_1 s}\right]$$

$$-\frac{1}{\delta_1 s}\sin\frac{s}{a_2}\left(\frac{\pi\left(a_2-a_1\right)}{2a_1}+x\right)\left[a_1\cos\alpha+\frac{B\left(\frac{\pi}{2},s,\Delta\left(\tau\right)\right)\sin\alpha}{a_1}\right]$$

$$-\frac{D\left(x,s,\Delta\left(\tau\right)\right)\sin\alpha\sin\frac{s}{a_2}\left(\frac{\pi(a_2-a_1)}{2a_1}+x\right)+}{a_2\delta_1 s}$$
$$-\frac{C\left(x,s,\Delta\left(\tau\right)\right)\sin\alpha\cos\frac{s}{a_2}\left(\frac{\pi(a_2-a_1)}{2a_1}+x\right)}{a_2\delta_1 s}+O\left(\frac{1}{s^2}\right). \tag{46}$$

Now replacing $s$ by $s_n$ and using (39) we get

$$\phi_{2n}(x)=\frac{\sin\alpha}{\delta_1}\cos\left(\frac{2na_1}{a_1+a_2}\left(\frac{\pi\left(a_2-a_1\right)}{2a_1}+x\right)\right)\left[1+\frac{\left(a_1+a_2\right)A\left(\frac{\pi}{2},\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)}{2na_1^2a_2}\right]$$

$$-\frac{\sin\alpha}{\delta_1}\sin\left(\frac{2na_1}{a_1+a_2}\left(\frac{\pi\left(a_2-a_1\right)}{2a_1}+x\right)\right)\left\{\frac{x}{n\pi}\left[-\frac{\cos\beta\sin\alpha}{\delta_1}+\frac{\sin\beta\cos\alpha}{\delta_2}\right.\right.$$

$$\left.\left.+\frac{B\left(\frac{\pi}{2},\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)\sin\beta}{a_1a_2\delta_1}+\frac{D\left(\pi,\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)\sin\alpha\sin\beta}{a_2^2\delta_1}\right]\right\}$$

$$-\frac{a_1+a_2}{2na_1a_2\delta_1}\sin\left(\frac{2na_1}{a_1+a_2}\left(\frac{\pi\left(a_2-a_1\right)}{2a_1}+x\right)\right)\left[a_1\cos\alpha+\frac{B\left(\frac{\pi}{2},\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)\sin\alpha}{a_1}\right]$$

$$-\frac{\left(a_1+a_2\right)D\left(x,\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)\sin\alpha}{2na_1a_2^2\delta_1}\sin\left(\frac{2na_1}{a_1+a_2}\left(\frac{\pi\left(a_2-a_1\right)}{2a_1}+x\right)\right)$$

$$+\frac{\left(a_1+a_2\right)C\left(x,\frac{2na_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)\sin\alpha}{2na_1a_2^2\delta_1}\cos\left(\frac{2na_1}{a_1+a_2}\left(\frac{\pi\left(a_2-a_1\right)}{2a_1}+x\right)\right)+O\left(\frac{1}{n^2}\right).$$

**Case 2.** Replacing $s$ by $s_n$ in (44), we find, by use of (40), that

$$\phi_{1n}(x) = \sin\alpha\cos\frac{(2n+1)\,a_2x}{a_1+a_2}\left[1+\frac{(a_1+a_2)\,A\left(x,\frac{(2n+1)a_1a_2}{a_1+a_2},\Delta(\tau)\right)}{(2n+1)\,a_1^2a_2}\right]$$
$$-\sin\alpha\sin\frac{(2n+1)\,a_2x}{a_1+a_2}$$
$$\times\left\{-\frac{2x}{(2n+1)\,\pi}\left[\frac{a_1\cos\alpha}{\delta_1}+\frac{B\left(\frac{\pi}{2},\frac{(2n+1)a_1a_2}{a_1+a_2},\Delta(\tau)\right)}{a_1\delta_1}+\frac{D\left(\pi,\frac{(2n+1)a_1a_2}{a_1+a_2},\Delta(\tau)\right)}{a_2\delta_1}\right]\right\}$$
$$-\frac{a_1+a_2}{(2n+1)\,a_1a_2}\sin\frac{(2n+1)\,a_2x}{a_1+a_2}\left[a_1\cos\alpha+\frac{\sin\alpha B\left(x,\frac{(2n+1)a_1a_2}{a_1+a_2},\Delta(\tau)\right)}{a_1}\right]+O\left(\frac{1}{n^2}\right).$$

Now replacing $s$ by $s_n$ in (46), we find, by use of (40), that

$$\phi_{2n}(x) = \frac{\sin\alpha}{\delta_1}\cos\left(\frac{(2n+1)\,a_1}{a_1+a_2}\left(\frac{\pi\,(a_2-a_1)}{2a_1}+x\right)\right)$$
$$\times\left[1+\frac{(a_1+a_2)\,A\left(\frac{\pi}{2},\frac{(2n+1)a_1a_2}{a_1+a_2},\Delta(\tau)\right)}{(2n+1)\,a_1^2a_2}\right]$$
$$-\frac{\sin\alpha}{\delta_1}\sin\left(\frac{(2n+1)\,a_1}{a_1+a_2}\left(\frac{\pi\,(a_2-a_1)}{2a_1}+x\right)\right)\left\{-\frac{2x}{(2n+1)\,\pi}\right.$$
$$\left.\times\left[\frac{a_1\cos\alpha}{\delta_1}+\frac{B\left(\frac{\pi}{2},\frac{(2n+1)a_1a_2}{a_1+a_2},\Delta(\tau)\right)}{a_1\delta_1}+\frac{D\left(\pi,\frac{(2n+1)a_1a_2}{a_1+a_2},\Delta(\tau)\right)}{a_2\delta_1}\right]\right\}$$

$$-\frac{a_1+a_2}{(2n+1)\,a_1a_2\delta_1}\sin\left(\frac{(2n+1)\,a_1}{a_1+a_2}\left(\frac{\pi\,(a_2-a_1)}{2a_1}+x\right)\right)$$
$$\times\left[a_1\cos\alpha+\frac{B\left(\frac{\pi}{2},\frac{(2n+1)a_1a_2}{a_1+a_2},\Delta(\tau)\right)\sin\alpha}{a_1}\right]$$
$$-\frac{(a_1+a_2)\,D\left(x,\frac{(2n+1)a_1a_2}{a_1+a_2},\Delta(\tau)\right)\sin\alpha}{(2n+1)\,a_1a_2^2\delta_1}\sin\left(\frac{(2n+1)\,a_1}{a_1+a_2}\left(\frac{\pi\,(a_2-a_1)}{2a_1}+x\right)\right)$$
$$+\frac{(a_1+a_2)\,C\left(x,\frac{(2n+1)a_1a_2}{a_1+a_2},\Delta(\tau)\right)\sin\alpha}{(2n+1)\,a_1a_2^2\delta_1}$$
$$\times\cos\left(\frac{(2n+1)\,a_1}{a_1+a_2}\left(\frac{\pi\,(a_2-a_1)}{2a_1}+x\right)\right)+O\left(\frac{1}{n^2}\right).$$

**Case 3.** By use of (8), (36) and (38) and making necessary arrangements we have

$$\phi_1(x,\lambda) = -\frac{B\left(x,s,\Delta\left(\tau\right)\right)\cos\frac{s}{a_1}x}{s^2} - \frac{\sin\frac{s}{a_1}x}{s}\left[a_1\cos\alpha + \frac{A\left(x,s,\Delta\left(\tau\right)\right)}{s}\right] + O\left(\frac{1}{s^3}\right). \tag{47}$$

Now replacing $s$ by $s_n$ and using (41) we get

$$\begin{aligned}\phi_{1n}\left(x\right) = &-\frac{\left(a_1+a_2\right)^2 B\left(x,\frac{(2n+1)a_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)\cos\frac{(2n+1)a_2x}{a_1+a_2}}{\left(2n+1\right)^2 a_1^2a_2^2}\\ &-\frac{\left(a_1+a_2\right)\sin\frac{(2n+1)a_2x}{a_1+a_2}}{\left(2n+1\right)a_1a_2}\left[a_1\cos\alpha + \frac{\left(a_1+a_2\right)A\left(x,\frac{(2n+1)a_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)\sin\alpha}{\left(2n+1\right)a_1a_2}\right]\\ &+\frac{2\left(a_1+a_2\right)x\cos\alpha\cos\frac{(2n+1)a_2x}{a_1+a_2}}{\left(2n+1\right)^2 a_2\pi}\\ &\times\left[\frac{a_1\cos\beta}{\delta_1} - \frac{A\left(\frac{\pi}{2},\frac{(2n+1)a_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)\sin\beta}{a_2\delta_1} - \frac{a_1D\left(\pi,\frac{(2n+1)a_1a_2}{a_1+a_2},\Delta\left(\tau\right)\right)\sin\beta}{a_2^2\delta_1}\right]\\ &+O\left(\frac{1}{n^3}\right).\end{aligned}$$

From (12), (37) and (38), we have

$$\begin{aligned}\frac{\phi_1'(x,\lambda)}{s} = &\frac{a_1\left[D\left(x,s,\Delta\left(\tau\right)\right)\cos\frac{s}{a_2}\left(\frac{\pi(a_2-a_1)}{2a_1}+x\right)\right]}{s^2a_2\delta_1}\\ &+\frac{a_1\left[C\left(x,s,\Delta\left(\tau\right)\right)\sin\frac{s}{a_2}\left(\frac{\pi(a_2-a_1)}{2a_1}+x\right)\right]}{s^2a_2\delta_1} + O\left(\frac{1}{s^3}\right).\end{aligned} \tag{48}$$

From (9), (37), (38), (47) and (48)

$$\begin{aligned}\phi_2(x,\lambda) = &-\frac{B\left(\frac{\pi}{2},s,\Delta\left(\tau\right)\right)\cos\frac{s}{a_2}\left(\frac{\pi(a_2-a_1)}{2a_1}+x\right)}{s^2\delta_1}\\ &-\frac{\sin\frac{s}{a_2}\left(\frac{\pi(a_2-a_1)}{2a_1}+x\right)\left[a_1\cos\alpha+\frac{A\left(\frac{\pi}{2},s,\Delta(\tau)\right)}{s}\right]}{s\delta_1}\\ &+\frac{a_1\left[D\left(x,s,\Delta\left(\tau\right)\right)\cos\frac{s}{a_2}\left(\frac{\pi(a_2-a_1)}{2a_1}+x\right)\right]}{s^2a_2\delta_1}\\ &+\frac{a_1\left[C\left(x,s,\Delta\left(\tau\right)\right)\sin\frac{s}{a_2}\left(\frac{\pi(a_2-a_1)}{2a_1}+x\right)\right]}{s^2a_2\delta_1} + O\left(\frac{1}{s^3}\right).\end{aligned} \tag{49}$$

Now replacing $s$ by $s_n$ and using (41) we get

$$\phi_{2n}(x) = -\frac{(a_1+a_2)^2 B\left(\frac{\pi}{2}, \frac{(2n+1)a_1a_2}{a_1+a_2}, \Delta(\tau)\right)}{(2n+1)^2 a_1^2 a_2^2 \delta_1} \cos\left(\frac{(2n+1)a_1}{a_1+a_2}\left(\frac{\pi(a_2-a_1)}{2a_1}+x\right)\right)$$
$$-\frac{(a_1+a_2)}{(2n+1)a_1a_2\delta_1} \sin\left(\frac{(2n+1)a_1}{a_1+a_2}\left(\frac{\pi(a_2-a_1)}{2a_1}+x\right)\right)$$
$$\times\left[a_1\cos\alpha + \frac{(a_1+a_2)A\left(\frac{\pi}{2}, \frac{(2n+1)a_1a_2}{a_1+a_2}, \Delta(\tau)\right)\sin\alpha}{(2n+1)a_1a_2}\right] + \frac{2(a_1+a_2)x\cos\alpha}{(2n+1)^2 a_2\delta_1\pi}$$
$$\times\cos\left(\frac{(2n+1)a_1}{a_1+a_2}\left(\frac{\pi(a_2-a_1)}{2a_1}+x\right)\right)$$
$$\times\left[\frac{a_1\cos\beta}{\delta_1} - \frac{A\left(\frac{\pi}{2}, \frac{(2n+1)a_1a_2}{a_1+a_2}, \Delta(\tau)\right)\sin\beta}{a_2\delta_1} - \frac{a_1 D\left(\pi, \frac{(2n+1)a_1a_2}{a_1+a_2}, \Delta(\tau)\right)\sin\beta}{a_2^2\delta_1}\right]$$
$$+\frac{(a_1+a_2)^2}{(2n+1)^2 a_1 a_2^3 \delta_1}\left[D\left(x, \frac{(2n+1)a_1a_2}{a_1+a_2}, \Delta(\tau)\right)\cos\left(\frac{(2n+1)a_1}{a_1+a_2}\left(\frac{\pi(a_2-a_1)}{2a_1}+x\right)\right)\right.$$
$$\left.+C\left(x, \frac{(2n+1)a_1a_2}{a_1+a_2}, \Delta(\tau)\right)\sin\left(\frac{(2n+1)a_1}{a_1+a_2}\left(\frac{\pi(a_2-a_1)}{2a_1}+x\right)\right)\right] + O\left(\frac{1}{n^3}\right).$$

**Case 4.** Replacing $s$ by $s_n$ in (47), we find, by use of (42), that

$$\phi_{1n}(x) = -\frac{(a_1+a_2)^2 B\left(x, \frac{2na_1a_2}{a_1+a_2}, \Delta(\tau)\right)\cos\frac{2na_2x}{a_1+a_2}}{4n^2a_1^2a_2^2}$$
$$-\frac{(a_1+a_2)\sin\frac{2na_2x}{a_1+a_2}}{2na_1a_2}\left[a_1\cos\alpha + \frac{(a_1+a_2)A\left(x, \frac{2na_1a_2}{a_1+a_2}, \Delta(\tau)\right)}{2na_1a_2}\right]$$
$$+\frac{(a_1+a_2)x\cos\alpha\cos\frac{2na_2x}{a_1+a_2}}{2n^2a_2\pi}$$
$$\times\left[\frac{a_1\left[D\left(\pi, \frac{2na_1a_2}{a_1+a_2}, \Delta(\tau)\right) + C\left(\pi, \frac{2na_1a_2}{a_1+a_2}, \Delta(\tau)\right)\right]}{a_2\delta_1} - \frac{B\left(\frac{\pi}{2}, \frac{2na_1a_2}{a_1+a_2}, \Delta(\tau)\right)}{\delta_1}\right] + O\left(\frac{1}{n^3}\right).$$

Now replacing $s$ by $s_n$ in (49), we find, by use of (42), that

$$\phi_{2n}(x) = -\frac{(a_1+a_2)^2 B\left(\frac{\pi}{2}, \frac{2na_1a_2}{a_1+a_2}, \Delta(\tau)\right)}{4n^2a_1^2a_2^2\delta_1}\cos\left(\frac{2na_1}{a_1+a_2}\left(\frac{\pi(a_2-a_1)}{2a_1}+x\right)\right)$$
$$-\frac{(a_1+a_2)}{2na_1a_2\delta_1}\sin\left(\frac{2na_1}{a_1+a_2}\left(\frac{\pi(a_2-a_1)}{2a_1}+x\right)\right)$$

$$\times\left[a_1\cos\alpha+\frac{(a_1+a_2)\,A\left(\frac{\pi}{2},\frac{2na_1a_2}{a_1+a_2},\Delta(\tau)\right)\sin\alpha}{2na_1a_2}\right]$$

$$-\frac{(a_1+a_2)\,x\cos\alpha}{2n^2a_2\delta_1\pi}\cos\left(\frac{2na_1x}{a_1+a_2}\left(\frac{\pi(a_2-a_1)}{2a_1}+x\right)\right)$$

$$\times\left[\frac{a_1\left[D\left(\pi,\frac{2na_1a_2}{a_1+a_2},\Delta(\tau)\right)+C\left(\pi,\frac{2na_1a_2}{a_1+a_2},\Delta(\tau)\right)\right]}{a_2\delta_1}-\frac{B\left(\frac{\pi}{2},\frac{2na_1a_2}{a_1+a_2},\Delta(\tau)\right)}{\delta_1}\right]$$

$$+\frac{(a_1+a_2)^2}{4n^2a_1a_2^3\delta_1}\left[D\left(x,\frac{2na_1a_2}{a_1+a_2},\Delta(\tau)\right)\cos\left(\frac{2na_1}{a_1+a_2}\left(\frac{\pi(a_2-a_1)}{2a_1}+x\right)\right)\right.$$

$$\left.+C\left(x,\frac{2na_1a_2}{a_1+a_2},\Delta(\tau)\right)\sin\left(\frac{2na_1}{a_1+a_2}\left(\frac{\pi(a_2-a_1)}{2a_1}+x\right)\right)\right]+O\left(\frac{1}{n^3}\right).$$

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