Article electronically published on March 10, 2017

Commun.Fac.Sci.Univ.Ank.Series A1
Volume 66, Number 2, Pages 263–275 (2017)
DOI: 10.1501/Commua1_0000000817
ISSN 1303–5991

http://communications.science.ankara.edu.tr/index.php?series=A1

$\alpha\beta$-STATISTICAL CONVERGENCE OF MODIFIED q-DURRMEYER OPERATORS

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Abstract. In this work, we investigate weighted $\alpha\beta$-statistical approximation properties of q-Durrmeyer-Stancu operators. We also give some corrections in limit of q-Durrmeyer-Stancu operators defined in [1] and discuss their convergence properties.

1. Introduction

The concept of statistical convergence has been defined by Fast [2] and studied by many other authors. It is well known that every ordinary convergent sequence is statistically convergent but the converse is not true, examples and some related work can be found in [3, 4, 5, 6, 7]. The idea $\alpha\beta$-statistical convergence was introduced by Aktŭglu in [8] as follows:
Let $\alpha(n)$ and $\beta(n)$ be two sequences positive numbers which satisfy the following conditions

(i) α and β are both non-decreasing,
(ii) $\beta(n) \geq \alpha(n)$,
(iii) $\beta(n) - \alpha(n) \to \infty$ as $n \to \infty$

and let $\wedge$ denote the set of pairs (α, β) satisfying (i)-(iii). For each pair $(\alpha, \beta) \in \wedge$, $0 < \gamma \leq 1$ and $K \subset \mathbb{N}$, we define $\delta^{\alpha,\beta}(K, \gamma)$ in the following way

$$\delta^{\alpha,\beta}(K,\gamma) = \lim_{n\to\infty} \frac{|K \cap P_n^{\alpha,\beta}|}{(\beta(n) - \alpha(n) + 1)^{\gamma}},$$

where $P_n^{\alpha,\beta}$ is the closed interval $[\alpha(n), \beta(n)]$ and $|S|$ represents the cardinality of S. A sequence $x = (x_k)$ is said to be $\alpha\beta$-statistically convergent of order γ to ℓ or

Received by the editors: June 28, 2016; Accepted: January 12, 2017.
2010 *Mathematics Subject Classification.* 41A25, 41A30, 41A36.
Key words and phrases. Durrmeyer operators, Korovkin type theorems rate of the weighted $\alpha\beta$-statistical convergent.

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$S_{\alpha\beta}^{\gamma}$-convergent, if

$$\delta^{\alpha,\beta}\left(\{k : |x_k - \ell| \geq \epsilon\}, \gamma\right) = \lim_{n\to\infty} \frac{\left|\{k \in P_n^{\alpha,\beta} : |x_k - \ell| \geq \epsilon\}\right|}{(\beta(n) - \alpha(n) + 1)^{\gamma}} = 0.$$

The concept of weighted $\alpha\beta$-statistically convergence was developed by Karakaya and Karaisa [9]. Let $s = (s_k)$ be a sequence of non-negative real numbers such that $s_0 > 0$ and

$$S_n = \sum_{k\in P_n^{\alpha,\beta}} s_k \to \infty, \text{ as } n \to \infty \text{ and } z_n^{\gamma}(x) = \frac{1}{S_n^{\gamma}} \sum_{k\in P_n^{\alpha,\beta}} s_k x_k.$$

A sequence $x = (x_k)$ is said to be weighted $\alpha\beta$-statistically convergent of order γ to ℓ or $S_{\alpha\beta}^{\gamma}$-convergent, if for every $\epsilon > 0$

$$\delta^{\alpha,\beta}\left(\{k : s_k|x_k - \ell| \geq \epsilon\}, \gamma\right) = \lim_{n\to\infty} \frac{1}{S_n^{\gamma}} |\{k \leq S_n : s_k|x_k - \ell| \geq \epsilon\}| = 0$$

and denote $st_{\alpha\beta}^{\gamma} - \lim x = \ell$ or $x_k \to \ell[\bar{S}_{\alpha\beta}^{\gamma}]$, where $\bar{S}_{\alpha\beta}^{\gamma}$ denotes the set of all weighted $\alpha\beta$-statistically convergent sequences of order γ.

Definition 1 ([9]). *A sequence $x = (x_k)$ is said to be strongly weighted $\alpha\beta$-summable of order γ to a number ℓ if*

$$\lim_{n\to\infty} \frac{1}{S_n^{\gamma}} \sum_{k\in P_n^{\alpha,\beta}} s_k|x_k - \ell| \to 0 \text{ as } n \to \infty.$$

We denote it by $x_k \to \ell[\bar{N}_{\alpha,\beta}^{\gamma}, s]$. Similarly, for $\gamma = 1$ the sequence $x = (x_k)$ is said to be strongly weighted $\alpha\beta$-summable to ℓ . The set of all strongly weighted $\alpha\beta$-summable of order γ and strongly weighted $\alpha\beta$-summable sequence will be denoted by $[\bar{N}_{\alpha,\beta}^{\gamma}, s]$ and $[\bar{N}_{\alpha,\beta}, s]$, respectively.

Definition 2 ([9]). *A sequence $x = (x_k)$ is said to be weighted $\alpha\beta$-summable of order γ to a number ℓ, if $z_n^{\gamma}(x) \to \ell$ as $n \to \infty$. Similarly, for $\gamma = 1$ the sequence $x = (x_k)$ is said to be weighted $\alpha\beta$-summable of order γ $z_n(x) \to \ell$ as $n \to \infty$. The set of all weighted $\alpha\beta$-summable sequence of order γ and weighted $\alpha\beta$-summable sequence will be denoted by $(\bar{N}_{\alpha,\beta}^{\gamma}, s)$ and $(\bar{N}_{\alpha,\beta}, s)$, respectively.*

Remark 1. *If $\gamma = 1$, $\alpha(n) = 1$ and $\beta(n) = n$, weighted $\alpha\beta$-summability reduces to weighted mean summability and $[\bar{N}_{\alpha,\beta}, s]$ summable sequences coincide with $(\bar{N}, p_n)$ summable sequences introduced in* [10, 11]*. Also if $p_n = 1$, then $(\bar{N}, p_n)$ reduces to C_1-summability and called Cesàro summability.*

The q-Bernstein operators were introduced by Phillips [12]. A survey of the obtained results and references concerning q-Bernstein operators can be found in [13]. It is worth mentioning that the first generalization of the Bernstein operators based on q-integers was obtained by Lupaş [14]. The Durrmeyer type modification of q-Bernstein operators were established by Gupta [15] and its local approximation, global approximation and simultaneous approximation properties were discussed in

[16], we refer some of the important papers in this direction as [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. Stancu type generalization of the q-Durrmeyer operators were discussed by Mishra and Patel [1, 36], which defined for $f \in C([0,1])$ and $0 \le \varpi \le \vartheta$ as

$$\begin{aligned} D_{n,q}^{\varpi,\vartheta}(f,x) &= [n+1]_q \sum_{k=0}^{n} q^{-k} p_{nk}(q;x) \int_0^1 f\left(\frac{[n]_q t + \varpi}{[n]_q + \vartheta}\right) p_{nk}(q,qt) d_q t \quad (1.1) \\ &= \sum_{k=0}^{n} A_{n,k}^{\varpi,\vartheta}(f) p_{nk}(q;x); 0 \le x \le 1, \end{aligned}$$

where $p_{nk}(q;x) = \binom{n}{k}_q x^k (1-x)_q^{n-k}$.

We have used notations of q-calculus as given in [37]. Along the paper, $C([a,b])$ denotes the set all of continuous functions on interval $[a,b]$ and $\|h\|_{C([a,b])}$ represents the sup-norm of the function $h\,|_{[a,b]}$.

In this work, we establish $\alpha\beta$-statistical convergence for operators (1.1). In section 3, we discuss convergence results of limit of q-Durrmeyer-Stancu operators (1.1).

Lemma 1 ([1]). *We have*

$$D_{n,q}^{\varpi,\vartheta}(1;x) = 1, \;\; D_{n,q}^{\varpi,\vartheta}(t;x) = \frac{[n]_q + \varpi[n+2]_q + qx[n]_q^2}{[n+2]_q([n]_q + \vartheta)}$$

and

$$\begin{aligned} D_{n,q}^{\varpi,\vartheta}(t^2;x) &= \frac{q^3[n]_q^3([n]_q - 1)x^2 + \left(\left(q(1+q)^2 + 2\varpi q^4\right)[n]_q^3 + 2\varpi q[3]_q[n]_q^2\right)x}{([n]_q+\vartheta)^2[n+2]_q[n+3]_q} \\ &\quad + \frac{(1+q+2\varpi q^3)[n]_q^2 + 2\varpi[3]_q[n]_q}{([n]_q+\vartheta)^2[n+2]_q[n+3]_q} + \frac{\varpi^2}{([n]_q+\vartheta)^2}. \end{aligned}$$

Remark 2. *By simple computation, we can find the central moments*

$$\delta_n(x) = D_{n,q}^{\varpi,\vartheta}(t-x;x) = \left(\frac{q[n]_q^2}{[n+2]_q([n]_q+\vartheta)} - 1\right)x + \frac{[n]_q + \varpi[n+2]_q}{[n+2]_q([n]_q+\vartheta)},$$

$$\begin{aligned} \gamma_n(x) &= D_{n,q}^{\varpi,\vartheta}((t-x)^2;x) \\ &= \frac{q^4[n]_q^4 - q^3[n]_q^3 - 2q[n]_q^2[n+3]_q([n]_q+\vartheta) + [n+2]_q[n+3]_q([n]_q+\vartheta)^2}{([n]_q+\vartheta)^2[n+2]_q[n+3]_q}x^2 \\ &+ \frac{q(1+q)^2[n]_q^3 + 2q\varpi[n]_q^2[n+3]_q - (2[n]_q + 2\varpi[n+2]_q)[n+3]_q([n]_q+\vartheta)}{([n]_q+\vartheta)^2[n+2]_q[n+3]_q}x \\ &\qquad + \frac{(1+q)[n]_q^2 + 2\varpi[n]_q[n+3]_q}{([n]_q+\vartheta)^2[n+2]_q[n+3]_q}. \end{aligned}$$

2. $\alpha\beta$-Statistical Convergence

Theorem 1 ([9])**.** *Let (L_k) be a sequence of positive linear operators from $C([a,b])$ into $C([a,b])$. Then for all $f \in C([a,b])$*

$$\bar{S}^{\gamma}_{\alpha\beta} - \lim_{k\to\infty} \|L_k(f,x) - f(x)\|_{C([a,b])} = 0$$

if and only if

$$\bar{S}^{\gamma}_{\alpha\beta} - \lim_{k\to\infty} \|L_k(x^i,x) - x^i\|_{C([a,b])} = 0, \ i = 0,1,2.$$

Let $\{q_n\}$ be a sequence in the interval $[0,1]$ satisfying

$$\bar{S}^{\gamma}_{\alpha\beta} - \lim_{n\to\infty} q_n = 1, \ \bar{S}^{\gamma}_{\alpha\beta} - \lim_{n\to\infty} (q_n)^n = a(a<1), \ \bar{S}^{\gamma}_{\alpha\beta} - \lim_{n\to\infty} \frac{1}{[n]_q} = 0. \tag{2.1}$$

For example, take $\gamma = \frac{1}{2}$, $\alpha(n) = n$ and $\beta(n) = n^{\frac{1}{\gamma}}$ and define the sequence , (q_n) by

$$q_n = \begin{cases} 0 & \text{if } n = m^2 (m = 1,2,3,\ldots); \\ 1 - \dfrac{e^{-n}}{n} & \text{if } n \neq m^2. \end{cases}$$

Now, we note that

$$\begin{aligned} \delta^{\alpha,\beta}\left\{k \in P_n^{\alpha,\beta} : |q_k - 1| \geq \epsilon\right\} &= \lim_{n\to\infty} \frac{|\left\{k \in P_n^{\alpha,\beta} : |q_k - 1| \geq \epsilon\right\}|}{(\beta(n) - \alpha(n) - 1)^{\gamma}} \\ &= \lim_{n\to\infty} \frac{|\left\{k \in [n,n^2] : |q_k - 1| \geq \epsilon\right\}|}{(\beta(n) - \alpha(n) - 1)^{\gamma}} \\ &= \lim_{n\to\infty} \frac{|\left\{k \in [n,n^2] : |q_k - 1| \geq \epsilon\right\}|}{(n^2 - n - 1)^2} \\ &\leq \lim_{n\to\infty} \frac{n}{(n^2 - n - 1)^2} = 0. \end{aligned}$$

Therefore, $\bar{S}^{\gamma}_{\alpha\beta} - \lim_{n\to\infty} q_n = 1$.
Also, for $a < 1$

$$\begin{aligned} \delta^{\alpha,\beta}\left\{k \in P_n^{\alpha,\beta} : |q_k^k - a| \geq \epsilon\right\} &= \lim_{n\to\infty} \frac{|\left\{k \in P_n^{\alpha,\beta} : |q_k^k - a| \geq \epsilon\right\}|}{(\beta(n) - \alpha(n) - 1)^{\gamma}} \\ &= \lim_{n\to\infty} \frac{|\left\{k \in [n,n^2] : |q_k^k - a| \geq \epsilon\right\}|}{(\beta(n) - \alpha(n) - 1)^{\gamma}} \\ &= \lim_{n\to\infty} \frac{|\left\{k \in [n,n^2] : |q_k^k - a| \geq \epsilon\right\}|}{(n^2 - n - 1)^2} \\ &\leq \lim_{n\to\infty} \frac{n^2 - n + 1}{(n^2 - n - 1)^2} = 0. \end{aligned}$$

Thus, $\bar{S}^{\gamma}_{\alpha\beta} - \lim_{n\to\infty} q_n^n = a, (a < 1)$.

Theorem 2. *Let $\{q_n\}$ be a sequence satisfying (2.1) and $D_{n,q}^{\varpi,\vartheta}$ as defined in* (1.1)*. For any $f \in C([0,1])$, we have*

$$\bar{S}_{\alpha\beta}^{\gamma} - \lim_{n\to\infty} \|D_{n,q_n}^{\varpi,\vartheta}(f,x) - f(x)\|_{C([0,1])} = 0.$$

Proof: By Theorem 1, it is enough to prove that

$$\bar{S}_{\alpha\beta}^{\gamma} - \lim_{n\to\infty} \|D_{n,q_n}^{\varpi,\vartheta}(t^j,x) - x^j\|_{C([0,1])} = 0, \ j = 0,1,2 \tag{2.2}$$

From the $D_{n,q_n}^{\varpi,\vartheta}(1,x) = 1$, it is easy to obtain that

$$\bar{S}_{\alpha\beta}^{\gamma} - \lim_{n\to\infty} \|D_{n,q_n}^{\varpi,\vartheta}(1,x) - 1\|_{C([0,1])} = 0.$$

Now,

$$\begin{aligned} |D_{n,q_n}^{\varpi,\vartheta}(t;x) - x| &\le \left| \frac{q_n[n]_{q_n}^2 - [n+2]_{q_n}([n]_{q_n}+\vartheta)}{[n+2]_{q_n}([n]_{q_n}+\vartheta)} \right| + \left| \frac{[n]_{q_n} + \varpi[n+2]_{q_n}}{[n+2]_{q_n}([n]_{q_n}+\vartheta)} \right| \\ &= \left| \frac{[n]_{q_n}\left(q_n[n]_{q_n} - [n+2]_{q_n}\right) - \vartheta[n+2]_{q_n}}{[n+2]_{q_n}([n]_{q_n}+\vartheta)} \right| + \left| \frac{[n]_{q_n} + \varpi[n+2]_{q}}{[n+2]_{q_n}([n]_{q_n}+\vartheta)} \right| \\ &\le \left| \frac{[n]_{q_n}(1+q_n^{n+1})}{[n+2]_{q_n}([n]_{q_n}+\vartheta)} \right| + \left| \frac{\vartheta}{[n]_{q_n}+\vartheta} \right| + \left| \frac{[n]_{q_n} + \varpi[n+2]_{q_n}}{[n+2]_{q_n}([n]_{q_n}+\vartheta)} \right| \end{aligned}$$

Using equation (2.1), we get

$$\bar{S}_{\alpha\beta}^{\gamma} - \lim_{n\to\infty} \frac{[n]_{q_n}(1+q_n^{n+1})}{[n+2]_{q_n}([n]_{q_n}+\vartheta)} = 0; \ \bar{S}_{\alpha\beta}^{\gamma} - \lim_{n\to\infty} \left| \frac{\vartheta}{[n]_{q_n}+\vartheta} \right| = 0$$

and

$$\bar{S}_{\alpha\beta}^{\gamma} - \lim_{n\to\infty} \frac{[n]_{q_n} + \varpi[n+2]_{q_n}}{[n+2]_{q_n}([n]_{q_n}+\vartheta)} = 0.$$

Define the following sets:

$$A = \left\{n \in \mathbb{N} : \|D_{n,q_n}^{\varpi,\vartheta}(\cdot;x) - x\|_{C([a,b])} \ge \epsilon\right\}; \quad A_1 = \left\{n \in \mathbb{N} : \frac{[n]_{q_n}(1+q_n^{n+1})}{[n+2]_{q_n}([n]_{q_n}+\vartheta)} \ge \frac{\epsilon}{3}\right\};$$

$$A_2 = \left\{n \in \mathbb{N} : \frac{\vartheta}{[n]_{q_n}+\vartheta} \ge \frac{\epsilon}{3}\right\}, \ A_3 = \left\{n \in \mathbb{N} : \frac{[n]_{q_n} + \varpi[n+2]_{q_n}}{[n+2]_{q_n}([n]_{q_n}+\vartheta)} \ge \frac{\epsilon}{3}\right\}.$$

Then, we obtain $A \subset A_1 \cup A_2 \cup A_3$, which implies that $\delta_{\gamma}^{\alpha,\beta}(A) \le \delta_{\gamma}^{\alpha,\beta}(A_1) + \delta_{\gamma}^{\alpha,\beta}(A_2) + \delta_{\gamma}^{\alpha,\beta}(A_3)$ and hence

$$\bar{S}_{\alpha\beta}^{\gamma} - \lim_{n\to\infty} \|D_{n,q_n}^{\varpi,\vartheta}(t,x) - x\|_{C([0,1])} = 0.$$

Similarly, we have

$$\begin{aligned}
|D^{\varpi,\vartheta}_{n,q_n}(t^2;x)-x^2| &\leq \left|\frac{q_n^3[n]^3_{q_n}\left([n]_{q_n}-1\right)}{([n]_{q_n}+\vartheta)^2[n+2]_{q_n}[n+3]_{q_n}}-1\right| \\
&\quad+\left|\frac{\left(\left(q_n(1+q_n)^2+2\varpi q_n^4\right)[n]^3_{q_n}+2\varpi q_n[3]_{q_n}[n]^2_{q_n}\right)}{([n]_{q_n}+\vartheta)^2[n+2]_{q_n}[n+3]_{q_n}}\right| \\
&\quad+\left|\frac{(1+q_n+2\varpi q_n^3)[n]^2_{q_n}+2\varpi[3]_{q_n}[n]_{q_n}}{([n]_{q_n}+\vartheta)^2[n+2]_{q_n}[n+3]_{q_n}}\right|+\left|\frac{\varpi^2}{([n]_{q_n}+\vartheta)^2}\right| \\
&\leq \left|\frac{q_n^3[n]^4_{q_n}(1-{q_n}^2)}{([n]_{q_n}+\vartheta)^2[n+2]_{q_n}[n+3]_{q_n}}\right|+\left|\frac{\left(q_n(1+q_n)^2+2\varpi q_n^4\right)[n]^3_{q_n}}{([n]_{q_n}+\vartheta)^2[n+2]_{q_n}[n+3]_{q_n}}\right| \\
&\quad+\left|\frac{2\varpi q_n[3]_{q_n}[n]^2_{q_n}}{([n]_{q_n}+\vartheta)^2[n+2]_{q_n}[n+3]_{q_n}}\right|+\left|\frac{(1+q_n+2q_n^3\varpi)[n]^2_{q_n}}{([n]_{q_n}+\vartheta)^2[n+2]_{q_n}[n+3]_{q_n}}\right| \\
&\quad+\left|\frac{2\varpi[3]_{q_n}[n]_{q_n}}{([n]_{q_n}+\vartheta)^2[n+2]_{q_n}[n+3]_{q_n}}\right|+\left|\frac{\varpi^2}{([n]_{q_n}+\vartheta)^2}\right|.
\end{aligned}$$

Again, using $\bar{S}^{\gamma}_{\alpha\beta}-\lim\limits_{n\to\infty} q_n=1,\ \bar{S}^{\gamma}_{\alpha\beta}-\lim\limits_{n\to\infty}(q_n)^n=a\in(0,1),\ \bar{S}^{\gamma}_{\alpha\beta}-\lim\limits_{n\to\infty}\frac{1}{[n]_{q_n}}=0,$ we get

$$\begin{aligned}
\bar{S}^{\gamma}_{\alpha\beta}-\lim_{n\to\infty}\frac{q_n^3[n]^4_{q_n}(1-q_n^2)}{([n]_{q_n}+\vartheta)^2[n+2]_{q_n}[n+3]_{q_n}} &= 0, \\
\bar{S}^{\gamma}_{\alpha\beta}-\lim_{n\to\infty}\frac{\left(q_n(1+q_n)^2+2\varpi q_n^4\right)[n]^3_{q_n}}{([n]_{q_n}+\vartheta)^2[n+2]_{q_n}[n+3]_{q_n}} &= 0, \\
\bar{S}^{\gamma}_{\alpha\beta}-\lim_{n\to\infty}\frac{2\varpi q_n[3]_{q_n}[n]^2_{q_n}}{([n]_{q_n}+\vartheta)^2[n+2]_{q_n}[n+3]_{q_n}} &= 0, \\
\bar{S}^{\gamma}_{\alpha\beta}-\lim_{n\to\infty}\frac{(1+q_n+2q_n^3\varpi)[n]^2_{q_n}}{([n]_{q_n}+\vartheta)^2[n+2]_{q_n}[n+3]_{q_n}} &= 0, \\
\bar{S}^{\gamma}_{\alpha\beta}-\lim_{n\to\infty}\frac{2\varpi[3]_{q_n}[n]_{q_n}}{([n]_{q_n}+\vartheta)^2[n+2]_{q_n}[n+3]_{q_n}} &= 0, \\
\bar{S}^{\gamma}_{\alpha\beta}-\lim_{n\to\infty}\frac{\varpi^2}{([n]_{q_n}+\vartheta)^2} &= 0.
\end{aligned}$$

Now, consider the following sets:

$$B_1 := \left\{ n \in \mathbb{N} : \frac{q_n^3 [n]_{q_n}^4 (1-q_n^2)}{([n]_{q_n}+\vartheta)^2 [n+2]_{q_n} [n+3]_{q_n}} \geq \frac{\epsilon}{6} \right\},$$

$$B_2 := \left\{ n \in \mathbb{N} : \frac{\left(q_n(1+q_n)^2 + 2\varpi q_n^4\right) [n]_{q_n}^3}{([n]_{q_n}+\vartheta)^2 [n+2]_{q_n} [n+3]_{q_n}} \geq \frac{\epsilon}{6} \right\},$$

$$B_3 := \left\{ n \in \mathbb{N} : \frac{2\varpi q_n [3]_{q_n} [n]_{q_n}^2}{([n]_{q_n}+\vartheta)^2 [n+2]_{q_n} [n+3]_{q_n}} \geq \frac{\epsilon}{6} \right\},$$

$$B_4 := \left\{ n \in \mathbb{N} : \frac{(1+q_n+2q_n^3\varpi) [n]_{q_n}^2}{([n]_{q_n}+\vartheta)^2 [n+2]_{q_n} [n+3]_{q_n}} \geq \frac{\epsilon}{6} \right\},$$

$$B_5 := \left\{ n \in \mathbb{N} : \frac{2\varpi [3]_{q_n} [n]_{q_n}}{([n]_{q_n}+\vartheta)^2 [n+2]_{q_n} [n+3]_{q_n}} \geq \frac{\epsilon}{6} \right\},$$

$$B_6 := \left\{ n \in \mathbb{N} : \frac{\varpi^2}{([n]_{q_n}+\vartheta)^2} \geq \frac{\epsilon}{6} \right\}.$$

Consequently, we obtain $B \subset B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6$, which implies that $\delta(B) \leq \sum_{i=1}^{6} \delta(B_i)$. Hence, we get

$$\bar{S}_{\alpha\beta}^{\gamma} - \lim_{n\to\infty} \| D_{n,q_n}^{\varpi,\vartheta}(t^2, x) - x^2 \|_{C([0,1])} = 0.$$

This completes the proof of Theorem 2.

3. Limit of q-Durrmeyer-Stancu operators

In [36, Sec.4], the operators $D_{\infty,q}^{\varpi,\vartheta}$ [36, Eq.(4.2)], which depend on $[n]_q$, have been defined and studied. Since [36, Theorem 2] is incorrect, we have modified the operators Eq.(4.2) and proof of Theorem 2 of [36] with this note.
Here, we define the limit q-Durrmeyer-Stancu operators (1.1) as:
Let $q \in (0,1)$ be fixed and $x \in [0,1]$, the operators $D_{\infty,q}^{\varpi,\vartheta}(f;x)$ is defined by

$$\begin{aligned} D_{\infty,q}^{\varpi,\vartheta}(f;x) &= \frac{1}{1-q} \sum_{k=0}^{\infty} p_{\infty k}(q;x) q^{-k} \int_0^1 f\left(\frac{t+(1-q)\vartheta}{1+(1-q)\varpi}\right) p_{\infty k}(q;qt) d_q t \\ &= \sum_{k=0}^{\infty} A_{\infty k}^{\varpi,\vartheta}(f) p_{\infty k}(q;x), \end{aligned} \tag{3.1}$$

where $p_{\infty k}(q;x) = \frac{x^k}{(1-q)^k [k]!}(1-x)_q^{\infty}$.
Using the fact that (see [38]), we have

$$\sum_{k=0}^{\infty} p_{\infty k}(q;x) = 1, \ \sum_{k=0}^{\infty} (1-q^k) p_{\infty k}(q;x) = x, \tag{3.2}$$

and

$$\sum_{k=0}^{\infty}(1-q^k)^2 p_{\infty k}(q;x) = x^2 + (1-q)x(1-x). \tag{3.3}$$

Also, (see [39, eq.4])

$$\int_0^1 t^s p_{\infty k}(q;qt)d_qt = (1-q)^{s+1}\frac{q^k[k+s]!}{[k]!}, s = 0,1,2\dots. \tag{3.4}$$

Using (3.1) and (3.2)-(3.4), it is easy to prove that

$$D^{\varpi,\vartheta}_{\infty,q}(1;x) = 1,\ D^{\varpi,\vartheta}_{\infty,q}(t;x) = \frac{1+q(x-1)+\vartheta(1-q)}{1+\varpi(1-q)},$$

$$D^{\varpi,\vartheta}_{\infty,q}(t^2;x) = \frac{q^4x^2 + \left(q(1+q)(1-q^2)+2(1-q)q\vartheta\right)x}{(1+\varpi(1-q))^2} + \frac{((1+q)+2\vartheta+\vartheta^2)(1-q)^2}{(1+\varpi(1-q))^2}.$$

For $f \in C[0,1], t > 0$, we define the modulus of continuity $\omega(f,t)$ as follows:

$$\omega(f,t) = \sup\{|f(x)-f(y)| : |x-y| \le t,\ x,y \in [0,1]\}.$$

Theorem 3. *Let $0 < q < 1$ then for each $f \in C[0,1]$ the sequence $\{D^{\varpi,\vartheta}_{n,q}(f;x)\}$ converges to $D^{\varpi,\vartheta}_{\infty,q}(f;x)$ uniformly on $[0,1]$. Furthermore,*

$$\|D^{\varpi,\vartheta}_{n,q}(f) - D^{\varpi,\vartheta}_{\infty,q}(f)\| \le C^{\varpi,\vartheta}_q \omega(f,q^n).$$

Proof: $D^{\varpi,\vartheta}_{\infty,q}(f;x)$ and $D^{\varpi,\vartheta}_{n,q}(f;x)$ reproduce constant function that is $D^{\varpi,\vartheta}_{n,q}(1;x) = D^{\varpi,\vartheta}_{\infty,q}(1;x) = 1$. Hence for all $x \in [0,1)$, by definition of $D^{\varpi,\vartheta}_{n,q}(f;x)$ and $D^{\varpi,\vartheta}_{\infty,q}(f;x)$, we know that

$$\begin{aligned}
|D^{\varpi,\vartheta}_{n,q}(f;x) - D^{\varpi,\vartheta}_{\infty,q}(f;x)| &= \left|\sum_{k=0}^{n} A^{\varpi,\vartheta}_{nk}(f)p_{nk}(q;x) - \sum_{k=0}^{\infty} A^{\varpi,\vartheta}_{\infty k}(f)p_{\infty k}(q;x)\right| \\
&= \left|\sum_{k=0}^{n} A^{\varpi,\vartheta}_{nk}\left(f-f(1)\right)p_{nk}(q;x)\right. \\
&\quad \left. - \sum_{k=0}^{\infty} A^{\varpi,\vartheta}_{\infty k}\left(f-f(1)\right)p_{\infty k}(q;x)\right| \\
&\le \sum_{k=0}^{n}\left|A^{\varpi,\vartheta}_{nk}\left(f-f(1)\right) - A^{\varpi,\vartheta}_{\infty k}\left(f-f(1)\right)\right|p_{nk}(q;x) \\
&\quad + \sum_{k=0}^{n}|A^{\varpi,\vartheta}_{\infty k}\left(f-f(1)\right)||p_{nk}(q;x) - p_{\infty k}(q;x)| \\
&\quad + \sum_{k=n+1}^{\infty}|A^{\varpi,\vartheta}_{\infty k}\left(f-f(1)\right)|p_{\infty k}(q;x) = I_1 + I_2 + I_3.
\end{aligned}$$

By the well known property of modulus of continuity (see [40]), $\omega(f,\lambda t)\leq(1+\lambda)\omega(f,t),\lambda>0$, we get

$$|f(t)-f(1)|\leq\omega(f,1-t)\leq\omega(f,q^n)\left(1+\frac{1-t}{q^n}\right).$$

Thus

$$\begin{aligned}
|A_{nk}^{\vartheta,\varpi}(f-f(1))| &= \left|[n+1]_q\int_0^1 q^{-k}\left(f\left(\frac{[n]_qt+\vartheta}{[n]_q+\varpi}\right)-f(1)\right)p_{nk}(q;qt)d_qt\right| \\
&\leq [n+1]_q\int_0^1 q^{-k}\left|f\left(\frac{[n]_qt+\vartheta}{[n]_q+\varpi}\right)-f(1)\right|p_{nk}(q;qt)d_qt \\
&\leq [n+1]_q\int_0^1 q^{-k}\omega(f,q^n)\left(1+\frac{1}{q^n}\left(1-\frac{[n]_qt+\vartheta}{[n]_q+\varpi}\right)\right)p_{nk}(q;qt)d_qt \\
&\leq \omega(f,q^n)\left(1+q^{-n}\left(1-\frac{[n]_q[k+1]_q+\vartheta[n+2]_q}{[n+2]_q([n]_q+\varpi)}\right)\right) \\
&\leq \omega(f,q^n)\left(1+\frac{q^{-n}[n]_q}{[n]_q+\varpi}\left(1-\frac{[k+1]_q}{[n+2]_q}\right)+\frac{q^{-n}(\varpi-\vartheta)}{[n]_q+\varpi}\right) \\
&= \omega(f,q^n)\left(1+q^{k+1-n}+\frac{q^{-n}(\varpi-\vartheta)}{[n]_q+\varpi}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
|A_{\infty k}^{\vartheta,\varpi}(f-f(1))| &= \frac{q^{-k}}{1-q}\left|\int_0^1\left(f\left(\frac{t+\vartheta(1-q)}{1+\varpi(1-q)}\right)-f(1)\right)p_{\infty k}(q;qt)d_qt\right| \\
&\leq \frac{q^{-k}}{1-q}\int_0^1\omega(f,q^n)\left(1+\frac{1}{q^n}\left(1-\frac{t+\vartheta(1-q)}{1+\varpi(1-q)}\right)\right)p_{\infty k}(q;qt)d_qt \\
&\leq \frac{q^{-k}}{1-q}\int_0^1\omega(f,q^n)\left(1+\frac{1}{q^n}(1-t)+\frac{1}{q^n}\frac{\varpi-\vartheta}{1+\varpi(1-q)}\right)p_{\infty k}(q;qt)d_qt \\
&\leq \omega(f,q^n)\left(1+q^{k+1-n}+\frac{q^{-n}(\varpi-\vartheta)}{1+\varpi(1-q)}\right).
\end{aligned}$$

From [36, Eq.4.5] and [39, Eq.8], we have

$$|p_{nk}(q;x)-p_{\infty k}(q;x)|\leq\frac{q^{n-k}}{1-q}\left(p_{nk}(q;x)+p_{\infty k}(q;x)\right). \tag{3.5}$$

Hence by using (3.5), we have

$$\begin{aligned}
& |A_{nk}^{\vartheta,\varpi}(f-f(1)) - A_{\infty k}^{\vartheta,\varpi}(f-f(1))| \\
\leq\ & [n+1]_q \int_0^1 q^{-k} \left| f\left(\frac{[n]_q t+\vartheta}{[n]_q+\varpi}\right) - f(1) \right| p_{nk}(q;qt) d_q t \\
& + \frac{1}{1-q} \int_0^1 q^{-k} \left| f\left(\frac{t+\vartheta(1-q)}{1+\varpi(1-q)}\right) - f(1) \right| p_{\infty k}(q;qt) d_q t \\
\leq\ & [n+1]_q \int_0^1 q^{-k} \left| f\left(\frac{[n]_q t+\vartheta}{[n]_q+\varpi}\right) - f(1) \right| |p_{nk}(q;qt) - p_{\infty k}(q;qt)| d_q t \\
+\ & \frac{1}{1-q} \int_0^1 q^{-k} \left| f\left(\frac{t+\vartheta(1-q)}{1+\varpi(1-q)}\right) - f(1) \right| p_{\infty k}(q;qt) d_q t \\
& + [n+1]_q \int_0^1 q^{-k} \left| f\left(\frac{[n]_q t+\vartheta}{[n]_q+\varpi}\right) - f(1) \right| p_{\infty k}(q;qt) d_q t \\
\leq\ & [n+1]_q \frac{q^{n-k}}{1-q} \int_0^1 q^{-k} \left| f\left(\frac{[n]_q t+\vartheta}{[n]_q+\varpi}\right) - f(1) \right| |p_{nk}(q;qt) + p_{\infty k}(q;qt)| d_q t \\
+\ & \frac{1}{1-q} \int_0^1 q^{-k} \left| f\left(\frac{t+\vartheta(1-q)}{1+\varpi(1-q)}\right) - f(1) \right| p_{\infty k}(q;qt) d_q t \\
& + [n+1]_q \int_0^1 q^{-k} \left| f\left(\frac{[n]_q t+\vartheta}{[n]_q+\varpi}\right) - f(1) \right| p_{\infty k}(q;qt) d_q t \\
\leq\ & \omega(f,q^n) \left[\begin{array}{c} 2\frac{q^{n-k}}{1-q}\left(1+q^{k+1-n}+\frac{q^{-n}(\varpi-\vartheta)}{[n]_q+\varpi}\right) + \left(1+q^{k+1-n}+\frac{q^{-n}(\varpi-\vartheta)}{1+\varpi(1-q)}\right) \\ +\left(1+q^{k+1-n}+\frac{q^{-n}(\varpi-\vartheta)}{[n]_q+\varpi}\right) \end{array} \right].
\end{aligned}$$

To estimate I_1, I_2 and I_3, we have

$$\begin{aligned}
I_1 \quad \leq\ & \frac{\omega(f,q^n)}{1-q}\left(8+\frac{3(\varpi-\vartheta)}{q^n([n]_q+\varpi)} + \frac{(\varpi-\vartheta)}{q^n(1+\varpi(1-q))}\right)\sum_{k=0}^{n} p_{nk}(q;x) \\
=\ & \frac{\omega(f,q^n)}{1-q}\left(8+\frac{3(\varpi-\vartheta)}{q^n([n]_q+\varpi)} + \frac{(\varpi-\vartheta)}{q^n(1+\varpi(1-q))}\right);
\end{aligned}$$

$$\begin{aligned}
I_3 &= \sum_{k=n+1}^{\infty} |A_{\infty k}^{\varpi,\vartheta}(f-f(1))| p_{\infty k}(q;x) \\
&\leq \omega(f,q^n) \sum_{k=n+1}^{\infty} \left(1+q^{k+1-n}+\frac{q^{-n}(\varpi-\vartheta)}{1+\varpi(1-q)}\right) p_{\infty k}(q;x) \\
&\leq \omega(f,q^n)\left(2+\frac{q^{-n}(\varpi-\vartheta)}{1+\varpi(1-q)}\right); \\
I_2 &= \sum_{k=0}^{n} A_{\infty k}^{\varpi,\vartheta}(f-f(1))\,|p_{nk}(q;x)-p_{\infty k}(q;x)| \\
&\leq \sum_{k=0}^{n}\left[\omega(f,q^n)\left(1+q^{k+1-n}+\frac{q^{-n}(\varpi-\vartheta)}{1+\varpi(1-q)}\right)\right]\left[\frac{q^{n-k}}{1-q}|p_{nk}(q;x)+p_{\infty k}(q;x)|\right] \\
&\leq \frac{2\omega(f,q^n)}{1-q}\left(2+\frac{q^{-n}(\varpi-\vartheta)}{1+\varpi(1-q)}\right).
\end{aligned}$$

Combining the estimates I_1–I_3, we conclude that $\|D_{n,q}^{\varpi,\vartheta}(f)-D_{\infty,q}^{\varpi,\vartheta}(f)\| \leq C_q^{\varpi,\vartheta}\omega(f,q^n)$. This completes the proof of Theorem 3.

Lemma 2 ([41])**.** *Let L be a positive linear operator on $C([0,1])$ which reproduces constant functions.*
If $L(t,x) > x$ for all $x \in [0,1)$, then $L(f)=f$ if and only if f is a constant.

Remark 3. *Since* $D_{\infty,q}^{\varpi,\vartheta}(t;x) = \dfrac{(1+q(x-1))+\vartheta(1-q)}{1+\varpi(1-q)} > x$ *for* $0<q<1$, *as a consequence of Lemma 2, we have the following:*

Theorem 4. *Let $0<q<1$ be fixed and let $f \in C([0,1])$. Then $D_{\infty,q}^{\varpi,\vartheta}(f;x)=f(x)$ for all $x\in[0,1]$ if and only if f is constant.*

Theorem 5. *Let $0<q<1$ be fixed and let $f \in C([0,1])$. Then $\{D_{\infty,q}^{\vartheta,\varpi}(f)\}$ converges to f uniformly on $[0,1]$ as $q \to 1^-$.*

Proof: We know that the operators $D_{\infty,q}^{\vartheta,\varpi}$ is positive linear operator on $C([0,1])$ for $0<q<1$ and reproduce constant functions. Also, $D_{\infty,q}^{\varpi,\vartheta}(t;x) \to x$ uniformly on $[0,1]$ as $q\to 1^-$ and $D_{\infty,q}^{\vartheta,\varpi}(t^2;x)\to x^2$ uniformly on $[0,1]$ as $q \to 1^-$. Thus, Theorem 5 follows from Korovkin Theorem.

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