Article electronically published on March 15, 2017

Commun.Fac.Sci.Univ.Ank.Series A1
Volume 66, Number 2, Pages 289–296 (2017)
DOI: 10.1501/Commua1_0000000819
ISSN 1303–5991
http://communications.science.ankara.edu.tr/index.php?series=A1

ON THE FABER POLYNOMIAL COEFFICIENT BOUNDS OF BI-BAZILEVIČ FUNCTIONS

ŞAHSENE ALTINKAYA AND SIBEL YALÇIN

Abstract. In this work, considering bi-Bazilevič functions and using the Faber polynomials, we obtain coefficient expansions for functions in this class. In certain cases, our estimates improve some of those existing coefficient bounds.

1. Introduction

Let A denote the class of functions f which are analytic in the open unit disk $\mathbf{U}=\{z : z\in\mathbb{C} \text{ and } |z|<1\}$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n. \tag{1.1}$$

Let S be the subclass of A consisting of functions f which are also univalent in $\mathbf{U}$ and let P be the class of functions

$$\varphi(z)=1+\sum_{n=1}^{\infty}\varphi_n z^n$$

that are analytic in $\mathbf{U}$ and satisfy the condition $\Re(\varphi(z))>0$ in $\mathbf{U}$. By the Caratheodory's lemma (e.g., see [11]) we have $|\varphi_n|\leq 2$.

For $f(z)$ and $F(z)$ analytic in $\mathbf{U}$, we say that $f(z)$ is subordinate to $F(z)$, written $f\prec F$, if there exists a Schwarz function

$$u(z)=\sum_{n=1}^{\infty}c_n z^n$$

with $|u(z)|<1$ in $\mathbf{U}$, such that $f(z)=F(u(z))$. For the Schwarz function $u(z)$ we note that $|c_n|<1$. (e.g. see Duren [11]).

Received by the editors: September 28, 2016; Accepted: February 04, 2017.
2010 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C50.
Key words and phrases. Analytic and univalent functions, bi-univalent functions, faber polynomials.

For $0 \leq \alpha < 1$ and $0 \leq \beta < 1$, $f \in \Sigma$ and $g = f^{-1}$, let $B(\alpha, \beta)$ denote the class of bi-Bazilevič functions of order α and type β (see Bazilevič [7]) if and only if

$$\Re\left(\left(\frac{z}{f(z)}\right)^{1-\beta} f'(z)\right) > \alpha, \qquad z \in \mathbf{U}$$

and

$$\Re\left(\left(\frac{w}{g(w)}\right)^{1-\beta} g'(w)\right) > \alpha, \qquad w \in \mathbf{U}.$$

It is well known that every function $f \in S$ has an inverse f^{-1}, satisfying $f^{-1}(f(z)) = z$, $(z \in \mathbf{U})$ and $f\left(f^{-1}(w)\right) = w$, $\left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right)$, where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots .$$

A function $f \in A$ is said to be bi-univalent in $\mathbf{U}$ if both f and f^{-1} are univalent in $\mathbf{U}$. For a brief history and interesting examples in the class Σ, see [24].

Historically, Lewin [17] studied the class of bi-univalent functions, obtaining the bound 1.51 for the modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [8] conjectured that $|a_2| \leqq \sqrt{2}$ for $f \in \Sigma$. Later on, Netanyahu [20] showed that $\max |a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Brannan and Taha [9] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^\star(\beta)$ and $\mathcal{K}(\beta)$ of starlike and convex functions of order β $(0 \leqq \beta < 1)$ in $\mathbb{U}$, respectively (see [20]). The classes $\mathcal{S}_\Sigma^\star(\beta)$ and $\mathcal{K}_\Sigma(\beta)$ of bi-starlike functions of order β in $\mathbb{U}$ and bi-convex functions of order β in $\mathbb{U}$, corresponding to the function classes $\mathcal{S}^\star(\beta)$ and $\mathcal{K}(\beta)$, were also introduced analogously. For each of the function classes $\mathcal{S}_\Sigma^\star(\beta)$ and $\mathcal{K}_\Sigma(\beta)$, they found non-sharp estimates for the initial coefficients. Recently, motivated substantially by the aforementioned pioneering work on this subject by Srivastava *et al.* [24], many authors investigated the coefficient bounds for various subclasses of bi-univalent functions (see, for example, [5], [13], [15], [18], [19], [25]).

The Faber polynomials introduced by Faber [12] play an important role in various areas of mathematical sciences, especially in geometric function theory. Grunsky [14] succeeded in establishing a set of conditions for a given function which are necessary and in their totality sufficient for the univalency of this function, and in these conditions the coefficients of the Faber polynomials play an important role. Schiffer [22] gave a differential equations for univalent functions solving certain extremum problems with respect to coefficients of such functions; in this differential equation appears again a polynomial which is just the derivative of a Faber polynomial (Schaeffer-Spencer [23]).

Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([6], [10], [15], [16]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N} \backslash \{1, 2\}$; $\mathbb{N} = \{1, 2, 3, ...\}$) is still an open problem.

Definition 1. *A function $f \in \Sigma$ is said to be in the class* $\mathrm{B}_{\Sigma}(\beta, \varphi)$, $0 \leq \beta < 1$, *if the following subordination holes*

$$\left(\frac{z}{f(z)}\right)^{1-\beta} f'(z) \prec \varphi(z) \tag{1.2}$$

and

$$\left(\frac{w}{g(w)}\right)^{1-\beta} g'(w) \prec \varphi(w) \tag{1.3}$$

where $g(w) = f^{-1}(w)$.

Remark 1. *From among the many choices of β and φ which would provide the following known subclasses:*

1) $B_{\Sigma}(1, \varphi) = H_{\Sigma}^{\varphi}$ (see [21]).
2) $B_{\Sigma}(0, \varphi) = S_{\Sigma}^{*}(\varphi)$ (see [21]).

We note that, for different choices of the function φ, we get known subclasses of the function class A. For example (see [26])

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}; \quad 0 < \alpha \leq 1 \qquad \text{and} \qquad \varphi(z) = \frac{1+(1-2\lambda)z}{z}; \quad 0 \leq \lambda < 1 .$$

In this paper, we use the Faber polynomial expansions to obtain bounds for the general coefficients $|a_n|$ of bi-Bazilevič functions in $B_{\Sigma}(\beta, \varphi)$ as well as we provide estimates for the initial coefficients of these functions.

2. Main Results

Using the Faber polynomial expansion of functions $f \in A$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as, [3],

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots) w^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} = {} & \frac{(-n)!}{(-2n+1)!\,(n-1)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!\,(n-3)!} a_2^{n-3} a_3 \\ & + \frac{(-n)!}{(-2n+3)!\,(n-4)!} a_2^{n-4} a_4 \\ & + \frac{(-n)!}{[2(-n+2)]!\,(n-5)!} a_2^{n-5} \left[a_5 + (-n+2) a_3^2\right] \\ & + \frac{(-n)!}{(-2n+5)!\,(n-6)!} a_2^{n-6} \left[a_6 + (-2n+5) a_3 a_4\right] \\ & + \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned} \tag{2.1}$$

such that V_j with $7 \le j \le n$ is a homogeneous polynomial in the variables $a_2, a_3, ..., a_n$ [4]. In particular, the first three terms of K_{n-1}^{-n} are

$$\begin{aligned} \frac{1}{2}K_1^{-2} &= -a_2, \\ \frac{1}{3}K_2^{-3} &= 2a_2^2 - a_3, \\ \frac{1}{4}K_3^{-4} &= -\left(5a_2^3 - 5a_2a_3 + a_4\right). \end{aligned} \tag{2.2}$$

In general, for any $p \in \mathbb{N}$ and $n \ge 2$, an expansion of K_{n-1}^p is as, [3],

$$K_{n-1}^p = pa_n + \frac{p(p-1)}{2}E_{n-1}^2 + \frac{p!}{(p-3)!3!}E_{n-1}^3 + ... + \frac{p!}{(p-n+1)!(n-1)!}E_{n-1}^{n-1}, \tag{2.3}$$

where $E_{n-1}^p = E_{n-1}^p(a_2, a_3, ...)$ and by [1],

$$E_{n-1}^m(a_2, ..., a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1} ... (a_n)^{\mu_{n-1}}}{\mu_1! ... \mu_{n-1}!}, \quad \text{for } m \le n$$

while $a_1 = 1$, and the sum is taken over all nonnegative integers $\mu_1, ..., \mu_n$ satisfying

$$\begin{aligned} \mu_1 + \mu_2 + \ ... \ + \mu_{n-1} &= m, \\ \mu_1 + 2\mu_2 + \ ... \ + (n-1)\mu_{n-1} &= n-1. \end{aligned}$$

Evidently, $E_{n-1}^{n-1}(a_2, ..., a_n) = a_2^{n-1}$,(see [2]); while $a_1 = 1$, and the sum is taken over all nonnegative integers $\mu_1, ..., \mu_n$ satisfying

$$\begin{aligned} \mu_1 + \mu_2 + \ ... \ + \mu_n &= m, \\ \mu_1 + 2\mu_2 + \ ... \ + n\mu_n &= n. \end{aligned}$$

It is clear that $E_n^n(a_1, a_2, ..., a_n) = a_1^n$. The first and the last polynomials are:

$$E_n^1 = a_n \qquad E_n^n = a_1^n.$$

Theorem 1. *For* $0 \le \beta < 1$, *let* $f \in \mathrm{B}_\Sigma(\beta, \varphi)$. *If* $a_m = 0$; $2 \le m \le n-1$, *then*

$$|a_n| \le \frac{2}{\beta + (n-1)}; \qquad n \ge 4. \tag{2.4}$$

Proof. Let f be given by (1.1). We have

$$\left(\frac{f(z)}{z}\right)^\beta \left(\frac{zf'(z)}{f(z)}\right) = 1 + \sum_{n=2}^{\infty}\left[1 + \frac{(n-1)}{\beta}\right] K_{n-1}^{-\beta}(a_2, a_3, ..., a_n) z^{n-1}, \tag{2.5}$$

and for its inverse map, $g = f^{-1}$, we have

$$\left(\frac{g(w)}{w}\right)^{\beta}\left(\frac{wg'(w)}{g(w)}\right) = 1 + \sum_{n=2}^{\infty}\left[1 + \frac{(n-1)}{\beta}\right] K_{n-1}^{-\beta}\left(A_2, A_3, ..., A_n\right) w^{n-1}. \quad (2.6)$$

where

$$A_n = \frac{1}{n} K_{n-1}^{-n}\left(a_2, a_3, ..., a_n\right), \qquad n \geq 2.$$

On the other hand, for $f \in B_{\Sigma}(\beta, \varphi)$ and $\varphi \in P$ there are two Schwarz functions

$$u(z) = \sum_{n=1}^{\infty} c_n z^n$$

and

$$v(w) = \sum_{n=1}^{\infty} d_n w^n$$

such that

$$\left(\frac{f(z)}{z}\right)^{\beta}\left(\frac{zf'(z)}{f(z)}\right) = \varphi(u(z)) \quad (2.7)$$

and

$$\left(\frac{g(w)}{w}\right)^{\beta}\left(\frac{wg'(w)}{g(w)}\right) = \varphi(v(w)) \quad (2.8)$$

where

$$\varphi(u(z)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \varphi_k E_n^k\left(c_1, c_2, ..., c_n\right) z^n, \quad (2.9)$$

and

$$\varphi(v(w)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \varphi_k E_n^k\left(d_1, d_2, ..., d_n\right) w^n. \quad (2.10)$$

Comparing the corresponding coefficients of (2.7) and (2.9) yields

$$\left[\beta + (n-1)\right] a_n = \sum_{k=1}^{n-1} \varphi_k E_{n-1}^k\left(c_1, c_2, ..., c_{n-1}\right), \; n \geq 2 \quad (2.11)$$

and similarly, from (2.8) and (2.10) we obtain

$$\left[\beta + (n-1)\right] A_n = \sum_{k=1}^{n-1} \varphi_k E_{n-1}^k\left(d_1, d_2, ..., d_{n-1}\right), \quad n \geq 2. \quad (2.12)$$

Note that for $a_m = 0 \; ; \; 2 \leq m \leq n-1$ we have $A_n = -a_n$ and so

$$\begin{aligned} \left[\beta + (n-1)\right] a_n &= \varphi_1 c_{n-1} \\ \left[\beta + (n-1)\right] a_n &= \varphi_1 d_{n-1} \end{aligned} \quad (2.13)$$

Now taking the absolute values of either of the above two equations in (2.13) and using the facts that $|\varphi_1| \leq 2$, $|c_{n-1}| \leq 1$and $|d_{n-1}| \leq 1$, we obtain

$$|a_n| \leq \frac{|\varphi_1 c_{n-1}|}{|\beta + (n-1)|} = \frac{|\varphi_1 d_{n-1}|}{|\beta + (n-1)|} \leq \frac{2}{\beta + (n-1)}. \tag{2.14}$$

□

Theorem 2. *Let* $f \in \mathrm{B}_\Sigma(\beta, \varphi)$, *and* $0 \leq \beta < 1$. *Then*

$$\begin{aligned}
&(i) && |a_2| \leq \min\left\{\frac{2}{\beta+1}, \sqrt{\frac{8}{(\beta+1)(\beta+2)}}\right\} = \frac{2}{\beta+1} \\
&(ii) && |a_3| \leq \min\left\{\frac{4}{(\beta+1)^2} + \frac{2}{\beta+2}, \frac{8}{(\beta+1)(\beta+2)} + \frac{2}{\beta+2}\right\}. \\
& && \qquad = \frac{4}{(\beta+1)^2} + \frac{2}{\beta+2} \\
&(iii) && \left|a_3 - a_2^2\right| \leq \frac{2}{\beta+2}
\end{aligned} \tag{2.15}$$

Proof. Replacing n by 2 and 3 in (2.11) and (2.12), respectively, we find that

$$(\beta+1)\, a_2 = \varphi_1 c_1, \tag{2.16}$$

$$\frac{(\beta-1)(\beta+2)}{2} a_2^2 + (2+\beta)\, a_3 = \varphi_1 c_2 + \varphi_2 c_1^2, \tag{2.17}$$

$$-(\beta+1)\, a_2 = \varphi_1 d_1, \tag{2.18}$$

$$\frac{(\beta+2)(\beta+3)}{2} a_2^2 - (2+\beta)\, a_3 = \varphi_1 d_2 + \varphi_2 d_1^2 \tag{2.19}$$

From (2.16) or (2.18) we obtain

$$|a_2| \leq \frac{|\varphi_1 c_1|}{\beta+1} = \frac{|\varphi_1 d_1|}{\beta+1} \leq \frac{2}{\beta+1}. \tag{2.20}$$

Adding (2.17) to (2.19) implies

$$(\beta+1)(\beta+2)\, a_2^2 = \varphi_1 (c_2 + d_2) + \varphi_2 \left(c_1^2 + d_1^2\right)$$

or, equivalently,

$$|a_2| \leq \sqrt{\frac{8}{(\beta+1)(\beta+2)}}. \tag{2.21}$$

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (2.19) from (2.17). We thus get

$$2(\beta+2)\left(a_3 - a_2^2\right) = \varphi_1 (c_2 - d_2) + \varphi_2 \left(c_1^2 - d_1^2\right) \tag{2.22}$$

or

$$a_3 = a_2^2 + \frac{\varphi_1 (c_2 - d_2)}{2(\beta+2)} \tag{2.23}$$

Upon substituting the value of a_2^2 from (2.20) and (2.21) into (2.23), it follows that

$$|a_3| \leq \frac{4}{(\beta+1)^2} + \frac{2}{\beta+2}$$

and

$$|a_3| \leq \frac{8}{(\beta+1)(\beta+2)} + \frac{2}{\beta+2}.$$

Solving the equation (2.22) for $\left(a_3 - a_2^2\right)$, we obtain

$$\left|a_3 - a_2^2\right| = \frac{\left|\varphi_1 (c_2 - d_2) + \varphi_2 \left(c_1^2 - d_1^2\right)\right|}{2(\beta+2)} \leq \frac{2}{\beta+2}$$

□

Putting $\beta = 0$ in Theorem 2, we obtain the following corollary for analytic bi-starlike functions.

Corollary 1. *If* $f \in \mathrm{S}_\Sigma^*(\varphi)$, *then*

$$\begin{array}{ll} (i) & |a_2| \leq 2 \\ (ii) & |a_3| \leq 5 \\ (iii) & \left|a_3 - a_2^2\right| \leq 1 \end{array}.$$

Putting $\beta = 1$ in Theorem 1, we obtain the following corollary.

Corollary 2. *If* $f \in \mathrm{H}_\Sigma^\varphi$, *then*

$$\begin{array}{ll} (i) & |a_2| \leq 1 \\ (ii) & |a_3| \leq \frac{5}{3} \\ (iii) & \left|a_3 - a_2^2\right| \leq \dfrac{2}{3} \end{array}.$$

References

[1] Airault, H., Symmetric sums associated to the factorization of Grunsky coefficients, in Conference, Groups and Symmetries, Montreal, Canada, April 2007.

[2] Airault, H., Remarks on Faber polynomials, *Int. Math. Forum* 3 (2008), 449-456.

[3] Airault, H. and Bouali, H., Differential calculus on the Faber polynomials, *Bulletin des Sciences Mathematiques* 130 (2006), 179-222.

[4] Airault, H. and Ren, J., An algebra of differential operators and generating functions on the set of univalent functions, *Bulletin des Sciences Mathematiques* 126 (2002), 343-367.

[5] Altınkaya, Ş. and Yalçın, S., Coefficient Estimates for Two New Subclasses of Bi-univalent Functions with respect to Symmetric Points, *Journal of Function Spaces* Article ID 145242, (2015), 5 pp.

[6] Altınkaya, Ş. and Yalçın, S., Faber polynomial coefficient bounds for a subclass of bi-univalent function, *C. R. Acad. Sci. Paris, Ser. I* 353 (2015), 1075-1080.

[7] Bazilevič, I. E., On a case of integrability in quadratures of the Loewner-Kufarev equation, *Matematicheskii Sbornik* 37 (1955), 471-476.

[8] Brannan, D. A. and Clunie, J., Aspects of contemporary complex analysis, Proceedings of the NATO Advanced Study Instute Held at University of Durham, New York: Academic Press, 1979.

[9] Brannan, D. A. and Taha, T. S., On some classes of bi-univalent functions, *Studia Universitatis Babeş-Bolyai Mathematica* 31 (1986), 70-77.
[10] Bulut, S., Magesh N. and Balaji, V. K., Faber polynomial coefficient estimates for certain subclasses of meromorphic bi-univalent functions, *C. R. Acad. Sci. Paris, Ser. I*, 353 (2015), 113-116.
[11] Duren, P. L., Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Springer, New York, USA, 259, 1983.
[12] Faber, G., Über polynomische entwickelungen, *Math. Ann.* 57 (1903), 1569-1573.
[13] Frasin, B. A. and Aouf, M. K., New subclasses of bi-univalent functions, *Appl. Math. Lett.* 24 (2011), 1569-1573.
[14] Grunsky, H., Koeffizientenbedingungen für schlicht abbildende meromorphe funktionen, *Math. Zeit.* 45 (1939), 29-61.
[15] Hamidi, S. G. and Jahangiri, J. M., Faber polynomial coefficient estimates for analytic bi-close-to-convex functions, *C. R. Acad. Sci. Paris, Ser. I* 352 (2014), 17–20.
[16] Hamidi, S. G. and Jahangiri, J. M., Faber polynomial coefficients of bi-subordinate function, *C. R. Acad. Sci. Paris, Ser. I* 354 (2016), 365-370.
[17] Lewin, M., On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc.* 18 (1967), 63-68.
[18] Magesh, N., Rosy, T. and Varma, S., Coefficient estimate problem for a new subclass of bi-univalent functions, *Journal of Complex Analysis*, Volume 2013, Article ID: 474231, 3 pages.
[19] Magesh, N. and Yamini, J., Coefficient bounds for certain subclasses of bi-univalent functions, *Int. Math. Forum*, 8 (27) (2013), 1337-1344.
[20] Netanyahu, E., The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, *Archive for Rational Mechanics and Analysis* 32 (1969), 100-112.
[21] Rosihan, M. A., Lee, S. K., Ravichandran, V. and Supramaniama, S., Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, *Appl. Math. Lett.* 25 (2012), 344–351.
[22] Schiffer, M., A method of variation within the family of simple functions, *Proc. London Math. Soc.* 44 (1938), 432-449.
[23] Schaeffer, A. C. and Spencer, D. C., The coefficients of schlicht functions, *Duke Math. J.* 10 (1943), 611-635.
[24] Srivastava, H. M., Mishra, A. K. and Gochhayat, P., Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.* 23 (2010), 1188-1192.
[25] Srivastava, H. M., Joshi, S. B., Joshi, S. S. and Pawar, H., Coefficient estimates for certain subclasses of meromorphically bi-univalent functions, *Palest. J. Math.* 5 (Special Issue: 1) (2016), 250-258.
[26] Zaprawa, P., On Fekete-Szegö problem for classes of bi-univalent functions, *Bull. Belg. Math. Soc. Simon Stevin* 21 (2014), 169-178.

Current address: Şahsene Altınkaya: Department of Mathematics, Faculty of Arts and Science, Uludag University, 16059 Bursa, Turkey.
E-mail address: sahsene@uludag.edu.tr
Current address: Sibel Yalçın: Department of Mathematics, Faculty of Arts and Science, Uludag University, 16059 Bursa, Turkey.
E-mail address: syalcin@uludag.edu.tr