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MORE ON α-TOPOLOGICAL SPACES

SHYAMAPADA MODAK AND MD. MONIRUL ISLAM

Abstract. The aim of this paper is to introduce a new topology with the help of a-open sets. For this job, we shall define two new types of set and discuss its properties in detail and characterize Njastad's α-open sets and Levine's semi-open sets through these new types of set.

1. Introduction

The study of ideal in topological space was introduced and studied by Kuratowski [15] and Vaidyanathaswamy [22] but in this study Jankovic and Hamlett gave a new dimension through their paper "New topologies from old via ideals" [14]. Now a days the authors like Navaneethakrishnan et al. [19], Hamlett and Jankovic [12], Arenas et al. [4], Nasef and Mahmoud [18], Mukherjee et al. [17] Dontchev et al. [6] and many others have enriched this study. The authors Al-Omari et al. [1, 2] in their papers "a-local function and its properties in ideal topological spaces" and "The $\Re_a$ operator in ideal topological spaces", have studied Ekici's [7, 8, 9] a-open sets in terms of ideals. They have obtained a new topology with the help of two operators viz. $\Re_a$ and $()^{a^*}$, and have shown that this topology is finer than Ekici's a - topology.

In this paper, we have further considered the space which is the joint venture of a-topology and an ideal as like Al-omari et al. have considered in [2, 1]. Through this paper we will solve the question "how much finer is Noiri's et al.'s topology than Ekici's topology?" For solution of this question we have considered Njastad's α-open sets [20] from literature.

2. Preliminaries

In this section we have discussed some preliminary concepts of literature and introduce some prime results for discussing the paper.

Let A be a subset of a topological space (X, τ), then '$Int(A)$' and '$Cl(A)$' will denote 'interior of A' and 'closure of A' respectively.

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We define following as a mathematical tool for this research article:

Definition 1. *Let A be a subset of a topological space (X, τ). A is said to be regular open* [21] *(resp. semi-open* [16, 11]*, semi-pre open* [3]*, α-open* [20]*) if $A = Int(Cl(A))$ (resp. $A \subseteq Cl(Int(A))$, $A \subseteq Cl(Int(Cl(A)))$, $A \subseteq Int(Cl(Int(A)))$).*

Definition 2. [23] *A subset A of a topological space (X, τ) is said to be δ-open if, for each $x \in A$, there exists a regular open set G such that $x \in G \subseteq A$.*

The complement of δ-open set is called δ-closed. Let (X, τ) be a topological space, then the point $x \in X$ is called δ-cluster point of A if $Int(Cl(V)) \cap A \neq \emptyset$, for each open set V containing x.

The δ-closure of A is denoted as $Cl_\delta(A)$ [23] and it is a set of all δ-cluster point of A. In this regards, $Int_\delta(A)$ [23] is the δ-interior of A and it is the union of all regular open sets of (X, τ) contained in A. If $Int_\delta(A) = A$ for a topological space (X, τ), then A is δ-open and conversely [23]. It is remarkable that the collection of all δ-open sets in a topological space (X, τ) forms a topology and it is denoted as τ^δ [23].

Definition 3. [8, 9, 10] *A subset A of (X, τ) is said to be a-open (resp. a-closed) if $A \subseteq Int(Cl(Int_\delta(A)))$ (resp. $Cl(Int(Cl_\delta(A))) \subseteq A$).*

The family of a-open sets in (X, τ) forms a topology on X. This collection is denoted as τ^a [8], and $\tau^a(x)$ is denoted as the collection of all a-open sets containing x.

In this paper we also denote 'aCl' by the means of closure operator of Ekici's $a-$topology [7, 8].

Hereditary class and a-local function are also the mathematical tool for this paper:

Definition 4. [15] *A collection $I \subset \wp(X)$ is said to be an ideal on X if $B \subseteq A \in I$ implies $B \in I$ and $A,\ B \in I$ implies $A \cup B \in I$.*

Let I be an ideal on the topological space (X, τ), then (X, τ, I) is called an ideal topological space.

According to Al-Omari et al. [2, 1], we give the following:

The a-local function $()^{a^*} : \wp(X) \to \wp(X)$ for a subset A of an ideal topological space (X, τ, I) is defined as $(A)^{a^*} = \{x \in X : U \cap A \notin I, \text{ for every } U \in \tau^a(x)\}$, and as like complement operator of $()^{a^*}$, $\Re_a : \wp(X) \to \wp(X)$ is defined as $\Re_a(A) = X \setminus (X \setminus A)^{a^*} = \{x \in X : \text{ there exists } U_x \in \tau^a(x) \text{ such that } U_x \setminus A \in I\}$. Due to the operator $()^{a^*}$, we have a topology τ^{a^*} [1] whose one of the basis is $\beta(I, \tau) = \{V \setminus I : V \in \tau^a,\ I \in I\}$ [1]. In this respect, we will denote 'Int^{a^*}' and 'Cl^{a^*}' as 'interior' operator and 'closure' operator of (X, τ^{a^*}) respectively.

Following results help us for repairing the paper:

Theorem 1. [1] *Let (X, τ, I) be an ideal topological space and $U \in \tau^a$. Then $U \subseteq \Re_a(U)$.*

Corollary 2. *Let A be a subset of an ideal topological space (X, τ, I), then $aInt(A) \subseteq \Re_a(A)$.*

Theorem 3. [1] *Let A be a subset of an ideal topological space (X, τ, I) with $\tau^a \cap I = \emptyset$. Then $\Re_a(A) \subseteq (A)^{a^*}$.*

Corollary 4. *Let A be a subset of an ideal topological space (X, τ, I) with $\tau^a \cap I = \emptyset$. Then $\Re_a(A) \subseteq aCl(A)$.*

Lemma 5. *Let (X, τ, I) be an ideal topological space and $O \in \tau^a$. Then $\tau^a \cap I = \emptyset$ if and only if $(O)^{a^*} = aCl(O)$.*

Proof. Let $\tau^a \cap \mathcal{I} = \emptyset$ and $\emptyset \neq O \in \tau^{a^*}$. Now $O^{a^*} \subseteq aCl(O)$ always. For reverse inclusion, let $x \in aCl(O)$. Therefore all neighbourhoods $U_x \in \tau^a(x)$, $U_x \cap O \neq \emptyset$ implies $U_x \cap O \notin \mathcal{I}$, since $\tau^a \cap \mathcal{I} = \emptyset$. Therefore $x \in (O)^{a^*}$. Hence $(O)^{a^*} = aCl(O)$.

Conversely let $O \in \tau^a$, $(O)^{a^*} = aCl(O)$. Then $X^{a^*} = X$ and this implies $\mathcal{I} \cap \tau^{a^*} = \emptyset$ [2]. □

Proposition 6. *Let (X, τ, I) be an ideal topological space with $\tau^a \cap I = \emptyset$. Then following hold:*

(1) *For $A \subseteq X$, $\Re_a(A) \subseteq aInt(aCl(A))$.*
(2) *For a-closed subset A, $\Re_a(A) \subseteq A$.*
(3) *For $A \subseteq X$, $aInt(aCl(A)) = \Re_a(aInt(aCl(A)))$.*
(4) *For any τ^a-regular open subset A, $A = \Re_a(A)$.*
(5) *For any $O \in \tau^a$, $\Re_a(O) \subseteq aInt(aCl(O)) \subseteq (O)^{a^*}$.*

Proof. (1) From Theorem 3, $\Re_a(A) \subseteq (A)^{a^*}$. Then $\Re_a(A) \subseteq aCl(A)$, and since $\Re_a(A)$ is open, $\Re_a(A) \subseteq aInt(aCl(A))$.

(3) $\Re_a(aInt(aCl(A))) \subseteq (aInt(aCl(A))^{a^*} = aCl(aInt(aCl(A)))$ (from Lemma 5) $\subseteq aCl(A)$. Thus $\Re_a(aInt(aCl(A))) \subseteq aInt(aCl(A))$.

Reverse inclusion: $aInt(aCl(A)) \subseteq \Re_a(aInt(aCl(A)))$ (from Theorem 1).

Thus $aInt(aCl(A)) = \Re_a(aInt(aCl(A)))$. □

3. $\Re_a - aCl$ SETS

Definition 5. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$, A is said to be a $\Re_a - aCl$ set if $A \subseteq aCl(\Re_a(A))$.*

The collection of all $\Re_a - aCl$ sets in (X, τ, I) is denoted by $\Re_a(X, \tau^a)$.

Note 3.1. *Let (X, τ, I) be an ideal topological space. If $A \in \tau^a$, then $A \in \Re_a(X, \tau^a)$.*

Later, we shall given the example for the converse of this note.

Theorem 7. *Let $\{A_i : i \in \Lambda\}$ be a collection of nonempty $\Re_a - aCl$ sets in an ideal topological space (X, τ, I), then $\bigcup_{i \in \Lambda} A_i \in \Re_a(X, \tau^a)$.*

Proof. For each i, $A_i \subseteq aCl(\Re_a(A_i)) \subseteq aCl(\Re_a(\bigcup_{i\in\Lambda} A_i))$. This implies that $\bigcup_{i\in\Lambda} A_i \subseteq aCl(\Re_a(\bigcup_{i\in\Lambda} A_i))$. Thus $\bigcup_{i\in\Lambda} A_i \in \Re_a(X, \tau^a)$. □

For intersecting of two $\Re_a - aCl$ sets, we give following example:

Example 1. *Let* $X = \{e, b, c, d\}$, $\tau = \{\emptyset, X, \{e\}, \{c\}, \{e, b\}, \{e, c\}, \{e, b, c\}, \{e, c, d\}\}$, $I = \{\emptyset, \{b\}\}$. *Regular open sets are:* $\emptyset$, X, $\{c\}$, $\{e, b\}$. *Then* $\tau^a = \{\emptyset, X, \{c\}, \{e, b\}, \{e, b, c\}\}$. *Therefore* $\Re_a(\{c, d\}) = X \setminus (\{e, b\})^{a^*} = X \setminus \{e, b, d\} = \{c\}$ *and* $aCl(\Re_a(\{c, d\})) = \{c, d\}$. *Again* $\Re_a(\{e, b, d\}) = X\setminus(\{c\})^{a^*} = X\setminus\{c, d\} = \{e, b\}$ *and* $aCl(\Re_a(\{e, b, d\})) = \{e, b, d\}$. *Now* $\Re_a(\{d\}) = X \setminus (\{e, b, c\})^{a^*} = X \setminus \{e, b, c, d\} = \emptyset$. *Hence we have* $\{c, d\}$ *and* $\{e, b, d\}$ *are* $\Re_a - aCl$ *sets but they are not a-open sets. Again their intersection* $\{d\}$ *is not a* $\Re_a - aCl$ *set.*

We show that the intersecting of a $\Re_a - aCl$ set and an α - set of τ^a is also a $\Re_a - aCl$ set.

Theorem 8. *Let* (X, τ, I) *be an ideal topological space and* $A \in \Re_a(X, \tau^a)$. *If* $U \in \tau^{a^\alpha}$, *then* $U \cap A \in \Re_a(X, \tau^a)$ *(*τ^{a^α} *denotes the collection of all* α*-open sets in* (X, τ^a)*).*

Proof. Let G be a-open, and $A \subseteq X$, then it is obvious that
$G \cap aCl(A) \subseteq aCl(G \cap A)$ .... (i).
If V is a-open, then $V \subseteq aInt(aCl(V))$ and it is obvious that $aCl(aInt(aCl(V))) \subseteq aCl(V)$. Hence
$aCl(V) = aCl(aInt(aCl(A)))$ ...... (ii).
Again for A and B subsets of X,
$\Re_a(A \cap B) = \Re_a(A) \cap \Re_a(B)$ [1] ...... (iii).
Let $U \in \tau^{a^\alpha}$ and $A \in \Re_a(X, \tau^a)$, then we have $U \cap A \subseteq aInt(aCl(aInt(U))) \cap aCl(\Re_a(A)) \subseteq aInt(aCl(\Re_a(U))) \cap aCl(\Re_a(A))$ (Corollary 2). Since $aInt(aCl(\Re_a(U)))$ is a-open, from (i) we have
$U \cap A \subseteq aCl[aInt(aCl(\Re_a(U))) \cap \Re_a(A)] = aCl[aInt[aCl(\Re_a(U)) \cap \Re_a(A)]]$, since $\Re_a(A)$ is a-open. Now by again from (i), we have $U \cap A \subseteq aCl[aInt[aCl(\Re_a(U) \cap \Re_a(A))]]$. Since $\Re_a(U) \cap \Re_a(A)$ is a-open then from (ii), we get $U \cap A \subseteq aCl(\Re_a(U) \cap \Re_a(A)) = aCl(\Re_a(U \cap A))$ (using (iii)). Therefore, $U \cap A \in \Re_a(X, \tau^a)$. □

As $\tau^a \subseteq \tau^{a^\alpha}$ for a topological space (X, τ), then we have following corollary:

Corollary 9. *Let* (X, τ, I) *be a topological space and* $A \in \Re_a(X, \tau^a)$. *If* $U \in \tau^a$, *then* $U \cap A \in \Re_a(X, \tau^a)$.

For next, we recall that, a subset A of an ideal topological space (X, τ, I) is said to be I_a-dense [1] if $(A)^{a^*} = X$.

Theorem 10. $A \notin \Re_a(X, \tau^a)$ *if and only if there exists* $x \in A$ *such that there is a* τ^a *- neighbourhood* V_x *of* x *for which* $X \setminus A$ *is relatively* I_a *- dense in* V_x.

Proof. Let $A \notin \Re_a(X, \tau^a)$. We are to show that there exists an element $x \in A$ and a τ^a - neighbourhood V_x of x satisfying that $X \setminus A$ is relatively I_a - dense in V_x. Since $A \nsubseteq aCl(\Re_a(A))$, there exists $x \in X$ such that $x \in A$ but $x \notin aCl(\Re_a(A))$. Hence there exists a τ^a - neighbourhood V_x of x such that $V_x \cap \Re_a(A) = \emptyset$. This implies that $V_x \cap (X \setminus (X \setminus A)^{a^*} = \emptyset$ and so $V_x \subseteq (X \setminus A)^{a^*}$. Let U be any nonempty a-open set in V_x. Since $V_x \subseteq (X \setminus A)^{a^*}$, therefore $U \cap (X \setminus A) \notin \mathcal{I}$. This implies that $(X \setminus A)$ is relatively I_a - dense in V_x.

The converse part is obvious by reversing process. □

Relations between $\Re_a - aCl$ set with generalized open sets:

Theorem 11. *Let (X, τ, I) be a topological space, then $SO(X, \tau^a) \subseteq \Re_a(X, \tau^a)$ ($SO(X, \tau^a)$ denotes the collection of all semi-open sets in (X, τ^a)).*

Proof. For $A \subseteq aCl(aInt(A))$, $A \subseteq aCl(aInt(A)) \subseteq aCl(\Re_a(aInt(A))) \subseteq aCl(\Re_a(A))$. Thus $SO(X, \tau^a) \subseteq \Re_a(X, \tau^a)$. □

Theorem 12. *Let A be a $\Re_a - aCl$ set in a topological space (X, τ, I), where $\tau^a \cap I = \emptyset$. Then $A \in SPO(X, \tau^a)$ ($SPO(X, \tau^a)$ denotes the collection of all semi-preopen sets in (X, τ^a)).*

Let (X, τ, I) be an ideal topological space, and $I_n(\tau^a)$ is denoted as the collection of all nowhere dense subsets of (X, τ^a).

Lemma 13. *Let (X, τ, I) be an ideal topological space, where $I = I_n(\tau^a)$, then for $A \subseteq X$, $\Re_a(A) = aInt(aCl(aInt(A)))$.*

Proof. Proof is obvious from the fact that $\Re_a(A) = X \setminus (X \setminus A)^{a^*}$ and $(A)^{a^*} = aInt(aCl(aInt(A)))$. □

Theorem 14. *Let (X, τ, I) be an ideal topological space, where $I = I_n(\tau^a)$, then $\Re_a(X, \tau^a) = SO(X, \tau^a)$.*

Proof. Let $A \in \Re_a(X, \tau^a)$, therefore $A \subseteq aCl(\Re_a(A)) = aCl(aInt(aCl(aInt(A))))$ (from Lemma 13) $= aCl(aInt(A))$. Thus $A \in SO(X, \tau^a)$.

Suppose that $A \in SO(X, \tau^a)$. Then $A \subseteq aCl(aInt(A))$, so $aInt(A) \neq \emptyset$. We know that $aInt(A) \subseteq \Re_a(A)$ by Corollary 2. Thus $A \subseteq aCl(aInt(A)) \subseteq aCl(\Re_a(A))$. Hence the Theorem. □

In o.h.i. space two concepts semi-preopen set and $\Re_a - aCl$ set are synonymous, where o.h.i. space is defined as follows:

A space (X, τ) is said to be resolvable [13] if there is a dense subset D of X such that $X \setminus D$ are dense in (X, τ). Otherwise it is said to irresolvable [13]. Real line with usual topology is an example of a resolvable space. A space (X, τ) is called open hereditarily irresolvable (in short o.h.i.) [5] if every nonempty open subset of it is irresolvable.

Theorem 15. *Let (X,τ,I) be an ideal topological space, where (X,τ^a) is an o.h.i. space, $\tau^a \cap I = \emptyset$. Then $\Re_a(X,\tau^a) = SPO(X,\tau^a)$.*

Proof. We shall prove only the inclusion $SPO(X,\tau^a) \subseteq \Re_a(X,\tau_a)$, reverse inclusion has already been done. Let $A \in SPO(X,\tau^a)$. Then $A \subseteq aCl(aInt(aCl(A)))$. Let $x \in aInt(aCl(A))$. Therefore there exists a nonempty a-open set O_x (containing x) such that $O_x \subseteq aCl(A)$. Now it is obvious that $O_x \cap A$ is dense in O_x. Since the space is o.h.i., therefore $aInt(O_x \cap A)$ is dense in O_x, that is $O_x \subseteq aCl(aInt(A))$ and hence $x \in aCl(aInt(A))$. Thus $aInt(aCl(A)) \subseteq aCl(aInt(A))$. Now $A \subseteq aCl(aInt(aCl(A))) \subseteq aCl(aInt(A))$. But $aInt(A) \subseteq \Re_a(aInt(A)) \subseteq \Re_a(A)$, thus $A \subseteq aCl(\Re_a(A))$. Therefore $A \in \Re_a(X,\tau^a)$. □

4. α - TOPOLOGY OF τ^a

Definition 6. *Let (X,τ,I) be an ideal topological space and $A \subseteq X$. A is said to be a $\Re_a$ - set if $A \subseteq aInt(aCl(\Re_a(A)))$.*

The collection of all $\Re_a$ sets in (X,τ,I) is denoted as $\tau^{a^{\Re_a}}$.
It is obvious that $\tau^a \subseteq \tau^{a^{\Re_a}} \subseteq \Re_a(X,\tau^a)$.

Theorem 16. *Let (X,τ,I) be an ideal topological space with $\tau^a \cap I = \emptyset$, then the collection $\tau^{a^{\Re_a}} = \{A \subseteq X : A \subseteq aInt(aCl(\Re_a(A)))\}$ forms a topology on X.*

Proof. We shall prove only finite intersection property:

Let $A_1,\ A_2 \in \tau^{a^{\Re_a}}$. We are to show that $A_1 \cap A_2 \in \tau^{a^{\Re_a}}$. If $A_1 \cap A_2 = \emptyset$, we are done. Let $A_1 \cap A_2 \neq \emptyset$. Let $x \in A_1 \cap A_2$. Now $A_1 \subseteq aInt(aCl(\Re_a(A_1)))$ and $A_2 \subseteq aInt(aCl(\Re_a(A_2)))$, implies that $x \in aInt(aCl(\Re_a(A_1))) \cap aInt(aCl(\Re_a(A_2)))$. So $x \in aInt[aCl(\Re_a(A_1)) \cap aCl(\Re_a(A_2))]$. Therefore there exists an a-open set V_x containing x such that $V_x \subseteq aCl(\Re_a(A_1)) \cap aCl(\Re_a(A_2))$. Let U_x be any a-neighbourhood of x. Then $\emptyset \neq V_x \cap U_x \subseteq aCl(\Re_a(A_1))$ and $V_x \cap U_x \subseteq aCl(\Re_a(A_2))$. Let $y \in V_x \cap U_x$. Consider any open set G_y containing y. Without loss of generality we may suppose that $G_y \subseteq V_x \cap U_x$. So $G_y \cap \Re_a(A_1) \neq \emptyset$. From definition of $\Re_a(A_1)$ there exists a nonempty a-open set U such that $U \subseteq G_y$ and $U \setminus A_1 \in \mathcal{I}$. Again $U \subseteq aCl(\Re_a(A_2))$, so there exists a nonempty a-open set $U' \subseteq U$ such that $U' \setminus A_2 \in \mathcal{I}$. Now $U' \setminus (A_1 \cap A_2) = (U' \setminus A_1) \cup (U' \setminus A_2) \subseteq (U \setminus A_1) \cup (U' \setminus A_2) \in \mathcal{I}$ (finite additivity). Hence from definition $U' \subseteq \Re_a(A_1 \cap A_2)$. Since $U' \subseteq G_y,\ G_y \cap \Re_a(A_1 \cap A_2) \neq \emptyset$, therefore $y \in aCl(\Re_a(A_1 \cap A_2)$. Since y was any point of $U_x \cap V_x$, it follows that $U_x \cap V_x \subseteq aCl(\Re_a(A_1 \cap A_2))$, implies that $x \in aInt(aCl(\Re_a(A_1 \cap A_2)))$. Thus $A_1 \cap A_2 \subseteq aInt(aCl(\Re_a(A_1 \cap A_2)))$. Hence $A_1 \cap A_2 \in \tau^{a^{\Re_a}}$. □

Theorem 17. *Let (X,τ,I) be an ideal topological space, where $\tau^a \cap I = \emptyset$, then $\tau^{a^{\alpha}} \subseteq \tau^{a^{\Re_a}}$.*

Corollary 18. *Let (X,τ,I) be an ideal topological space, where $I = I_n(\tau^a)$, then $\tau^{a^{\alpha}} = \tau^{a^{\Re_a}}$.*

Lemma 19. *Let (X, τ, I) be an ideal topological space, where $\tau^a \cap I = \emptyset$. Then $\Re_a(A) \neq \emptyset$ if and only if A contains a nonempty τ^{a^*}-interior.*

Proof. Let $\Re_a(A) \neq \emptyset$. Therefore $\Re_a(A) = \cup\{M : M \in \tau^a, M \setminus A \in \mathcal{I}\} \neq \emptyset$, implies that there exists $\emptyset \neq M \in \tau^a$ such that $M \setminus A \in \mathcal{I}$. Let $M \setminus A = P$, where $P \in \mathcal{I}$. So $M \setminus P \subseteq A$ where $M \setminus P \neq \emptyset$, since $\tau^a \cap \mathcal{I} = \emptyset$. Since $M \setminus P \in \tau^{a^*}$, so that A contains a nonempty τ^{a^*} - interior.

Conversely suppose that A contains a τ^{a^*} - interior $M \setminus P$ (say), where $M \in \tau^a$, $P \in \mathcal{I}$. Thus $M \setminus P \subseteq A$, that is $M \setminus A \subseteq P$. Hence $M \setminus A \in \mathcal{I}$. So $\cup\{M : M \in \tau^a, M \setminus A \in \mathcal{I}\} \neq \emptyset$. This implies that $\Re_a(A) \neq \emptyset$. □

Corollary 20. *Let $x \in X$. Then $\{x\}$ is open in (X, τ^{a^*}) if and only if $\{x\} \in \Re_a(X, \tau^a)$.*

Corollary 21. *Let $x \in X$, then $\{x\} \in \tau^{a^{\Re_a}}$ if and only if $\{x\} \in \Re_a(X, \tau^a)$.*

Theorem 22. *$\tau^{a^{\Re_a}}$ is exactly the collection such that $A \in \tau^{a^{\Re_a}}$ and $B \in \Re_a(X, \tau^a)$ implies $A \cap B \in \Re_a(X, \tau^a)$.*

Proof. Let $A \in \tau^{a^{\Re_a}}$ and $B \in \Re_a(X, \tau^a)$. Now we are to show that $A \cap B \in \Re_a(X, \tau^a)$. If $A \cap B = \emptyset$, we are done. Let $A \cap B \neq \emptyset$. Let $x \in A \cap B$. This implies that $x \in aInt(aCl(\Re_a(A)))$, therefore $aInt(aCl(\Re_a(A)))$ is a a-neighbourhood of x. Consider any a-neighbourhood U_x of x, then $U_x \cap aInt(aCl(\Re_a(A)))$ is a a-neighbourhood of x. Since $x \in B \subseteq aCl(\Re_a(B))$, then $U_x \cap aInt(aCl(\Re_a(A))) \cap \Re_a(B) \neq \emptyset$. Let $V = U_x \cap aInt(aCl(\Re_a(A))) \cap \Re_a(B)$, then $V \subseteq aCl(\Re_a(A))$. This implies that $U_x \cap \Re_a(A) \cap \Re_a(B) = V \cap \Re_a(A) \neq \emptyset$, since $\Re_a(A)$ is a-open. Therefore $x \in aCl(\Re_a(A) \cap \Re_a(B))$, that is $x \in aCl(\Re_a(A \cap B))$. Hence $A \cap B \subseteq aCl(\Re_a(A \cap B))$, therefore $A \cap B \in \Re_a(X, \tau^a)$.

Next we consider a subset A of X such that $A \cap B \in \Re_a(X, \tau^a)$ for each $B \in \Re_a(X, \tau^a)$. We show that $A \in \tau^{a^{\Re_a}}$, that is $A \subseteq aInt(aCl(\Re_a(A)))$. If possible suppose that $x \in A$ but $x \notin aInt(aCl(\Re_a(A)))$. Therefore $x \in A \cap [X \setminus aInt(aCl(\Re_a(A)))] = A \cap aCl(X \setminus aCl(\Re_a(A))) = A \cap aCl(G)$ (say). It is obvious that $G = X \setminus aCl(\Re_a(A))$ is a nonempty $a-$open set. Since $x \in aCl(G)$ then for all a-open sets V_x containing x, $V_x \cap G \neq \emptyset$. Therefore $V_x \cap \Re_a(G) \neq \emptyset$, since $G \subseteq \Re_a(G)$. This implies that

$x \in aCl(\Re_a(G)) \subseteq aCl(\Re_a(\{x\} \cup G))$ ...... (i).

Again

$G \subseteq aCl(\Re_a(G)) \subseteq aCl(\Re_a(\{x\} \cup G))$ ......(ii).

From (i) and (ii) $\{x\} \cup G \subseteq aCl(\Re_a(\{x\} \cup G))$. Thus $\{x\} \cup G \in \Re_a(X, \tau^a)$. Now by given condition $A \cap (\{x\} \cup G)$ is a $\Re_a - aCl$ set.

We shall prove that $A \cap (\{x\} \cup G) = \{x\}$.

If possible suppose that there exists $y \in X$ and $x \neq y$ such that $y \in A \cap (\{x\} \cup G)$. So $y \in A$ and $y \in G$. Now $A = A \cap X$ and $X \in \Re_a(X, \tau^a)$, again by given condition $A \in \Re_a(X, \tau^a)$. Since $y \in A$, and $y \in aCl(\Re_a(A))$ - a contradiction to the fact that $y \in G = X \setminus aCl(\Re_a(A))$. Thus $A \cap (\{x\} \cup G) = \{x\}$. Since $\{x\} \in \Re_a(X, \tau^a)$, then

$\{x\} \in \tau^{a^{\Re_a}}$. So $\{x\} \subseteq aInt(aCl(\Re_a(\{x\}))) = aInt(aCl(\Re_a(A \cap (\{x\} \cup G)))) \subseteq aInt(aCl(\Re_a(A)))$. But $x \in aInt(aCl(\Re_a(A)))$, that is $A \in \tau^{a^{\Re_a}}$. □

Theorem 23. *Let (X, τ, I) be an ideal topological space, where $\tau^a \cap I = \emptyset$. Then $SO(X, \tau^{a^*}) = \Re_a(X, \tau^a)$.*

Proof. Let $A \in SO(X, \tau^{a^*})$. Then $A \subseteq Cl^{a^*}(Int^{a^*}(A)) = Cl^{a^*}[\Re_a(A) \cap A] \subseteq aCl(\Re_a(A) \cap A) \subseteq aCl(\Re_a(A))$. Thus $A \in \Re_a(X, \tau^a)$. For reverse inclusion, let $A \in \Re_a(X, \tau^a)$. We show that $A \in SO(X, \tau^{a^*})$. Take $x \in A$. Consider $G_1 \in \beta(I, \tau)$ [2] such that $x \in G_1$. Then G_1 is of the form $G_1 = G \setminus E$, where $G \in \tau^a$, $E \in I$. So $x \in G$. Since $A \subseteq aCl(\Re_a(A))$ and $G \in \tau^a$, $G \cap (\Re_a(A)) \neq \emptyset$. Let $y \in G \cap (\Re_a(A))$. Thus there exists $O_y \in \tau^a$ such that $O_y \setminus A \in \mathcal{I}$ by definition of $\Re_a(A)$. Consider $\emptyset \neq G \cap O_y$. So $(G \cap O_y) \setminus A \in I$ (by heredity). Let $G' = G \cap O_y$. Then $G' \neq \emptyset$, $G' \in \tau^a$ and $G' \setminus A = P$ say where $P \in \mathcal{I}$ and so $G' \setminus P \subseteq A$. Hence $G' \setminus (E \cup P) \subseteq A$ where $G' \setminus (E \cup P) \neq \emptyset$, since $\tau^a \cap \mathcal{I} = \emptyset$. Write $M = G' \setminus (E \cup P)$. Then $\emptyset \neq M \in \tau^{a^*}$ such that $M \subseteq A \cap (G \setminus E)$. Hence A contains a nonempty τ^{a^*}-open set M contained in $G \setminus E = G_1$. Since x is an arbitrary point of A, we get $A \subseteq Cl^{a^*}(Int^{a^*}(A))$. Therefore $A \in SO(X, \tau^{a^*})$. □

Corollary 24. *Let $x \in X$, then $\{x\} \in SO(X, \tau^{a^*})$ if and only if $\{x\} \in \tau^{a^{\Re_a}}$.*

Theorem 25. *$\tau^{a^{\Re_a}}$ is exactly the collection such that $A \in \tau^{a^{\Re_a}}$ and $B \in SO(X, \tau^{a^*})$ imply $A \cap B \in SO(X, \tau^{a^*})$, where $\tau^{a^*} \cap I = \emptyset$.*

Theorem 26. [20] *Let (X, τ) be a topological space. τ^α consists of exactly those sets A for which $A \cap B \in SO(X, \tau)$ for all $B \in SO(X, \tau)$.*

From above Theorem we get the representation of α - sets of (X, τ^a):

Theorem 27. *Let (X, τ, I) be an ideal topological space with $\tau^a \cap I = \emptyset$. Then $\tau^{a^{\Re_a}} = \tau^{{a^*}^{\alpha}}$.*

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Current address: Shyamapada Modak: Department of Mathematics, University of Gour Banga P.O. Mokdumpur, Malda - 732103, India

E-mail address: spmodak2000@yahoo.co.in

Current address: Md. Monirul Islam: Department of Mathematics, University of Gour Banga P.O. Mokdumpur, Malda - 732103, India

E-mail address: moni.math007@gmail.com