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$\mathcal{N}$-FUZZY IDEALS OF LATTICES

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Abstract. In this paper, the new concepts of $\mathcal{N}$-fuzzy ideals and $\mathcal{N}$-fuzzy prime ideals of lattices have been introduced. Also, some of theirs basic properties are investigated. Hence, some results about the homomorphic $\mathcal{N}$-image and pre-image of $\mathcal{N}$-fuzzy ideals of lattices are established.

1. Introduction

The concept of fuzzy sets was firstly introduced by Zadeh [16]. Rosenfeld [12] used this concept to formulate the notion of fuzzy groups. Since then many other fuzzy algebraic concepts had been studied by several authors. Yuan and Wu [15] introduced the concepts of fuzzy sublattice and fuzzy ideals of a lattice. Biswas [1] introduced the concept of anti fuzzy subgroups of groups. Shabir and Nawaz [13] introduced the concept of anti fuzzy ideals in semigroups. Khan and Asif [6] characterized different classes of semigroups by the properties of their anti fuzzy ideals. Lekkoksung [10] introduced the concept of an anti fuzzy bi-ideal of ordered Γ-semigroups. Kim and Jun [7] studied the notion of anti fuzzy ideals of a near-ring. Datta [2] introduced the concept of anti fuzzy bi-ideals in rings. Anti fuzzy ideals of Γ-rings were studied by Zhou et al. [17]. Srinivas et al. [14] introduced the concept of anti fuzzy ideals of Γ-near-ring. Dheena and Mohanraaj [3] introduced the notion of anti fuzzy right ideal, anti fuzzy right k-ideal and intuitionistic fuzzy right k-ideal in semiring. Hong and Jun [5] introduced the notion of anti fuzzy ideals of BCK algebras. In this paper, we introduce the concepts of $\mathcal{N}$-fuzzy ideals and $\mathcal{N}$-fuzzy prime ideals of lattices and investigate some related properties. Also, we give some results about the homomorphic $\mathcal{N}$-image and pre-image of $\mathcal{N}$-fuzzy ideals of lattices.

2. Preliminaries

In this section, let X denotes a bounded lattice with the least element 0 and the greatest element 1 unless otherwise specified.

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Definition 1. [4] *A non-empty subset I of X is called an ideal of X if, for any $a, b \in I$, $a \vee b \in I, a \wedge b \in I$ and, for any $a \in I$ and $x \in X$, $x \wedge a = x$ implies $x \in I$. A non-empty subset D of X is called a dual ideal of X if, for any $a, b \in D$, $a \wedge b \in D$, $a \vee b \in D$ and, for $a \in D$ and $x \in X$, $x \vee a = x$ implies $x \in D$. An ideal P of X is called a proper ideal if $P \neq X$. A proper ideal P of X is called a prime ideal of X if, for any $a, b \in X$, $a \wedge b \in P$ implies $a \in P$ or $b \in P$.*

Definition 2. [11] *Let X, Y be two sets and f be a mapping from X to Y. A fuzzy set μ of X (see* [16]*) is a map from X to $[0,1]$. If $\mathcal{F}(X)$ is the family of all fuzzy sets of X, then, for all $\mu, \nu \in \mathcal{F}(X)$, $\omega \in \mathcal{F}(Y)$ and $x \in X, y \in Y$, the following operations are defined:*

$$\begin{aligned}
(\mu \vee \nu)(x) &= max\{\mu(x), \nu(x)\} \\
(\mu \wedge \nu)(x) &= min\{\mu(x), \nu(x)\} \\
(\mu \times \omega)(x, y) &= min\{\mu(x), \omega(y)\} \\
f(\mu)(y) &= \begin{cases} \bigvee_{f(a)=y} \mu(a) & if \quad f^{-1}(y) \neq \emptyset, \\ 0 & otherwise, \end{cases} \\
f^{-1}(\omega)(x) &= \omega(f(x))
\end{aligned}$$

where $f(\mu)$ and $f^{-1}(\omega)$ are called, respectively, the image of μ under f and the pre-image of ω under f.

We denote $\mu \leq \nu$ if and only if $\mu(x) \leq \nu(x)$ for every $x \in X$. For $T \subseteq X$, $\chi_T \in \mathcal{F}(X)$ is called characteristic function of T, and defined by $\chi_T(x) = 1$ if $x \in T$ and $\chi_T(x) = 0$ otherwise for all $x \in X$.

Definition 3. [8] *Let μ be a fuzzy set of X. Then the complement of μ, denoted by μ^c, is the fuzzy set of X given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$. For $t \in [0,1]$, the set $\mu_t^- = \{x \in X | \mu(x) \leq t\}$ is called a lower t-level cut of μ and $\mu_t^+ = \{x \in X | \mu(x) \geq t\}$ is called an upper t-level cut of μ.*

It is clearly seen that $\mu_t^- = \left(\mu^c\right)_{1-t}^+$ for all $t \in [0,1]$.

Definition 4. [8] *Let $\{\mu_i | i \in \Lambda\}$ be a family of fuzzy sets in X, then the union $(\vee \mu_i)_{i \in \Lambda}$ is defined by $(\vee \mu_i)_{i \in \Lambda}(x) = sup\{\mu_i(x) | i \in \Lambda\}$ for each $x \in X$.*

Definition 5. [4] *A fuzzy set μ of X is proper if it is a non constant function.*

Definition 6. [9] *A fuzzy set μ of X is called a fuzzy sublattice of X if $\mu(x \wedge y) \geq min\{\mu(x), \mu(y)\}$ and $\mu(x \vee y) \geq min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.*

Definition 7. [9] *A fuzzy sublattice μ of X is called a fuzzy ideal of X if $\mu(x \vee y) = min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.*

Definition 8. [9] *A proper fuzzy ideal μ of X is called fuzzy prime ideal of X if $\mu(x \vee y) \leq max\{\mu(x), \mu(y)\}$ for all $x, y \in X$.*

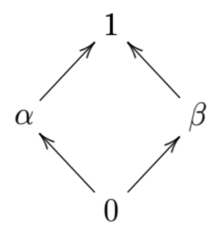

FIGURE 1. .

Theorem 1. [9] *A nonempty subset P of X is a prime ideal of X if and only if χ_p is a fuzzy prime ideal of X.*

3. $\mathcal{N}$-Fuzzy Ideals of Lattices

Definition 9. *A fuzzy set μ of X is called an $\mathcal{N}$-fuzzy sublattice of X if $\mu(x\wedge y) \leq max\{\mu(x), \mu(y)\}$ and $\mu(x \vee y) \leq max\{\mu(x), \mu(y)\}$ for all $x, y \in X$.*

Example 1. *Consider the lattice $X = \{0, \alpha, \beta, 1\}$ given by as follows*
Let define a fuzzy set μ of X by $\mu(0) = 0.1$, $\mu(\alpha) = 0.2$, $\mu(\beta) = 0.3$ and $\mu(1) = 0.3$. Then μ is an $\mathcal{N}$-fuzzy sublattice of X.

Theorem 2. *A fuzzy set μ of X is an $\mathcal{N}$-fuzzy sublattice of X if and only if μ^c is a fuzzy sublattice of X.*

Proof. Let μ be an $\mathcal{N}$-fuzzy sublattice of X. Then for $x, y \in X$,

$$\begin{aligned}
\mu^c(x\wedge y) &= 1-\mu(x\wedge y)\\
&\geq 1-max\{\mu(x),\mu(y)\}\\
&= min\{1-\mu(x),1-\mu(y)\}\\
&= min\{\mu^c(x),\mu^c(y)\}
\end{aligned}$$

$$\begin{aligned}
\mu^c(x\vee y) &= 1-\mu(x\vee y)\\
&\geq 1-max\{\mu(x),\mu(y)\}\\
&= min\{1-\mu(x),1-\mu(y)\}\\
&= min\{\mu^c(x),\mu^c(y)\}
\end{aligned}$$

Hence μ^c is a fuzzy sublattice of X. Similarly, the converse can be proved. □

Definition 10. *Let μ be an $\mathcal{N}$-fuzzy sublattice of X. Then μ is an $\mathcal{N}$-fuzzy ideal of X if $\mu(x \vee y) = max\{\mu(x), \mu(y)\}$ for all $x, y \in X$.*

Example 2. *Let consider the lattice $X = \{0, \alpha, \beta, 1\}$ given by in the Figure 1. Let define a fuzzy set μ of X by $\mu(0) = 0.1$, $\mu(\alpha) = 0.2$, $\mu(\beta) = 0.3$ and $\mu(1) = 0.3$. Then μ is an $\mathcal{N}$-fuzzy ideal of X.*

Every $\mathcal{N}$-fuzzy sublattice of X need not be $\mathcal{N}$-fuzzy ideal of X.

Example 3. *Let consider the lattice $X = \{0, \alpha, \beta, 1\}$ given by in the Figure 1. Let define a fuzzy set μ of X by $\mu(0) = 0$, $\mu(\alpha) = 0.2$, $\mu(\beta) = 0.3$ and $\mu(1) = 0.2$. Then μ is an $\mathcal{N}$-fuzzy sublattice of X, but μ is not an $\mathcal{N}$-fuzzy ideal of X as $\mu(\alpha \vee \beta) = \mu(1) \neq max\{\mu(\alpha), \mu(\beta)\}$.*

Remark 1. *Let μ be an $\mathcal{N}$-fuzzy ideal of X. As $\mu(x) = \mu(x \vee 0) = max\{\mu(x), \mu(0)\}$, then we get $\mu(0) \leq \mu(x)$ for any $x \in X$.*

Theorem 3. *A fuzzy set μ of X is an $\mathcal{N}$-fuzzy ideal of X if and only if μ^c is a fuzzy ideal of X.*

Proof. Let μ be an $\mathcal{N}$-fuzzy ideal of X. By Theorem 2, μ^c is a fuzzy sublattice of X. For $x, y \in X$,

$$\begin{aligned} \mu^c(x \vee y) &= 1 - \mu(x \vee y) \\ &= 1 - max\{\mu(x), \mu(y)\} \\ &= min\{1 - \mu(x), 1 - \mu(y)\} \\ &= min\{\mu^c(x), \mu^c(y)\} \end{aligned}$$

Thus, μ^c is a fuzzy ideal of X. Similarly, the converse can be proved. □

Theorem 4. *A fuzzy set μ of X is an $\mathcal{N}$-fuzzy ideal of X if and only if μ_t^- is an ideal of X for each $t \in [\mu(0), 1]$.*

Proof. Let μ be an $\mathcal{N}$-fuzzy ideal of X and $t \in [\mu(0), 1]$. Then by Theorem 3, μ^c is a fuzzy ideal of X. Hence $\mu_t^- = \left(\mu^c\right)_{1-t}^+$ is an ideal of X.

Conversely, μ, μ_t^- is an ideal of X for each $t \in [\mu(0), 1]$ and $s \in [0,\ 1 - \mu(0)] = [0,\ \mu^c(0)]$. Then $1 - s \in [\mu(0), 1]$ and $\left(\mu^c\right)_s^+ = \mu_{1-s}^-$ is an ideal of X. Hence $\left(\mu^c\right)_s^+$ is an ideal of X for all $s \in [0, \mu^c(0)]$, and μ^c is a fuzzy ideal of X. This shows that μ is an $\mathcal{N}$-fuzzy ideal of X. □

Theorem 5. *If I is an ideal of X, then for each $t \in [0, 1]$, there exists an $\mathcal{N}$-fuzzy ideal μ of X such that $\mu_t^- = I$.*

Proof. Let I be an ideal of X and $t \in [0, 1]$. Let define a fuzzy set μ of X by

$$\mu(x) = \begin{cases} t, & if\ x \in I \\ 1, & if\ x \notin I \end{cases}$$

for each $x \in X$. Then $\mu_s^- = I$ for any $s \in [t, 1) = [\mu(0), 1)$, and $\mu_1^- = X$. Thus μ_s^- is an ideal of X for all $s \in [\mu(0), 1]$. Hence, from Theorem 4, μ is $\mathcal{N}$-fuzzy ideal of X and $\mu_t^- = I$.

□

Let μ be a fuzzy set of X and let define $X_\mu = \{x \in X | \mu(x) = \mu(0)\}$. We then get the following theorem.

Theorem 6. *If μ is an $\mathcal{N}$-fuzzy ideal of X, then X_μ is an ideal of X.*

Proof. Let μ be an $\mathcal{N}$-fuzzy ideal of X and $x, y \in X_\mu$. Then $\mu(x) = \mu(0)$ and $\mu(y) = \mu(0)$. So $\mu(x \vee y) = max\{\mu(x), \mu(y)\} = \mu(0)$. Hence $x \vee y \in X_\mu$. Now let $x \leq a$, $x \in X$ and $a \in X_\mu$. Then $x \vee a = a$ and $\mu(a) = \mu(0)$. As μ is an $\mathcal{N}$-fuzzy ideal of X, $\mu(x \vee a) = max\{\mu(x)\mu(a)\}$. Thus $\mu(a) = max\{\mu(x), \mu(a)\}$. Therefore $\mu(x) \leq \mu(a) = \mu(0)$. Also by Remark 1, $\mu(0) \leq \mu(x)$. So we get $\mu(x) = \mu(0)$. Hence $x \in X_\mu$. This shows that X_μ is an ideal of X. □

Theorem 7. *If $\{\mu_i | i \in \Lambda\}$ a family of $\mathcal{N}$-fuzzy ideals of X, then so is $(\vee\mu_i)_{i\in\Lambda}$.*

Proof. Let $\{\mu_i, | \, i \in \Lambda\}$ be a family of $\mathcal{N}$-fuzzy ideals of X. Let $x, y \in X$.

$$\begin{aligned}
(\vee\mu_i)_{i\in\Lambda}(x \wedge y) &= sup\{\mu_i(x \wedge y) | i \in \Lambda\} \\
&\leq sup\big\{max\{\mu_i(x), \mu_i(y)\}\big\} \\
&= max\big\{sup\{\mu_i(x)\}, sup\{\mu_i(y)\}\big\} \\
&= max\{\vee\mu_i)_{i\in\Lambda}(x), \vee\mu_i)_{i\in\Lambda}(y)\}
\end{aligned}$$

$$\begin{aligned}
(\vee\mu_i)_{i\in\Lambda}(x \vee y) &= sup\{\mu_i(x \vee y) | i \in \Lambda\} \\
&\leq sup\big\{max\{\mu_i(x), \mu_i(y)\}\big\} \\
&= max\big\{sup\{\mu_i(x)\}, sup\{\mu_i(y)\}\big\} \\
&= max\{\vee\mu_i)_{i\in\Lambda}(x), \vee\mu_i)_{i\in\Lambda}(y)\}
\end{aligned}$$

Hence $(\vee\mu_i)_{i\in\Lambda}$ is an $\mathcal{N}$-fuzzy sublattice of X. Also,

$$\begin{aligned}
(\vee\mu_i)_{i\in\Lambda}(x \vee y) &= sup\{\mu_i(x \vee y) | i \in \Lambda\} \\
&= sup\big\{max\{\mu_i(x), \mu_i(y)\}\big\} \\
&= max\big\{sup\{\mu_i(x)\}, sup\{\mu_i(y)\}\big\} \\
&= max\{(\vee\mu_i)_{i\in\Lambda}(x), (\vee\mu_i)_{i\in\Lambda}(y)\}
\end{aligned}$$

Hence $(\vee\mu_i)_{i\in\Lambda}$ is an $\mathcal{N}$-fuzzy ideal of X. □

Theorem 8. *Let $f : X \to Y$ be a lattice homomorphism where Y is a bounded lattice. Let μ be an $\mathcal{N}$-fuzzy ideal of Y. Then $f^{-1}(\mu)$ is an $\mathcal{N}$-fuzzy ideal of X.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned}
f^{-1}(\mu)(x \wedge y) &= \mu(f(x \wedge y)) \\
&= \mu(f(x) \wedge f(y)) \\
&\leq max\{\mu(f(x)), \mu(f(y))\} \\
&= max\{f^{-1}(\mu)(x), f^{-1}(\mu)(y)\}
\end{aligned}$$

Thus $f^{-1}(\mu)(x \wedge y) \leq max\{f^{-1}(\mu)(x), f^{-1}(\mu)(y)\}$.

Similarly we can prove $f^{-1}(\mu)(x \vee y) \leq max\{f^{-1}(\mu)(x), f^{-1}(\mu)(y)\}$. Hence $f^{-1}(\mu)$ is an $\mathcal{N}$-fuzzy sublattice of X. Also for $x, y \in X$

$$\begin{aligned} f^{-1}(\mu)(x \vee y) &= \mu(f(x \vee y)) \\ &= \mu(f(x) \vee f(y)) \\ &= max\{\mu(f(x)), \mu(f(y))\} \\ &= max\{f^{-1}(\mu)(x), f^{-1}(\mu)(y)\} \end{aligned}$$

Thus $f^{-1}(\mu)(x \vee y) = max\{f^{-1}(\mu)(x), f^{-1}(\mu)(y)\}$. This shows that $f^{-1}(\mu)$ is an $\mathcal{N}$-fuzzy ideal of X. □

Definition 11. *Let $f : X \to Y$ be a mapping where Y is a non-empty set. Let μ be a fuzzy set of X. Then $\mathcal{N}$-image of μ under f is a fuzzy set $f(\mu)$ of Y defined by $f(\mu)(y) = inf\{\mu(x) | x \in X$ and $f(x) = y\}$ for all $y \in Y$.*

Theorem 9. *Let $f : X \to Y$ be an onto lattice homomorphism where Y is a bounded lattice. Let μ be $\mathcal{N}$-fuzzy ideal of X. Then $f(\mu)$ is an $\mathcal{N}$-fuzzy ideal of Y.*

Proof. Let μ be an $\mathcal{N}$-fuzzy ideal of X and $a, b \in Y$. As f is onto, there exist $p, q \in X$ such that $f(p) = a$ and $f(q) = b$. Also $a \wedge b = f(p) \wedge f(q) = f(p \wedge q)$ and

$$\begin{aligned} f(\mu)(a \wedge b) &= inf\{\mu(z) | z \in X \text{ and } f(z) = a \wedge b\} \\ &\leq inf\{\mu(p \wedge q) | f(p) = a \text{ and } f(q) = b\} \\ &\leq inf\big\{max\{\mu(p), \mu(q)\} | f(p) = a \text{ and } f(q) = b\big\} \\ &= max\big\{inf\{\mu(p)/f(p) = a\}, inf\{\mu(q) | f(q) = b\}\big\} \\ &= max\{f(\mu)(a), f(\mu)(b)\} \end{aligned}$$

Thus $f(\mu)(a \wedge b) \leq max\{f(\mu)(a), f(\mu)(b)\}$. Similarly we can prove that $f(\mu)(a \vee b) \leq max\{f(\mu)(a), f(\mu)(b)\}$. Hence $f(\mu)$ is an $\mathcal{N}$-fuzzy sublattice of Y.

Again let $x, y \in Y$. As f is onto, there exist $r, s \in X$ such that $f(r) = x$ and $f(s) = y$. Also $x \vee y = f(r) \vee f(s) = f(r \vee s)$ and

$$\begin{aligned} f(\mu)(x \vee y) &= inf\{\mu(z) | z \in X \text{ and } f(z) = x \vee y\} \\ &= inf\{\mu(r \vee s) | f(r) = x \text{ and } f(s) = y\} \\ &= inf\big\{max\{\mu(r), \mu(s)\} | f(r) = x \text{ and } f(y) = s\big\} \\ &= max\big\{inf\{\mu(r) | f(r) = x\}, inf\{\mu(s) | f(s) = y\}\big\} \\ &= max\{f(\mu)(x), f(\mu)(y)\} \end{aligned}$$

Thus $f(\mu)(x \vee y) = max\{f(\mu)(x), f(\mu)(y)\}$. This shows that $f(\mu)$ is an $\mathcal{N}$-fuzzy ideal of Y. □

Theorem 10. *Every $\mathcal{N}$-fuzzy ideal of X is order preserving.*

Proof. Let μ be an $\mathcal{N}$-fuzzy ideal of X. Let $x, y \in X$ such that $x \leq y$. Then $\mu(y) = \mu(x \vee y) = max\{\mu(x), \mu(y)\}$. Thus $\mu(x) \leq \mu(y)$. □

Definition 12. *Let λ and μ be fuzzy sets of X. The $\mathcal{N}$ Cartesian product $\lambda \times \mu : X \times X \to [0,1]$ is defined by $\lambda \times \mu(x,y) = max\{\lambda(x), \mu(y)\}$ for all $x, y \in X$.*

Theorem 11. *If λ and μ are $\mathcal{N}$-fuzzy ideals of X, then $\lambda \times \mu$ is $\mathcal{N}$-fuzzy ideal of $X \times X$.*

Proof. Let (x_1, y_1) and $(x_2, y_2) \in X \times X$. Then

$$\begin{aligned}
\lambda \times \mu((x_1,y_1) \wedge (x_2,y_2)) &= \lambda \times \mu(x_1 \wedge x_2,\ y_1 \wedge y_2)\} \\
&= max\{\lambda(x_1 \wedge x_2),\ \mu(y_1, y_2)\} \\
&\leq max\big\{max\{\lambda(x_1), \lambda(x_2)\}, \{max\{\mu(y_1), \mu(y_2)\}\big\} \\
&= max\big\{max\{\lambda(x_1), \mu(y_1)\}, \{max\{\lambda(x_2), \mu(y_2)\} \\
&= max\{\lambda \times \mu(x_1, y_1),\ \lambda \times \mu(x_2, y_2)\}
\end{aligned}$$

$$\begin{aligned}
\lambda \times \mu((x_1,y_1) \vee (x_2,y_2)) &= \lambda \times \mu(x_1 \vee x_2,\ y_1 \vee y_2)\} \\
&= max\{\lambda(x_1 \vee x_2),\ \mu(y_1 \vee y_2) \\
&\leq max\big\{max\{\lambda(x_1), \lambda(x_2)\}, max\{\mu(y_1), \mu(y_2)\}\big\} \\
&= max\big\{max\{\lambda(x_1), \mu(y_1)\}, max\{\lambda(x_2), \mu(y_2)\} \\
&= max\{\lambda \times \mu(x_1, y_1),\ \lambda \times \mu(x_2, y_2)\}
\end{aligned}$$

Thus $\lambda \times \mu$ is an $\mathcal{N}$-fuzzy sublattice of $X \times X$. Also,

$$\begin{aligned}
\lambda \times \mu((x_1,y_1) \vee (x_2,y_2)) &= \lambda \times \mu(x_1 \vee x_2,\ y_1 \vee y_2) \\
&= max\{\lambda(x_1 \vee x_2),\ \mu(y_1 \vee y_2) \\
&= max\big\{max\{\lambda(x_1), \lambda(x_2)\}, max\{\mu(y_1), \mu(y_2)\}\big\} \\
&= max\big\{max\{\lambda(x_1), \mu(y_1)\}, max\{\lambda(x_2), \mu(y_2)\}\big\} \\
&= max\{\lambda \times \mu(x_1, y_1),\ \lambda \times \mu(x_2, y_2)\}
\end{aligned}$$

Hence $\lambda \times \mu$ is an $\mathcal{N}$-fuzzy ideal of $X \times X$. □

Definition 13. *Let μ be an $\mathcal{N}$-fuzzy ideal of X. Then μ is called an $\mathcal{N}$-fuzzy prime ideal of X if $\mu(x \wedge y) \geq min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.*

Example 4. *Consider the lattice $X = \{0, \alpha, \beta, 1\}$ given by in the Figure 1. Let define a fuzzy set μ of X by $\mu(0) = 0.2$, $\mu(\alpha) = 0.2$, $\mu(\beta) = 0.3$ and $\mu(1) = 0.3$. Then μ is an $\mathcal{N}$-fuzzy prime ideal of X.*

Every $\mathcal{N}$-fuzzy ideal of X need not be $\mathcal{N}$-fuzzy prime ideal of X.

Example 5. *Consider the lattice $X = \{0, \alpha, \beta, 1\}$ given by in the Figure 1. Let define a fuzzy set μ of X by $\mu(0) = 0.1$, $\mu(\alpha) = 0.2$, $\mu(\beta) = 0.3$ and $\mu(1) = 0.3$. Then μ is an $\mathcal{N}$-fuzzy ideal of X, but μ is not an $\mathcal{N}$-fuzzy prime ideal of X as $\mu(\alpha \wedge \beta) = \mu(0) \ngeq min\{\mu(\alpha), \mu(\beta)\}$.*

Theorem 12. *Let P be a non-empty subset of X and $r, t \in [0,1]$ such that $r < t$. Let μ_p be a fuzzy subset of X such that*

$$\mu_p(x) = \begin{cases} r, & if\ x \in P \\ t, & if\ x \notin P \end{cases}$$

for all $x \in X$. Then P is a prime ideal of X if and only if μ_p is an $\mathcal{N}$-fuzzy prime ideal of X.

Proof. Let P be a prime ideal of X and $x, y \in X$. If $x \wedge y \in P$, then $\mu_p = (x \wedge y) = r \leq max\{\mu_p(x), \mu_p(y)\}$. If $x \wedge y \notin P$, then $x \notin P$ and $y \notin P$. Then $\mu_p(x \wedge y) = t,\ \mu_p(x) = t$ and $\mu_p(y) = t$. Hence $\mu_p(x \wedge y) \leq max\{\mu_p(x), \mu_p(y)\}$. Therefore we have $\mu_p(x \wedge y) \leq max\{\mu_p(x), \mu_p(y)\}$. Similarly we can prove that $\mu_p(x \vee y) \leq max\{\mu_p(x), \mu_p(y)\}$. This shows that μ_p is an $\mathcal{N}$-fuzzy sublattice of X.

Now let $x, y \in X$. If $x \vee y \in P$, then $x \in P$ and $y \in P$. Therefore $\mu_p(x \wedge y) = r,\ \mu_p(x) = r$ and $\mu_p(y) = r$. Hence $\mu_p(x \vee y) = max\{\mu_p(x), \mu_p(y)\}$. If $x \vee y \notin P$, then $x \notin P$ or $y \notin P$. Therefore $\mu_p(x \wedge y) = t,\ \mu_p(x) = t$ or $\mu_p(y) = t$. Hence $\mu_p(x \vee y) = max\{\mu_p(x), \mu_p(y)\}$. This shows that μ_p is an $\mathcal{N}$-fuzzy ideal of X.

Again let $x, y \in X$. If $x \wedge y \in P$, then $x \in P$ or $y \in P$ (since P is a prime ideal of X). Therefore $\mu_p(x \wedge y) = r,\ \mu_p(x) = r$ or $\mu_p(y) = r$. Hence $\mu_p(x \vee y) \geq min\{\mu_p(x), \mu_p(y)\}$. If $x \wedge y \notin P$, then $x \notin P$ and $y \notin P$ (since P is an ideal of X). Therefore $\mu_p(x \wedge y) = t,\ \mu_p(x) = t$ and $\mu_p(y) = t$. Hence $\mu_p(x \wedge y) \geq min\{\mu_p(x), \mu_p(y)\}$. This shows that μ_p is $\mathcal{N}$-fuzzy prime ideal of X.

Conversely, let μ_p be $\mathcal{N}$-fuzzy prime ideal of X and $x, y \in P$. As μ_p is $\mathcal{N}$-fuzzy sublattice of X, $\mu_p(x \wedge y) \leq max\{\mu_p(x), \mu_p(y)\} = r$. Hence $x \wedge y \in P$. Similarly we can prove that $x \vee y \in P$. This shows that P is a sublattice of X. Now let $a \in P, x \in X$ such that $x \wedge a = x$. Thus $x \vee a = (x \wedge a) \vee a = a$. Therefore $r = \mu_p(a) = \mu_p(x \vee a) = max\{\mu_p(x), \mu_p(a)\}$. Hence $\mu_p(x) = r$ and so $x \in P$. This shows that P is an ideal of X.

Again let $x \wedge y \in P$. Then $\mu_p(x \wedge y) = r$. As μ_p is $\mathcal{N}$-fuzzy prime ideal of X, $\mu_p(x \wedge y) \geq min\{\mu_p(x), \mu_p(y)\}$. Therefore $\mu_p(x) = r$ or $\mu_p(y) = r$. Hence $x \in P$ or $y \in P$. This shows that P is a prime ideal of X. □

4. Conclusion

In this paper we defined the notions of $\mathcal{N}$-fuzzy sublattice, $\mathcal{N}$-fuzzy ideal and $\mathcal{N}$-fuzzy prime ideal of a bounded lattice. We showed that the complement of $\mathcal{N}$-fuzzy sublattice of a bounded lattice is a fuzzy sublattice. We also showed that the union a family of $\mathcal{N}$-fuzzy ideals of a bounded lattice is also $\mathcal{N}$-fuzzy ideal of a bounded lattice. We stated how the homomorphic $\mathcal{N}$-images and inverse images of $\mathcal{N}$-fuzzy ideals of a bounded lattice become $\mathcal{N}$-fuzzy ideal of a bounded lattice. We also investigated how the $\mathcal{N}$-Cartesian product of $\mathcal{N}$-fuzzy ideals of a bounded lattice becomes $\mathcal{N}$-fuzzy ideal of a bounded lattice. Our future work will focus on studying the intuitionistic $\mathcal{N}$-fuzzy ideals of a bounded lattice.

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