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ABUNDANCE OF EQUIVALENT NORMS ON c_0 WITH FIXED POINT PROPERTY FOR AFFINE NONEXPANSIVE MAPPINGS

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Abstract. In 1979, K. Goebel and T. Kuczumow showed that a large class of closed, bounded, convex (c.b.c.), non-weak*-compact subsets K of l^1 has the fixed point property for nonexpansive mappings. Later, in 2008, P.K. Lin proved that l^1 can be renormed to have the fixed point property for nonexpansive mappings. Then, Nezir recently worked on c_0-analogue of Goebel and Kuczumow's theorem with an equivalent norm and showed that there exists a large class of equivalent norms $\|\cdot\|$ on c_0 for which there exist non-weakly compact closed, bounded, convex subsets that have the fixed point property for affine $\|\cdot\|$-nonexpansive mappings. In fact, he sees that his examples are closed, convex hulls of some asymptotically isometric (ai) c_0-summing basic sequences whereas Lennard and Nezir in 2011 showed that the closed, convex hull of any ai c_0-summing basic sequence fails the fixed point property for affine $\|\cdot\|_\infty$-nonexpansive mappings. In this work, we show that equivalent norms with fixed point property for affine nonexpansive mappings are somewhat abundant. Firstly, we construct many types of equivalent norms and even show some norms are exactly the same as the natural norm while it is not clear to see that in the beginning, and then we show with our new type of equivalent norms c_0 do not contain any asymptotically isometric copy of c_0. Next, we see that Nezir's equivalent norms are not the only ones with fixed point property for affine nonexpansive mappings on his sets.

1. Introduction

It is well-known that Banach space of sequences converging to 0, $(c_0, \|\cdot\|_\infty)$ has the weak fixed point property, so does the space of absolutely summable sequences, $(\ell^1, \|\cdot\|_1)$. In other words, for every weakly compact, convex (nonempty) subset C of $(c_0, \|\cdot\|)$, for all nonexpansive mappings $T\colon C \to C$ [i.e.,

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$\|Tx - Ty\|_\infty \leq \|x - y\|_\infty$, for all $x, y \in C$], T has a fixed fixed point in C. It is also well-known that both spaces fail the fixed point property (ℓ^1 fails so does c_0). Indeed, let $C := \{$sequences $(t_n)_{n\in\mathbb{N}}$: each $t_n \geq 0$ and $\sum_{n=1}^{\infty} t_n = 1\}$. This is a closed, bounded, convex subset of ℓ^1. Let $T\colon C \to C$ be the right shift map on C; i.e., $T(t_1, t_2, t_3, \dots) := (0, t_1, t_2, t_3, \dots)$. T is clearly $\|\cdot\|_1$-nonexpansive (being an isometry) and fixed point free on C. We should note that these two spaces can be considered as the examples of nonreflexive Banach spaces failing the fixed point property for nonexpansive mappings (FPP(n.e.)). It is an open question as to whether or not all nonreflexive Banach spaces fail the FPP(n.e) and it is unknown if every reflexive Banach space has the FPP(n.e.). However, in 1965, Kirk [5] showed that all reflexive Banach spaces with normal structure (spaces such that all non-trivial closed, bounded, convex sets C have a smaller Chebyshev radius than diameter) have the FPP(n.e.). Recently, in a significant development, Lin [7] provided the first example of a non-reflexive Banach space $(X, \|\cdot\|)$ with the fixed point property for nonexpansive mappings. Lin verified this fact for $(\ell^1, \|\cdot\|_1)$ with the equivalent norm $|||\cdot|||$ given by

$$|||x||| = \sup_{k\in\mathbb{N}} \frac{8^k}{1+8^k} \sum_{n=k}^{\infty} |x_n|, \text{ for all } x = (x_n)_{n\in\mathbb{N}} \in \ell^1 .$$

What about $(c_0, \|\cdot\|_\infty)$ analogue of P.K. Lin's work? While this is a famous open question, c_0 analogue of Goebel & Kuczmunow's theory (with an equivalent norm of course) has also great importance since it would be the first step to find a candidate equivalent norm to work on c_0 analogue of P.K. Lin's work. For the readers who don't know Goebel and Kuczmunow's work [3], we can explain their study which was done before P.K. Lin's. They showed that while l^1 fails the FPP(n.e.) with its usual norm, there exists a large class of closed, convex, bounded and non-weak*-compact subsets K of $(\ell^1, \|\cdot\|_1)$ such that every $\|\cdot\|_1$-nonexpansive mappings $T\colon K \to K$ has a fixed point. In contrast to Goebel and Kuczmunow's result for l^1, Dowling, Lennard and Turett [2] showed that any closed infinite dimensional subspace of $(c_0, \|\cdot\|_\infty)$ also fails the FPP(n.e.). Thus, to think about Goebel and Kuczmunow's work's analogue for c_0, we have to think about it after renorming c_0. That is, we can work on a question "do there exist any renorming of c_0 and a nonempty closed, bounded and convex subset C so that every nonexpansive mapping has fixed point property?". Nezir [8] recently gave positive answer for this question when the mapping is also affine. His work is interesting because he invented an equivalent norm and he showed that the closed convex hull of an asymptotically isometric c_0-summing basis for the usual absolute sup norm has the fixed point property for affine nonexpansive mappings whereas in 2011, Lennard and Nezir [6] proved that if a Banach space contains an asymptotically isometric (ai) c_0-summing basic sequence $(x_n)_{n\in\mathbb{N}}$, then the closed convex hull of $(x_n)_{n\in\mathbb{N}}$, $E := \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$, fails the fixed point property for affine nonexpansive mappings.

In their paper, first of all, they work on some specific ai c_0-summing basic sequences.

For example, they fix $b \in (0,1)$ and define the sequence $(f_n)_{n\in\mathbb{N}}$ in c_0 by setting $f_1 := b\,e_1$, $f_2 := b\,e_2$, and $f_n := e_n$, for all integers $n \geq 3$ where $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of c_0; i.e., the scalar sequence e_n, with domain $\mathbb{N}$, is defined to be 1 in its nth coordinate, and 0 in all other coordinates and recall that the sequence $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis for both $(c_0, \|\cdot\|_\infty)$ and $(\ell^1, \|\cdot\|_1)$.

Next, they define the closed, bounded, convex subset $E = E_b$ of c_0 by

$$E := \left\{ \sum_{n=1}^{\infty} t_n\, f_n : 1 = t_1 \geq t_2 \geq \cdots \geq t_n \downarrow_n 0 \right\} .$$

Then, they define the sequence $(\eta_n)_{n\in\mathbb{N}}$ in E in the following way. Let $\eta_1 := f_1$ and $\eta_n := f_1 + \cdots + f_n$, for all integers $n \geq 2$. Note that

$$E := \left\{ \sum_{n=1}^{\infty} \alpha_n\, \eta_n : \text{ each } \alpha_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = 1 \right\} .$$

Next, they give the following theorem:

Theorem 1.1. *Let $b \in (0,1)$. Then $E = E_b$ is such that there exists an affine $\|\cdot\|_\infty$-nonexpansive mapping $U : E \longrightarrow E$ that is fixed point free.*

Easily, it can be seen that the set E is the closed convex hull the sequence $(\eta_n)_{n\in\mathbb{N}}$ and this sequence is an ai c_0-summing basic sequence.

In the work of Nezir [8], he showed that c_0 can be renormed so that the set E above with b restricted little bit ($\exists C \in (0,1) \ \ni \forall b \in (0,C)$) for all affine nonexpansive mappings $T\colon E \longrightarrow E$, T has a fixed point in E. In fact, he presented there is a large class of renormings to have this property.

In our work, we invent another large class of renormings that gives the same results what Nezir had done in his above mentioned work. That is why, it can be said that equivalent norms with fixed point property for affine nonexpansive mappings on a large class of subsets of c_0 are somewhat abundant. Firstly, we construct many types of equivalent norms and even show some norms are exactly the same as the natural norm while it is not clear to see that in the beginning, and then we show with our new type of equivalent norms c_0 do not contain any asymptotically isometric copy of c_0. Next, we see that Nezir's equivalent norms are not the only ones with fixed point property for affine nonexpansive mappings on his sets.

We believe that our results have great importance in terms of bringing new candidates to solve c_0 analogue of P.K. Lin's theorem [7]. In fact, using our equivalent norms, one can obtain more equivalent norms satisfying our results and even better results.

Now, we can give preliminiaries for our work such that some preliminiaries have been given in [8].

2. Preliminaries

Definition 2.1. Let C be a non-empty closed, bounded, convex (c.b.c.) subset of a Banach space $(X, \|\cdot\|)$. A mapping $T : C \longrightarrow C$ is called *nonexpansive* if $\|T(x) - T(y)\| \leq \|x - y\|$, for all $x, y \in C$.

We say that C has the *fixed point property for nonexpansive mappings* [FPP(n.e.)] if for all nonexpansive mappings $T : C \longrightarrow C$, there exists $z \in C$ with $T(z) = z$.

Definition 2.2. Let C be a non-empty closed, bounded, convex subset of a Banach space $(X, \|\cdot\|)$. A mapping $U : C \longrightarrow C$ is said to be *affine* if for all $\lambda \in [0, 1]$, for all $x, y \in C$,

$$U\big((1-\lambda)\,x + \lambda\,y\big) = (1-\lambda)\,U(x) + \lambda\,U(y)\ .$$

We say that C has the *fixed point property for affine nonexpansive mappings* [FPP(affine, n.e.)] if for all affine nonexpansive mappings $U : C \longrightarrow C$, there exists $z \in C$ with $U(z) = z$.

Let $(X, \|\cdot\|)$ be a Banach space and $E \subseteq X$. We will denote the closed, convex hull of E by $\overline{\text{co}}(E)$. As usual, $(c_0, \|\cdot\|_\infty)$ is given by

$$c_0 := \left\{ x = (x_n)_{n\in\mathbb{N}} : \text{each } x_n \in \mathbb{R} \text{ and } \lim_{n\longrightarrow\infty} x_n = 0 \right\}\ .$$

Further, $\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|$, for all $x = (x_n)_{n\in\mathbb{N}} \in c_0$; and $(\ell^1, \|\cdot\|_1)$ is defined by

$$\ell^1 := \left\{ x = (x_n)_{n\in\mathbb{N}} : \text{each } x_n \in \mathbb{R} \text{ and } \|x\|_1 := \sum_{n=1}^{\infty} |x_n| < \infty \right\}\ .$$

We denote by c_{00} the vector space of all scalar sequences that have only finitely many non-zero terms. In other words, c_{00} is the linear span of $\{e_n : n \in \mathbb{N}\}$ inside c_0 (and ℓ^1).

We recall now the definition of an *asymptotically isometric c_0-summing basic sequence* in a Banach space $(X, \|\cdot\|)$ from the work of Lennard and Nezir [6].

Definition 2.3. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Banach space $(X, \|\cdot\|)$. We define $(x_n)_{n\in\mathbb{N}}$ to be an *asymptotically isometric (ai) c_0-summing basic sequence* in $(X, \|\cdot\|)$ if there exists a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $[0, \infty)$ such that for all sequences $(t_n)_{n\in\mathbb{N}} \in c_{00}$,

$$\sup_{n\geq 1} \left(\frac{1}{1+\varepsilon_n}\right) \left|\sum_{j=n}^{\infty} t_j\right| \leq \left\|\sum_{j=1}^{\infty} t_j\, x_j\right\| \leq \sup_{n\geq 1}\, (1+\varepsilon_n) \left|\sum_{j=n}^{\infty} t_j\right|\ .$$

2.1. **Nezir's equivalent norm and his results.**

Theorem 2.4. [8] *For $x = (\xi_k)_k \in c_0$, define*

$$\begin{aligned} \|x\|^{\heartsuit} \;&=\; \|x\|_\infty + \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k \left|\xi_k - \alpha\xi_j\right| \quad \textit{where } \sum_{k=1}^{\infty} Q_k = 1,\ Q_k \downarrow_k 0 \\ &\quad\ \textit{and } Q_k > Q_{k+1},\ \forall k \in \mathbb{N}. \end{aligned}$$

Then, if $\alpha = 0$ *or if* $Q_1 > \frac{2|\alpha|}{1+2|\alpha|}$ *when* $|\alpha| > 1$*, then* $\left(c_0,\ \|\cdot\|^{\heartsuit}\right)$ *does not contain an asymptotically isometric copy of* c_0*.*

2.1.1. *For an equivalent norm, a set on* c_0 *like Goebel and Kuczumow's set on* l^1*.* In 1979, K. Goebel and T. Kuczumow showed that there exist some closed, bounded, convex and non-weak*-compact subsets K of l^1 and K have the FPP(n.e.). In their work, they use the following lemma as a main tool and here we will get the analogous idea for c_0 that will be a tool for us.

Lemma 2.5. [3] *If* $\{x_n\}$ *is a sequence in* l^1 *converging to* x *in weak-star topology, then for any* $y \in l^1$

$$r(y) = r(x) + \|y - x\|_1$$

where $r(y) = \limsup_n \|x_n - y\|_1$ *.*

2.1.2. *A Function Like Asymptotic Center Function.*

Lemma 2.6. [8] *Let* $(X,\ \|\ .\ \|)$ *be a Banach Space,* $(x_n)_n$ *be a bounded sequence in* X*. For any arbitrary subsequence* $(x_{n_k})_k$*, consider a function* $s : X \to [0, \infty)$ *given by*

$$s(y) = \limsup_m \left\| \frac{1}{m} \sum_{k=1}^{m} x_{n_k} - y \right\| \ , \ \forall y \in X \ .$$

Then, if $(y_m)_m$ *is a bounded sequence in* c_0 *converging to* x *in weak topology, there exists a subsequence* $(x_n)_n$ *whose Cesaro mean in norm approaches to* x *and so when* $X = c_0$ *and* s *is defined by this subsequence, we have*

$$s(y) = s(x) + \|y - x\| \ , \ \forall y \in c_0$$

where $\|\cdot\|$ *is any equivalent norm to* $\|\cdot\|_\infty$ *on* c_0*.*

2.1.3. *A Set in* c_0 *having FPP(for n.e. and affine mappings) for an equivalent norm.*

Example 2.7. Fix $b \in (0, 1)$. We define the sequence $(f_n)_{n\in\mathbb{N}}$ in c_0 by setting $f_1 := b\, e_1$, $f_2 := b\, e_2$, and $f_n := e_n$, for all integers $n \geq 3$. Next, define the closed, bounded, convex subset $E = E_b$ of c_0 by

$$E := \left\{ \sum_{n=1}^{\infty} t_n\, f_n : 1 = t_1 \geq t_2 \geq \cdots \geq t_n \downarrow_n 0 \right\} \ .$$

Let us define the sequence $(\eta_n)_{n\in\mathbb{N}}$ in E in the following way. Let $\eta_1 := f_1$ and $\eta_n := f_1 + \cdots + f_n$, for all integers $n \geq 2$. It is straightforward to check that

$$E := \left\{ \sum_{n=1}^{\infty} \alpha_n\, \eta_n : \text{ each } \alpha_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = 1 \right\} \ .$$

Then, in 2011, Lennard and Nezir [6] show that $E = E_b$ is the closed convex hull of $(\eta_n)_{n\in\mathbb{N}}$ which is an asymptotically isometric c_0-summing basis respect to $\|\cdot\|_\infty$

and that there exists an affine $\|\cdot\|_\infty$-nonexpansive mapping $U : E \longrightarrow E$ that is fixed point free.

Theorem 2.8. [8] *There exist constants $0 < C \leq 1$ and $\alpha \geq \frac{1}{2}$ such that for all $b \in (0, C)$ the set E defined as in the example above has the fixed point property for $\|.\|$-nonexpansive affine mappings where the norm $\|\cdot\|^{\heartsuit}$ on c_0 is given as below such that $Q_1 > \frac{2\alpha}{1+2\alpha}$:*

For $x = (\xi_k)_k \in c_0$ and $\alpha > 0$,

$$\begin{aligned}\|x\|^{\heartsuit} &= \|x\|_\infty + \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k \left|\xi_k - \alpha\xi_j\right| \quad \text{where } \sum_{k=1}^{\infty} Q_k = 1,\ Q_k \downarrow_k 0, \\ & Q_k > Q_{k+1},\ \forall k \in \mathbb{N}.\end{aligned}$$

3. A new look to the absolute sup norm of c_0

Theorem 3.1. *For any $x = (\xi_i)_{i\in\mathbb{N}} \in c_0$ and for any $n, m \in \mathbb{N}$,*

$$\|x\|_\infty = \lim_{p\to\infty} \left(\sum_{k=1}^{\infty} \frac{|\xi_k|^p}{k^n} \right)^{\frac{1}{p}} \tag{3.1}$$

and

$$\sqrt[m]{\|x\|_\infty} = \lim_{p\to\infty} \left(\sum_{k=1}^{\infty} \frac{|\xi_k|^p}{k^n} \right)^{\frac{1}{mp}}. \tag{3.2}$$

Proof. Let $x = (\xi_i)_{i\in\mathbb{N}} \in c_0$. We will consider $x \neq (0, 0, \cdots)$ otherwise proof of the claim is clear.

Then,

$$\exists N \in \mathbb{N} \ni \|x\|_\infty = \sup_{k\in\mathbb{N}} |\xi_k| = \max_{k\in\mathbb{N}} |\xi_k| = |\xi_N|.$$

Due to power mean inequalities formula (see eg. [4]),

$$\begin{aligned}\|x\|_\infty &= \max_{k\leq N} |\xi_k| \\ &= \max\{|\xi_1|, |\xi_2|, \cdots, |\xi_N|\} \\ &= \lim_{p\to\infty} \left(\frac{|\xi_1|^p + |\xi_2|^p + \cdots + |\xi_N|^p}{N} \right)^{\frac{1}{p}} \\ &= \lim_{p\to\infty} \left(\sum_{k=1}^{N} \frac{|\xi_k|^p}{N} \right)^{\frac{1}{p}}.\end{aligned}$$

(1) Case $n = 1,\ m = 1$

Claim 3.2.

$$\|x\|_\infty = \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k}\right)^{\frac{1}{p}} \tag{3.3}$$

Indeed,

$$\begin{aligned}\|x\|_\infty &\le \lim_{p\to\infty}\left(\sum_{k=1}^{N}\frac{|\xi_k|^p}{k}\right)^{\frac{1}{p}}\\ &\le \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k}\right)^{\frac{1}{p}}.\end{aligned}$$

On the other hand, $\exists s\in\mathbb{N}$ such that $|\xi_k| < \frac{1}{k}$, $\forall k\ge s$. Thus,

$$\begin{aligned}\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k}\right)^{\frac{1}{p}} &= \lim_{p\to\infty}\left(\sum_{k=1}^{s-1}\frac{|\xi_k|^p}{k}+\sum_{k=s}^{\infty}\frac{|\xi_k|^p}{k}\right)^{\frac{1}{p}}\\ &\le \lim_{p\to\infty}\left(\sum_{k=1}^{s-1}\frac{|\xi_k|^p}{k}+\frac{|\xi_s|^p}{s}+\int_s^\infty\frac{|\xi_k|^p}{k}\,dk\right)^{\frac{1}{p}}\\ &= \lim_{p\to\infty}\left(\sum_{k=1}^{s}\frac{|\xi_k|^p}{k}+\int_s^\infty\frac{|\xi_k|^p}{k}\,dk\right)^{\frac{1}{p}}\\ &\le \lim_{p\to\infty}\left(\sum_{k=1}^{s}\frac{|\xi_k|^p}{k}+\int_s^\infty\frac{1}{k^{p+1}}\,dk\right)^{\frac{1}{p}}\\ &\le \lim_{p\to\infty}\left(|\xi_N|^p\sum_{k=1}^{s}\frac{1}{k}+\frac{1}{p(s+1)^p}\right)^{\frac{1}{p}}\\ &\le \lim_{p\to\infty}\left(|\xi_N|^p\sum_{k=1}^{s}\frac{1}{k}+\frac{1}{p(s+1)^p}\right)^{\frac{1}{p}}\\ &\le \lim_{p\to\infty}\left(|\xi_N|^p\left[1+\int_1^s\frac{1}{k}\,dk\right]+\frac{1}{p(s+1)^p}\right)^{\frac{1}{p}}\\ &= |\xi_N|\\ &= \|x\|_\infty.\end{aligned}$$

(2) Case $n=2,\ m=1$

Claim 3.3.

$$\|x\|_\infty = \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^2}\right)^{\frac{1}{p}} \tag{3.4}$$

Indeed, first of all, (due to first case) it is clear that

$$\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^2}\right)^{\frac{1}{p}} \leq \|x\|_\infty.$$

On the other hand, for p large enough, $\exists j \in \mathbb{N}$ such that $|\xi_k|^p < \frac{1}{k^2}$, $\forall k \geq j$. (so $\sum_{k=1}^{\infty} |\xi_k|^p$ is convergent. We should note that one could be confused by taking a sequence e.g. $\xi_k = \frac{1}{k^{\frac{1}{p}}}$ but as p large enough, that would not be in c_0. We will not give more detailed proof for our statement above.) Thus,

$$\begin{aligned}
\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k}\right)^{\frac{1}{p}} &= \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^{\frac{p}{2}}}{k}|\xi_k|^{\frac{p}{2}}\right)^{\frac{1}{p}} \\
&\leq \lim_{p\to\infty}\left(\sqrt{\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^2}}\sqrt{\sum_{k=1}^{\infty}|\xi_k|^p}\right)^{\frac{1}{p}} \\
&\leq \sqrt{\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^2}\right)^{\frac{1}{p}}}\sqrt{\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}|\xi_k|^p\right)^{\frac{1}{p}}} \\
&= \sqrt{\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^2}\right)^{\frac{1}{p}}}\sqrt{\lim_{p\to\infty}\left(\sum_{k=1}^{j-1}|\xi_k|^p+\sum_{k=j}^{\infty}|\xi_k|^p\right)^{\frac{1}{p}}} \\
&\leq \sqrt{\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^2}\right)^{\frac{1}{p}}}\sqrt{\lim_{p\to\infty}\left((j-1)\max_{1\leq k\leq j-1}|\xi_k|^p+\sum_{k=j}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{p}}} \\
&\leq \sqrt{\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^2}\right)^{\frac{1}{p}}}\sqrt{\lim_{p\to\infty}\left((j-1)|\xi_N|^p+\frac{\pi^2}{6}\right)^{\frac{1}{p}}} \\
&\leq \sqrt{\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^2}\right)^{\frac{1}{p}}}\sqrt{|\xi_N|} \\
&= \sqrt{\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^2}\right)^{\frac{1}{p}}}\sqrt{\|x\|_\infty} \\
&= \sqrt{\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^2}\right)^{\frac{1}{p}}}\sqrt{\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k}\right)^{\frac{1}{p}}}.
\end{aligned}$$

Therefore,

$$\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k}\right)^{\frac{1}{p}} \leq \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^2}\right)^{\frac{1}{p}}.$$

(3) Case $n = 2s+1,\ s\in\mathbb{N},\ m=1$

Claim 3.4.

$$\|x\|_\infty = \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^n}\right)^{\frac{1}{p}} \tag{3.5}$$

By induction, first we show that

Claim 3.5.

$$\|x\|_\infty = \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^3}\right)^{\frac{1}{p}} \tag{3.6}$$

Indeed,

$$\begin{aligned}
\|x\|_\infty &= \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^2}\right)^{\frac{1}{p}}\\
&= \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^{\frac{p}{2}}}{k^{\frac{3}{2}}}\frac{|\xi_k|^{\frac{p}{2}}}{k^{\frac{1}{2}}}\right)^{\frac{1}{p}}\\
&\leq \lim_{p\to\infty}\left(\sqrt{\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^3}}\sqrt{\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k}}\right)^{\frac{1}{p}}\\
&= \sqrt{\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^3}\right)^{\frac{1}{p}}}\sqrt{\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k}\right)^{\frac{1}{p}}}\\
&= \sqrt{\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^3}\right)^{\frac{1}{p}}}\sqrt{\|x\|_\infty}
\end{aligned}$$

and so

$$\|x\|_\infty \leq \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^3}\right)^{\frac{1}{p}}.$$

Furthermore, it is clear that

$$\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^3}\right)^{\frac{1}{p}} \leq \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k}\right)^{\frac{1}{p}} = \sqrt{\|x\|_\infty}.$$

Now, fix $t\in\mathbb{N}$ and assume that for $s\leq t$,

$$\sqrt{\|x\|_\infty} = \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^{2s+1}}\right)^{\frac{1}{p}}.$$

Then, we claim that

Claim 3.6.

$$\sqrt{\|x\|_\infty} = \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^{2s+3}}\right)^{\frac{1}{p}}.$$

Indeed,

$$\begin{aligned}
\|x\|_\infty &= \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^{2s+1}}\right)^{\frac{1}{p}}\\
&= \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^{\frac{p}{2}}}{k^{\frac{2s+3}{2}}}\frac{|\xi_k|^{\frac{p}{2}}}{k^{\frac{2s-1}{2}}}\right)^{\frac{1}{p}}\\
&\leq \lim_{p\to\infty}\left(\sqrt{\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^{2s+3}}}\sqrt{\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^{2s-1}}}\right)^{\frac{1}{p}}\\
&= \sqrt{\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^{2s+3}}\right)^{\frac{1}{p}}}\sqrt{\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k}\right)^{\frac{1}{p}}}\\
&= \sqrt{\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^{2s+3}}\right)^{\frac{1}{p}}}\sqrt{\|x\|_\infty}
\end{aligned}$$

and so

$$\|x\|_\infty \leq \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^{2s+3}}\right)^{\frac{1}{p}}.$$

Furthermore, it is clear that

$$\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^{2s+3}}\right)^{\frac{1}{p}} \leq \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k}\right)^{\frac{1}{p}} = \sqrt{\|x\|_\infty}.$$

Therefore, proof of this case is done.

(4) Case $n = 2s,\ s \in \mathbb{N},\ m = 1$

Proof of this case is similiar to the previous case.

In conclusion, we can say that proof of the theorem is complete since for any $m \in \mathbb{N}$ the rest (proof of the equation 3.2) is just application of the limit rules. □

Remark 3.7. Using the ideas above, we could extend examples and also check the connections between $\|x\|_\infty$ and

$$\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^k}\right)^{\frac{1}{p}} \text{ or } \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}a_k|\xi_k|^p\right)^{\frac{1}{p}} \text{ where } a_k \downarrow_k 0$$

for $x = (\xi_k)_k \in c_0$ but we like to leave those to readers and researchers due to keeping on our focus of research here.

4. An equivalent norm such that c_0 does not contain an asymptotically isometric copy of c_0 with that norm.

For $x = (\xi_k)_k \in c_0$ and for $\alpha \in \mathbb{R}$, define $\|x\|$ by

$$\begin{aligned}
\|x\| &= \frac{1}{\gamma_1} \lim_{p\to\infty} \sup_{k\in\mathbb{N}} \gamma_k \left(\sum_{j=k}^{\infty} \frac{|\xi_j|^p}{j} \right)^{\frac{1}{p}} + \gamma_1 \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k \left| {\xi^\star}_k - \alpha {\xi^\star}_j \right| \\
&\quad + \gamma_1 \sqrt{\sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} {Q_k}^2 \left|\xi_k - \alpha \xi_j\right|^2} \text{ where } \gamma_k \uparrow_k 1,\ \gamma_{k+1} > \gamma_k,\ \forall k \in \mathbb{N}, \\
&\quad x^\star := ({\xi^\star}_j)_{j\in\mathbb{N}} \text{ is the decreasing rearrangement of } x, \\
&\quad \sum_{k=1}^{\infty} Q_k = 1,\ Q_k \downarrow_k 0 \text{ and } Q_k > Q_{k+1},\ \forall k \in \mathbb{N}
\end{aligned}$$

such that from the definition of decreasing rearrangement, $\exists$ a 1-1 mapping $\pi : \mathbb{N} \longrightarrow \mathbb{N}$ and $\exists (\varepsilon_j)_{j\in\mathbb{N}}$ s.t. each $\varepsilon_{\pi(j)} \in \{-1, 1\}$ and then $(\xi^\star)_k = |\xi_{\pi(k)}| = \varepsilon_{\pi(k)}\, \xi_{\pi(k)}, \forall k \in \mathbb{N}$.

Then, using similiar ideas in [8] and due to the theorem above, clearly we can see $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$; furthermore, we give the following theorem. Note that the result below is important since we know that if a Banach space contains an asymptotically isometric copy of c_0 then it fails the FPP(n.e.). Thus, it is our initial step to say our equivalent norm is a good candidate to work on c_0 analogue of P.K. Lin's theorem; now that let's see our first theorem.

Theorem 4.1. *If $|\alpha| > 1$ and $Q_1 > \frac{1-\gamma_1+4|\alpha|}{1+4|\alpha|}$, then $(c_0,\ \|\cdot\|)$ does not contain an asymptotically isometric copy of c_0.*

Proof. By contradiction, assume $(c_0,\ \|\cdot\|)$ does contain an asymptotically isometric copy of c_0. That is, there exists a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in c_0 such that

$$\heartsuit \left[\begin{array}{l} \text{for every } n \in \mathbb{N} \text{ and every choice of scalars } t_1, t_2, \ldots, t_n, \\ \text{it follows that } \max\limits_{1\le k\le n} (1-\varepsilon_k)\,|t_k| \le \left\| \sum\limits_{k=1}^{n} t_k x_k \right\| \le \max\limits_{1\le k\le n} |t_k|\,. \end{array} \right]$$

Let $|\alpha| > 1$ and $Q_1 > \frac{1-\gamma_1+4|\alpha|}{1+4|\alpha|}$, then $Q_1 > \frac{2|\alpha|}{1+2|\alpha|}$.

Hence, $\frac{1}{\gamma_1}2|\alpha| > 2 > 1$, $|2\alpha - 1| > |\alpha|$ and we can assume for the following equivalent norm $\|.\|^{\sim}$, $(c_0,\ \|.\|^{\sim})$ contains an asymptotically isometric copy of c_0:

$$\begin{aligned}\|x\|^{\sim} &= \frac{1}{\gamma_1}\lim_{p\to\infty}\sup_{k\in\mathbb{N}}\gamma_k\left(\sum_{j=k}^{\infty}\frac{\left|\xi_j\right|^p}{j}\right)^{\frac{1}{p}} + \gamma_1\sup_{j\in\mathbb{N}}\sum_{k=1}^{\infty}Q_k\left|{\xi^\star}_k - \frac{2\alpha}{\gamma_1}{\xi^\star}_j\right| \\ &\quad +\gamma_1\sqrt{\sup_{j\in\mathbb{N}}\sum_{k=1}^{\infty}{Q_k}^2\left|\xi_k - \frac{2\alpha}{\gamma_1}\xi_j\right|^2}\ \text{ where } \gamma_k \uparrow_k 1,\ \gamma_{k+1} > \gamma_k,\ \forall k\in\mathbb{N}, \\ &\quad x^\star := ({\xi^\star}_j)_{j\in\mathbb{N}} \text{ is the decreasing rearrangement of } x, \\ &\quad \sum_{k=1}^{\infty}Q_k = 1,\ Q_k\downarrow_k 0 \text{ and } Q_k > Q_{k+1},\ \forall k\in\mathbb{N}\end{aligned}$$

Without loss of generality we can assume that the sequence $(x_n)_n$ converges pointwise to 0.

For each $n \in \mathbb{N}$, let $x_n = (\xi_j^n)_j$.

Note that, for every $x \in c_0$, there exists $L > 1$ such that $\|x\|_\infty \geq \left\|\frac{x}{L}\right\|^{\sim}$. Now, without loss of generality, by passing to a subsequence if necessary, we may assume there exists $s \in \mathbb{N}$ such that $\|x_s\|_\infty > \frac{1}{|2\alpha-\gamma_1|}$. We can do this since for $L > 1$, the sequence $(x_n)_n$ can be replaced with $(\frac{x_n}{L})_n$ so that the condition ♡ respect to newly defined norm is satisfied for null sequence $(\varepsilon_n)_n$ in $(0, 1)$ and so there exists $s \in \mathbb{N}$ such that $\varepsilon_s < 1 - \frac{1}{|2\alpha-\gamma_1|}$ and $\|x_s\|_\infty \geq \left\|\frac{x_s}{L}\right\|^{\sim} > 1 - \varepsilon_s > \frac{1}{|2\alpha-\gamma_1|}$.

Now , there exists $r \in \mathbb{N}$ s.t. $\xi_r^s \neq 0$ and, as previously, since $x_s \in c_0$, there exists $N^{(s)} \in \mathbb{N}$ such that $\|x_s\|_\infty = |\xi^s_{N^{(s)}}| \geq |\xi_r^s|$. Hence, take $p = \min\{r \mid |\xi_r^s| = |\xi_{N^{(s)}}|\}$.

Now, let $\delta = \left(Q_1 - \frac{1-\gamma_1+4|\alpha|}{1+4|\alpha|}\right)\frac{8|\alpha|(1+4|\alpha|)}{16|\alpha|^3+\left(68+\frac{4}{\gamma_1}\right)|\alpha|^2+\left(16+\frac{9}{\gamma_1}+8\gamma_1\right)|\alpha|+\frac{2}{\gamma_1}}$.

Now, choose $N_1 \geq p$ so that $\sum\limits_{k=1+N_1}^{\infty} Q_k < (\frac{4}{\gamma_1} + 4|\alpha|)\frac{\delta}{2}$. Choose $N_2 \in \mathbb{N}$ so that $\varepsilon_n < \min\left\{1 - \frac{1}{|2\alpha-\gamma_1|}\ ,\ \delta\right\}$ for all $n \geq \max\{s, N_2\}$. Choose $M \geq \max\{s, N_2\}$ so that $|2\alpha - \gamma_1|\left|\xi_j^n\right| < \frac{\left(\frac{1}{\gamma_1}+4\alpha\right)\delta}{8}$ and $\left|\xi_j^n\right| < \frac{\left(\frac{1}{\gamma_1}+4|\alpha|\right)\delta}{8|\alpha|}$ for $j = 1, 2, \ldots, N_1$ and for all $n \geq M$. Note that $1 \geq \|x_s\|^{\sim}$ and $1 \geq \|x_n\|^{\sim}$ and so $1 \geq \left|\xi_j^s\right|$ and $1 \geq \left|\xi_j^n\right|$ for all $j \in \mathbb{N}$.

Therefore, for each $n \geq M$,

$$\begin{aligned}
\|x_n\|_\infty &\leq \|x_n\|^\sim \\
&\leq \frac{1}{\gamma_1}\|x_n\|_\infty + \gamma_1 \sum_{k=1}^{\infty} Q_k \left|{\xi^\star}_k^n\right| + 2|\alpha| \sum_{k=1}^{\infty} Q_k \sup_{j\in\mathbb{N}} \left|{\xi^\star}_j^n\right| \\
&\quad + \gamma_1 \sqrt{\sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} {Q_k}^2 \left(|\xi_k^n| + \frac{2\alpha}{\gamma_1}|\xi_j^n|\right)^2} \\
&\leq \left(\frac{1}{\gamma_1} + 4|\alpha|\right)\|x_n\|_\infty + 2\gamma_1 \sum_{k=1}^{\infty} Q_k |\xi_k^n| \\
&\leq \left(\frac{1}{\gamma_1} + 4|\alpha|\right)\|x_n\|_\infty + 2\gamma_1 \sum_{k=1}^{N_1} Q_k |\xi_k^n| + 2\gamma_1 \sum_{k=1+N_1}^{\infty} Q_k |\xi_k^n| \\
&< \left(\frac{1}{\gamma_1} + 4|\alpha|\right)\|x_n\|_\infty + \frac{\frac{1}{\gamma_1}\left(\frac{1}{\gamma_1} + 4|\alpha|\right)\delta}{4|\alpha|} \sum_{k=1}^{N_1} Q_k + 2 \sum_{k=1+N_1}^{\infty} Q_k \\
&< \left(\frac{1}{\gamma_1} + 4|\alpha|\right)\|x_n\|_\infty + \frac{\frac{1}{\gamma_1}\left(\frac{1}{\gamma_1} + 4|\alpha|\right)\delta}{4|\alpha|} + \left(\frac{1}{\gamma_1} + 4|\alpha|\right)\delta \\
&< \left(\frac{1}{\gamma_1} + 4|\alpha|\right)\|x_n\|_\infty + \frac{\left(\frac{1}{\gamma_1} + 4|\alpha|\right)^2\delta}{4|\alpha|}.
\end{aligned}$$

By the triangle inequality $\|x_n\|_\infty \leq \frac{1}{2}\|x_n + x_s\|_\infty + \frac{1}{2}\|x_n - x_s\|_\infty$ and so either $\|x_n + x_s\|_\infty \geq \|x_n\|_\infty$ or $\|x_n - x_s\|_\infty \geq \|x_n\|_\infty$.

If $\|x_n + x_s\|_\infty \geq \|x_n\|_\infty$ then we have

$$\begin{aligned}
1 = \max\{1,1\} &\geq \|x_s + x_n\|^\sim \\
&\geq \|x_s + x_n\|_\infty + \gamma_1 \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k \left|(\xi_k^s + \xi_k^n)^\star - \frac{2\alpha}{\gamma_1}\left(\xi_j^s + \xi_j^n\right)^\star\right| \\
&\quad + \gamma_1 \sqrt{\sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} {Q_k}^2 \left|\xi_k^s + \xi_k^n - \frac{2\alpha}{\gamma_1}\left(\xi_j^s + \xi_j^n\right)\right|^2} \\
&\geq \|x_s + x_n\|_\infty + \gamma_1 \sum_{k=1}^{\infty} Q_k \left|(\xi_k^s + \xi_k^n)^\star - \frac{2\alpha}{\gamma_1}(\xi_1^s + \xi_1^n)^\star\right|.
\end{aligned}$$

Hence,

$$\begin{aligned}
1 &\geq \|x_s + x_n\|_\infty + Q_1 \left|2\alpha - \gamma_1\right| (\xi_1^s + \xi_1^n)^\star \\
&\geq \|x_n\|_\infty + Q_1 \left|2\alpha - \gamma_1\right| \left|\xi_p^s + \xi_p^n\right| \\
&\geq \|x_n\|_\infty + Q_1 \left|2\alpha - \gamma_1\right| \left|\xi_p^s\right| - Q_1 \left|2\alpha - \gamma_1\right| \left|\xi_p^n\right| \\
&> \frac{\gamma_1}{1 + 4\gamma_1|\alpha|} \|x_n\|^\sim - \frac{\left(\frac{1}{\gamma_1} + 4|\alpha|\right)\delta}{4|\alpha|} \\
&\quad + Q_1 \left|2\alpha - \gamma_1\right| \left|\xi_p^s\right| - Q_1 \left|2\alpha - \gamma_1\right| \left|\xi_p^n\right| \\
&> \frac{\gamma_1}{1 + 4|\alpha|} \|x_n\|^\sim - \frac{\left(\frac{1}{\gamma_1} + 4|\alpha|\right)\delta}{4|\alpha|} \\
&\quad + Q_1 - \left|2\alpha - \gamma_1\right| \left|\xi_p^n\right| \\
&> \frac{\gamma_1}{1 + 4|\alpha|}(1 - \varepsilon_n) - \frac{\left(\frac{1}{\gamma_1} + 4|\alpha|\right)\delta}{4|\alpha|} + Q_1 - \frac{\left(\frac{1}{\gamma_1} + 4|\alpha|\right)\delta}{8} \\
&> \frac{\gamma_1}{1 + 4|\alpha|}(1 - \delta) - \frac{\left(\frac{1}{\gamma_1} + 4|\alpha|\right)\delta}{4|\alpha|} + Q_1 - \frac{\left(\frac{1}{\gamma_1} + 4|\alpha|\right)\delta}{8} \\
&= 1 + Q_1 - \frac{1 - \gamma_1 + 4|\alpha|}{1 + 4|\alpha|} \\
&\quad -\delta \left(\frac{16|\alpha|^3 + \left(36 + \frac{4}{\gamma_1}\right)|\alpha|^2 + \left(8 + \frac{9}{\gamma_1} + 8\gamma_1\right)|\alpha| + \frac{2}{\gamma_1}}{8|\alpha|(1 + 4|\alpha|)} \right) \\
&> 1 + \delta \\
&\quad \text{which is not possible (contradiction).}
\end{aligned}$$

Similarly we arrive at a contradiction if we assume that $\|x_n - x_s\|_\infty \geq \|x_n\|_\infty$. □

Corollary 4.2. *For $x = (\xi_k)_k \in c_0$ and for $\alpha \in \mathbb{R}$, define $\|x\|^\sim$ by*

$$\begin{aligned}
\|x\|^\sim &= \frac{1}{\gamma_1} \lim_{p\to\infty} \sup_{k\in\mathbb{N}} \gamma_k \left(\sum_{j=k}^{\infty} \frac{\left|\xi_j\right|^p}{j} \right)^{\frac{1}{p}} + \gamma_1 \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k \left|\xi^\star{}_k - \alpha \xi^\star{}_j\right| \\
&\quad \textit{where } \gamma_k \uparrow_k 1,\ \gamma_{k+1} > \gamma_k,\ \forall k \in \mathbb{N}, \\
&\quad x^\star := (\xi^\star{}_j)_{j\in\mathbb{N}} \textit{ is the decreasing rearrangement of } x, \\
&\quad \sum_{k=1}^{\infty} Q_k = 1,\ Q_k \downarrow_k 0 \textit{ and } Q_k > Q_{k+1},\ \forall k \in \mathbb{N}
\end{aligned}$$

Clearly we can see $\|\cdot\|^{\sim}$ is equivalent to $\|\cdot\|_{\infty}$; moreover, if $|\alpha|>1$ and $Q_1>\frac{1-\gamma_1+2|\alpha|}{1+2|\alpha|}$, then $(c_0,\ \|\cdot\|^{\sim})$ does not contain an asymptotically isometric copy of c_0.

5. Main Result: A Set in c_0 having FPP(for n.e. and affine mappings) for an equivalent norm

Example 5.1. Fix $b\in(0,1)$. We define the sequence $(f_n)_{n\in\mathbb{N}}$ in c_0 by setting $f_1:=b\,e_1$, $f_2:=b\,e_2$, and $f_n:=e_n$, for all integers $n\geq 3$. Next, define the closed, bounded, convex subset $E=E_b$ of c_0 by

$$E:=\left\{\sum_{n=1}^{\infty} t_n\,f_n : 1=t_1\geq t_2\geq\cdots\geq t_n\downarrow_n 0\right\}\ .$$

Let us define the sequence $(\eta_n)_{n\in\mathbb{N}}$ in E in the following way. Let $\eta_1:=f_1$ and $\eta_n:=f_1+\cdots+f_n$, for all integers $n\geq 2$. It is straightforward to check that

$$E:=\left\{\sum_{n=1}^{\infty}\alpha_n\,\eta_n:\ \text{each }\alpha_n\geq 0\text{ and }\sum_{n=1}^{\infty}\alpha_n=1\right\}\ .$$

Then, in 2011, Lennard and Nezir [6] show that $E=E_b$ is the closed convex hull of $(\eta_n)_{n\in\mathbb{N}}$ which is an asymptotically isometric c_0-summing basis respect to $\|\cdot\|_\infty$ and that there exists an affine $\|\cdot\|_\infty$-nonexpansive mapping $U:E\longrightarrow E$ that is fixed point free.

Theorem 5.2. *Fix $b\in(0,1)$ and define the closed, bounded, convex subset $E=E_b$ of c_0 as in the example above. Define the equivalent norm $\|.\|$ on c_0 as below:*

For $x=(\xi_k)_k\in c_0$ and $\alpha>0$,

$$\begin{aligned}\|x\| &= \frac{1}{\gamma_1}\lim_{p\to\infty}\sup_{k\in\mathbb{N}}\gamma_k\left(\sum_{j=k}^{\infty}\frac{|\xi_j|^p}{j}\right)^{\frac{1}{p}}+\gamma_1\sup_{j\in\mathbb{N}}\sum_{k=1}^{\infty}Q_k\left|{\xi^\star}_k-\alpha{\xi^\star}_j\right|\\ &\quad+\gamma_1\sqrt{\sup_{j\in\mathbb{N}}\sum_{k=1}^{\infty}{Q_k}^2\left|\xi_k-\alpha\xi_j\right|^2}\ \ \text{where } \gamma_k\uparrow_k 1,\ \gamma_{k+2}>\gamma_{k+1},\ \forall k\in\mathbb{N},\\ &\quad x^\star:=({\xi^\star}_j)_{j\in\mathbb{N}} \text{ is the decreasing rearrangement of } x,\\ &\quad\sum_{k=1}^{\infty}Q_k=1,\ Q_k\downarrow_k 0\ \ Q_k>Q_{k+1},\ \forall k\in\mathbb{N} \text{ and } Q_1>\frac{1-\gamma_1+4|\alpha|}{1+4|\alpha|}.\end{aligned}$$

Now, let $\gamma_2=\gamma_1$. Then, there exist constants $0<C\leq 1$ and $\alpha\geq\frac{1}{2}$ such that for all $b\in(0,C)$ the set E defined as in the example above has the fixed point property for $\|.\|$-nonexpansive affine mappings.

Proof. Let $T:E\to E$ be an affine nonexpansive mapping. Then, there exists a sequence $\left(x^{(n)}\right)_{n\in\mathbb{N}}\in E$ such that $\left\|Tx^{(n)}-x^{(n)}\right\|\underset{n}{\longrightarrow}0$ and so

$\left\|Tx^{(n)} - x^{(n)}\right\|_\infty \underset{n}{\longrightarrow} 0$. Without loss of generality, passing to a subsequence if necessary, there exists $z \in c_0$ such that $x^{(n)}$ converges to z in weak topology. Then, by lemma 2.6, there exists a further subsequence $\left(x^{(n_k)}\right)_{k\in\mathbb{N}}$ such that we get a function $s : c_0 \to [0,\infty)$ given by $\forall y \in c_0$,

$$s(y) = \limsup_m \left\| \frac{1}{m}\sum_{k=1}^m x^{(k)} - y \right\| \text{. Then, } s(y) = s(z) + \|y - z\|,\ \forall y \in c_0.$$

Now, define $W := \overline{E}^{\sigma(l^\infty,\, l^1)} = \{\sum_{n=1}^\infty \alpha_n \eta_n :\ \text{each } \alpha_n \geq 0 \text{ and } \sum_{n=1}^\infty \alpha_n \leq 1\}$

Case 1: $z \in E$.

Then, we have $s(Tz) = s(z) + \|Tz - z\|$.

Also,

$$\begin{aligned} s(Tz) &= \limsup_m \left\| Tz - \frac{1}{m}\sum_{k=1}^m x^{(k)} \right\| \\ &\leq \limsup_m \left\| Tz - T\left(\frac{1}{m}\sum_{k=1}^m x^{(k)}\right) \right\| + \limsup_m \left\| \begin{array}{c} \frac{1}{m}\sum_{k=1}^m x^{(k)} \\ - \\ T\left(\frac{1}{m}\sum_{k=1}^m x^{(k)}\right) \end{array} \right\| \end{aligned}$$

Thus, since T is affine

$$\begin{aligned} s(Tz) &\leq \limsup_m \left\| Tz - T\left(\frac{1}{m}\sum_{k=1}^m x^{(k)}\right) \right\| + \limsup_m \left\| \begin{array}{c} \frac{1}{m}\sum_{k=1}^m x^{(k)} \\ - \\ \frac{1}{m}\sum_{k=1}^m Tx^{(k)} \end{array} \right\| \\ &\leq \limsup_m \left\| z - \frac{1}{m}\sum_{k=1}^m x^{(k)} \right\| \\ &= s(z). \end{aligned}$$

Therefore, $\|z - Tz\| \leq 0$ and so $Tz = z$.

Case 2: $z \in W \setminus E$.

Then, z is of the form $\sum_{n=1}^\infty \sigma_n \eta_n$ such that $\sum_{n=1}^\infty \sigma_n < 1$.

$$\text{Define } \delta := 1 - \sum_{n=1}^\infty \sigma_n \text{ and next define}$$

$$h_\lambda := (\sigma_1 + \lambda\delta)\eta_1 + (\sigma_2 + (1-\lambda)\delta)\eta_2 + \sum_{n=3}^\infty \sigma_n \eta_n.$$

We want h_λ to be in E, so we restrict values of λ to be in $\left[-\frac{\sigma_1}{\delta},\ \frac{\sigma_2}{\delta} + 1\right]$, then

$$\begin{aligned}
\|h_\lambda - z\| &= \frac{1}{\gamma_1} \lim_{p\to\infty} \max\left\{ \gamma_1 \left((b\delta)^p + \frac{[|1-\lambda|b\delta]^p}{2} \right)^{\frac{1}{p}} ,\ \gamma_2 |1-\lambda| b\delta \right\} \\
&\quad + \gamma_1 \max \left\{ \begin{array}{l} Q_1|1-\alpha|b\delta + Q_2|(1-\lambda)b\delta - \alpha b\delta| \\ +(1-Q_1-Q_2)\alpha b\delta, \\ Q_1|b\delta - \alpha(1-\lambda)b\delta| + Q_2|1-\alpha||1-\lambda|b\delta \\ +(1-Q_1-Q_2)\alpha|1-\lambda|b\delta, \\ Q_1 b\delta + Q_2|1-\lambda|b\delta \end{array} \right\} \\
&\quad + \gamma_1 \max \left\{ \begin{array}{l} \sqrt{\begin{array}{l} {Q_1}^2|1-\alpha|^2 b^2\delta^2 + {Q_2}^2|(1-\lambda)b\delta - \alpha b\delta|^2 \\ + \sum_{k=3}^\infty {Q_k}^2 \alpha^2 b^2 \delta^2 \end{array}}, \\ \sqrt{\begin{array}{l} {Q_1}^2|b\delta - \alpha(1-\lambda)b\delta|^2 + {Q_2}^2|1-\alpha|^2|1-\lambda|^2 b^2\delta^2 \\ + \sum_{k=3}^\infty {Q_k}^2 \alpha^2 |1-\lambda|^2 b^2\delta^2 \end{array}}, \\ \sqrt{ {Q_1}^2 b^2\delta^2 + {Q_2}^2|1-\lambda|^2 b^2\delta^2 } \end{array} \right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|h_\lambda - z\| &= \frac{1}{\gamma_1} \max\left\{ \gamma_1 |1-\lambda| b\delta ,\ \gamma_2 |1-\lambda| b\delta \right\} \\
&\quad + \gamma_1 \max \left\{ \begin{array}{l} Q_1|1-\alpha|b\delta + Q_2|(1-\lambda)b\delta - \alpha b\delta| \\ +(1-Q_1-Q_2)\alpha b\delta, \\ Q_1|b\delta - \alpha(1-\lambda)b\delta| + Q_2|1-\alpha||1-\lambda|b\delta \\ +(1-Q_1-Q_2)\alpha|1-\lambda|b\delta, \\ Q_1 b\delta + Q_2|1-\lambda|b\delta \end{array} \right\} \\
&\quad + \gamma_1 \max \left\{ \begin{array}{l} \sqrt{\begin{array}{l} {Q_1}^2|1-\alpha|^2 b^2\delta^2 + {Q_2}^2|(1-\lambda)b\delta - \alpha b\delta|^2 \\ + \sum_{k=3}^\infty {Q_k}^2 \alpha^2 b^2 \delta^2 \end{array}}, \\ \sqrt{\begin{array}{l} {Q_1}^2|b\delta - \alpha(1-\lambda)b\delta|^2 + {Q_2}^2|1-\alpha|^2|1-\lambda|^2 b^2\delta^2 \\ + \sum_{k=3}^\infty {Q_k}^2 \alpha^2 |1-\lambda|^2 b^2\delta^2 \end{array}}, \\ \sqrt{ {Q_1}^2 b^2\delta^2 + {Q_2}^2|1-\lambda|^2 b^2\delta^2 } \end{array} \right\}.
\end{aligned}$$

Define

$$\Gamma := \min_{\lambda \in \left[-\frac{\sigma_1}{\delta},\ \frac{\sigma_2}{\delta}+1\right]} \|h_\lambda - z\|.$$

Hence (ignoring some cases for α),

$$\Gamma = \begin{cases} b\delta \max\left\{1\ ,\ \frac{\gamma_2}{\gamma_1}\right\} + \gamma_1 Q_1 \delta b \\ +\gamma_1 Q_2 \delta b + \gamma_1 \delta b \sqrt{{Q_1}^2 + {Q_2}^2} & \text{if } \lambda \in \left[-\frac{\sigma_1}{\delta},\ 0\right) \text{ and } \frac{1}{2} \le \alpha \le 1 \\ b\delta \max\left\{1\ ,\ \frac{\gamma_2}{\gamma_1}\right\} \\ +\gamma_1(\alpha - Q_1 - Q_2)\delta b \\ +\gamma_1 \delta b \sqrt{\begin{array}{l} ({Q_1}^2 + {Q_2}^2)(\alpha-1)^2 \\ + \sum_{k=3}^{\infty} {Q_k}^2 \alpha^2 \end{array}} & \text{if } \lambda \in \left[-\frac{\sigma_1}{\delta},\ 0\right) \text{ and } \alpha > 1 \\ 2\gamma_1 Q_1 \delta b & \text{if } \lambda \in [0,\ 1] \text{ and } \alpha \le 2{Q_1}^2 + 2{Q_2}^2(1-\lambda) \\ b\delta \max\left\{1\ ,\ \frac{\gamma_2}{\gamma_1}\right\} \\ +\gamma_1(\alpha - Q_1 - Q_2)\delta b \\ +\gamma_1 \delta b \sqrt{\begin{array}{l} ({Q_1}^2 + {Q_2}^2)(\alpha-1)^2 \\ + \sum_{k=3}^{\infty} {Q_k}^2 \alpha^2 \end{array}} & \text{if } \lambda \in [0,\ 1] \text{ and } \alpha > 2Q_1 + 2Q_2(1-\lambda) \\ 2\gamma_1 Q_1 \delta b & \text{if } \lambda \in \left(1,\ \frac{\sigma_2}{\delta}+1\right] \text{ and } \alpha \le 2{Q_1}^2 \\ \gamma_1(\alpha - Q_1)\delta b \\ +\gamma_1 \delta b \sqrt{\begin{array}{l} {Q_1}^2(\alpha-1)^2 + {Q_2}^2\alpha^2 \\ + \sum_{k=3}^{\infty} {Q_k}^2 \alpha^2 \end{array}} & \text{if } \lambda \in \left(1,\ \frac{\sigma_2}{\delta}+1\right] \text{ and } \alpha > 2Q_1 \end{cases}$$

Thus,

$$\Gamma \le \Gamma^{\sim}$$

where

$$\Gamma^{\sim} = \begin{cases} b\delta + 2\gamma_1 Q_1 \delta b + 2\gamma_1 Q_2 \delta b & \text{if } \lambda \in \left[-\frac{\sigma_1}{\delta},\ 0\right) \text{ and } \frac{1}{2} \le \alpha \le 1 \\ b\delta + 2\gamma_1(\alpha - Q_1 - Q_2)\delta b & \text{if } \lambda \in \left[-\frac{\sigma_1}{\delta},\ 0\right) \text{ and } \alpha > 1 \\ 2\gamma_1 Q_1 \delta b & \text{if } \lambda \in [0,\ 1] \text{ and } \alpha \le 2{Q_1}^2 + 2{Q_2}^2(1-\lambda) \\ b\delta + 2\gamma_1(\alpha - Q_1 - Q_2)\delta b & \text{if } \lambda \in [0,\ 1] \text{ and } \alpha > 2Q_1 + 2Q_2(1-\lambda) \\ 2\gamma_1 Q_1 \delta b & \text{if } \lambda \in \left(1,\ \frac{\sigma_2}{\delta}+1\right] \text{ and } \alpha \le 2{Q_1}^2 \\ 2\gamma_1(\alpha - Q_1)\delta b & \text{if } \lambda \in \left(1,\ \frac{\sigma_2}{\delta}+1\right] \text{ and } \alpha > 2Q_1 \end{cases}$$

Note that, we can conclude that if $\alpha \ge 2$ while $\lambda \in \left[-\frac{\sigma_1}{\delta},\ 1\right]$ then $\|h_\lambda - z\|$ is minimized with unique minimizer such that its minimum value would be less than or equal to $b\delta + 2\gamma_1(\alpha - Q_1 - Q_2)\delta b$; on the other hand, if $2{Q_1}^2 < \alpha \le 2{Q_1}^2 + 2{Q_2}^2(1-\lambda)$ while $\lambda \in [0,\ 1]$ then $\|h_\lambda - z\|$ is minimized with unique minimizer such that its minimum value would be $2\gamma_1 Q_1 \delta b$. Therefore, firstly defining $\|x\|_{(j)}$ by

$$\begin{aligned} \|x\|_{(j)} : &= \frac{1}{\gamma_1} \lim_{p\to\infty} \sup_{k\in\mathbb{N}} \gamma_k \left(\sum_{j=k}^{\infty} \frac{\left|\xi_j\right|^p}{j} \right)^{\frac{1}{p}} + \gamma_1 \sum_{k=1}^{\infty} Q_k \left|{\xi^\star}_k - \alpha {\xi^\star}_j\right| \\ &\quad + \gamma_1 \sqrt{\sum_{k=1}^{\infty} {Q_k}^2 \left|\xi_k - \alpha \xi_j\right|^2} \end{aligned}$$

and next taking the following properties into consideration:

- $\sqrt{\sum_{k=1}^{\infty} {Q_k}^2 \left|\xi_k - \alpha\xi_j\right|^2} \geq \sum_{k=1}^{\infty} {Q_k}^{\frac{3}{2}} \left|\xi_k - \alpha\xi_j\right|,$
- if $x^\star = ({\xi^\star}_k)_{k\in\mathbb{N}}$ is the rearrangement of the sequence $x = (\xi_k)_{k\in\mathbb{N}}$, $|\alpha{\xi^\star}_j - {\xi^\star}_k| \geq (\alpha\xi_j - \xi_k)^\star \geq |\alpha\xi_j - \xi_k|$ for any $k, j \in \mathbb{N}$

then we have the following cases:

Subcase 2.1: Consider $\gamma_1 + \gamma_2 - t_1 - t_2 \geq 0$.

Now, (for now, if we consider $b \in (0,1)$) fix $y \in E$ of the form $\sum_{n=1}^{\infty} t_n\eta_n$ such that $\sum_{n=1}^{\infty} t_n = 1$ with $t_n \geq 0$, $\forall n \in \mathbb{N}$.

Then, using the definition of the equivalent norm

$$
\begin{aligned}
\|y - z\| &\geq \|y - z\|_{(1)} \\
&= \left\| \begin{pmatrix} (t_1 - \sigma_1)b + (t_2 - \sigma_2)b + (t_3 - \sigma_3) + (t_4 - \sigma_4) + \dots, \\ (t_2 - \sigma_2)b + (t_3 - \sigma_3) + (t_4 - \sigma_4) + \dots, \\ (t_3 - \sigma_3) + (t_4 - \sigma_4) + \dots, \\ (t_4 - \sigma_4) + (t_5 - \sigma_5) + \dots, \\ (t_5 - \sigma_5) + (t_6 - \sigma_6) + \dots, \dots \end{pmatrix} \right\|_{(1)} \\
&\geq |\sigma_1 + \sigma_2 + \delta - t_1 - t_2 + (t_1 - \sigma_1)b + (t_2 - \sigma_2)b| \\
&\quad + \gamma_1 \left|(Q_1 - \alpha)\left\{(1-b)\left[\sigma_1 + \sigma_2 + \delta - t_1 - t_2\right] + b\delta\right\}\right| - \gamma_1 Q_2 b \left|t_2 - \sigma_2\right| \\
&\quad - \gamma_1 Q_2 \sum_{j=3}^{\infty} |t_j - \sigma_j| - \gamma_1 Q_3 \sum_{j=3}^{\infty} |t_j - \sigma_j| - \gamma_1 Q_4 \sum_{j=4}^{\infty} |t_j - \sigma_j| - \dots \\
&\geq |\sigma_1 + \sigma_2 + \delta - t_1 - t_2 + (t_1 - \sigma_1)b + (t_2 - \sigma_2)b| \\
&\quad + \gamma_1 \left|(Q_1 - \alpha)\left\{(1-b)\left[\sigma_1 + \sigma_2 + \delta - t_1 - t_2\right] + b\delta\right\}\right| \\
&\quad - \gamma_1 (1 - Q_1) \sum_{j=2}^{\infty} |t_j - \sigma_j|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|y - z\| &\geq \delta \left(1 + \frac{{\gamma_1}^2}{1 + 4\alpha} + \gamma_1 (\alpha - Q_1)(1 - b)\right) \\
&\quad + (\sigma_1 - t_1)(1 + \gamma_1(\alpha - Q_1))(1 - b) \\
&\quad + (\sigma_2 - t_2)(1 + \gamma_1(\alpha - Q_1))(1 - b) - \frac{{\gamma_1}^2}{1 + 4\alpha}(2 - \sigma_1 - t_1).
\end{aligned}
$$

Subcase 2.1.1: $\sigma_1 - t_1 \geq 0$ and $\sigma_2 - t_2 \geq 0$.

Note that since $1 > \delta > 0$, there exists $c > 1$ such that $\delta > \frac{1}{c}$ and then choose b so that $b < \frac{\gamma_1}{2(4\alpha^2 - 3\alpha)}$ and $\alpha > \frac{2c}{1-b}$ (so $\alpha > 2$). Then,

$$\begin{aligned}
\|y-z\| - \Gamma^{\sim} \geq\ & \delta\left(1 + \frac{{\gamma_1}^2}{1+4\alpha} + \gamma_1(\alpha - Q_1)(1-b) - b(1+2\gamma_1[\alpha - Q_1 - Q_2])\right) \\
& +(\sigma_1 - t_1)(1+\gamma_1(\alpha - Q_1))(1-b) \\
& +(\sigma_2 - t_2)(1+\gamma_1(\alpha - Q_1))(1-b) - \frac{{\gamma_1}^2}{1+4\alpha}(2 - \sigma_1 - t_1) \\
\geq\ & (\delta + \sigma_1 - t_1 + \sigma_2 - t_2){\gamma_1}^2(1+\alpha - Q_1)(1-b) \\
& - \frac{{\gamma_1}^2 c}{1+4\alpha}\frac{1}{c}(2 - \sigma_1 - t_1) \\
\geq\ & (\delta + \sigma_1 - t_1 + \sigma_2 - t_2){\gamma_1}^2(1+\alpha - Q_1)(1-b) \\
& - \frac{{\gamma_1}^2 c}{1+4\alpha}(\delta + \sigma_1 - t_1 + \sigma_2 - t_2)(2 - \sigma_1 - t_1) \\
\geq\ & (\delta + \sigma_1 - t_1 + \sigma_2 - t_2){\gamma_1}^2\left[(1+\alpha - Q_1)(1-b) - \frac{2c}{1+4\alpha}\right] \\
\geq\ & \frac{{\gamma_1}^2 \alpha}{1+4\alpha}(\delta + \sigma_1 - t_1 + \sigma_2 - t_2)[\alpha(1-b) - 2c] \\
\geq\ & 0.
\end{aligned}$$

Subcase 2.1.2: Consider $\sigma_1 - t_1 \geq 0$ and $\sigma_2 - t_2 \leq 0$.

Note that $\sigma_1 - t_1 + \sigma_2 - t_2 \geq 0$ yields $\sigma_1 - t_1 \geq t_2 - \sigma_2 \geq 0$ and note:

$$\begin{aligned}
\|y-z\| \geq\ & \|y-z\|_{(2)} \\
=\ & \left\| \begin{pmatrix} (t_1 - \sigma_1)b + (t_2 - \sigma_2)b + (t_3 - \sigma_3) + (t_4 - \sigma_4) + \ldots, \\ (t_2 - \sigma_2)b + (t_3 - \sigma_3) + (t_4 - \sigma_4) + \ldots, \\ (t_3 - \sigma_3) + (t_4 - \sigma_4) + \ldots, \\ (t_4 - \sigma_4) + (t_5 - \sigma_5) + \ldots, \ldots \end{pmatrix} \right\|_{(2)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|y-z\| \geq\ & |\sigma_1+\sigma_2+\delta-t_1-t_2+(t_1-\sigma_1)b+(t_2-\sigma_2)b| \\
&+\gamma_1 \left| \begin{array}{l} (Q_1-\alpha)(1-b)[\sigma_1+\sigma_2+\delta-t_1-t_2] \\ +(Q_1-\alpha)b\delta+(Q_2-\alpha)(t_2-\sigma_2)b-\alpha b[\sigma_1+\sigma_2-t_1-t_2] \end{array} \right| \\
&-\gamma_1 Q_2 \sum_{j=3}^{\infty} |t_j-\sigma_j| - \gamma_1 Q_3 \sum_{j=3}^{\infty} |t_j-\sigma_j| - \gamma_1 Q_4 \sum_{j=4}^{\infty} |t_j-\sigma_j| \\
&-\gamma_1 Q_5 \sum_{j=5}^{\infty} |t_j-\sigma_j| - \dots \\
\geq\ & \delta+(1-b)[(\sigma_1-t_1)+(\sigma_2-t_2)] \\
&+\gamma_1 \left(\begin{array}{l} (\alpha-Q_1)(1-b)[\sigma_1+\sigma_2+\delta-t_1-t_2] \\ +(\alpha-Q_1)b\delta+\alpha(t_2-\sigma_2)b \\ +\alpha b[\sigma_1+\sigma_2-t_1-t_2] \end{array} \right) \\
&-\frac{{\gamma_1}^2}{1+4\alpha}(2-\sigma_1-\delta-t_1) \\
\geq\ & \delta+(1-b)[(\sigma_1-t_1)+(\sigma_2-t_2)] \\
&+\gamma_1 \left(\begin{array}{l} (\alpha-Q_1)(1-b)[\sigma_1+\sigma_2+\delta-t_1-t_2] \\ +(\alpha-Q_1)b\delta+\alpha(t_2-\sigma_2)b \end{array} \right) \\
&-\frac{{\gamma_1}^2}{1+4\alpha}(2-\sigma_1-\delta-t_1).
\end{aligned}$$

Now, similiarly to the subcase 2.1.1 there exists $c>1$ such that $\delta>\frac{1}{c}$ and then choose b so that $b<\frac{\gamma_1}{2(4\alpha^2-3\alpha)}$ and $\alpha>\frac{2c}{1-b}$ (so $\alpha>2$).

Thus,

$$\begin{aligned}
\|y-z\|-\Gamma^{\sim} \geq\ & \delta+(1-b)[(\sigma_1-t_1)+(\sigma_2-t_2)] \\
&+\gamma_1 \left(\begin{array}{l} (\alpha-Q_1)(1-b)[\sigma_1+\sigma_2+\delta-t_1-t_2] \\ +(\alpha-Q_1)b\delta+\alpha(t_2-\sigma_2)b \end{array} \right) \\
&-\frac{{\gamma_1}^2}{1+4\alpha}(2-\sigma_1-\delta-t_1)-(1+\alpha-Q_1-Q_2)b\delta \\
\geq\ & \delta\left(1+\frac{{\gamma_1}^2}{1+4\alpha}+\gamma_1(\alpha-Q_1)(1-b)-b(1+2\gamma_1[\alpha-Q_1-Q_2])\right) \\
&+(\sigma_1-t_1)(1+\gamma_1(\alpha-Q_1))(1-b) \\
&+(t_2-\sigma_2)[\gamma_1\alpha b-(1+\gamma_1(\alpha-Q_1))(1-b)]-\frac{{\gamma_1}^2}{1+4\alpha}(2-\sigma_1-t_1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|y-z\|-\Gamma^{\sim} &\geq \delta\left(1+\frac{{\gamma_1}^2}{1+4\alpha}+\gamma_1\left(\alpha-Q_1\right)(1-b)-b(1+2\gamma_1[\alpha-Q_1-Q_2])\right) \\
&\quad +(t_2-\sigma_2)\left(1+\gamma_1(\alpha-Q_1)\right)(1-b) \\
&\quad +(t_2-\sigma_2\left[\gamma_1\alpha b-\left(1+\gamma_1(\alpha-Q_1)\right)(1-b)\right]-\frac{{\gamma_1}^2}{1+4\alpha}\left(2-\sigma_1-t_1\right) \\
&\geq \delta\left(1+\gamma_1(\alpha-Q_1)\right)(1-b)-\frac{c{\gamma_1}^2}{1+4\alpha}\frac{1}{c}\left(2-\sigma_1-t_1\right) \\
&\geq \delta\left[{\gamma_1}^2\left(1+\alpha-Q_1\right)(1-b)-\frac{2{\gamma_1}^2c}{1+4\alpha}\right] \\
&\geq \frac{{\gamma_1}^2\alpha}{1+4\alpha}\delta\left[\alpha(1-b)-2c\right] \\
&\geq 0.
\end{aligned}$$

Subcase 2.1.3: Consider $\sigma_1-t_1\leq 0$ and $\sigma_2-t_2\geq 0$.

Then, since $\sigma_1-t_1+\sigma_2-t_2\geq 0$, $\sigma_2-t_2\geq t_1-\sigma_1\geq 0$ and again by the same assumptions of the previous two subcases; i.e., if $\delta>\frac{1}{c}$ and if we choose b so that $b<\frac{\gamma_1}{2(4\alpha^2-3\alpha)}$ and $\alpha>\frac{2c}{1-b}$ (so $\alpha>2$), then we get

$$\begin{aligned}
\|y-z\|-\Gamma^{\sim} &\geq \delta\left(1+\frac{{\gamma_1}^2}{1+4\alpha}+\gamma_1\left(\alpha-Q_1\right)(1-b)\right) \\
&\quad +(\sigma_1-t_1)\left(1+\gamma_1(\alpha-Q_1)\right)(1-b) \\
&\quad +(\sigma_2-t_2)\left(1+\gamma_1(\alpha-Q_1)\right)(1-b) \\
&\quad -\frac{{\gamma_1}^2}{1+4\alpha}\left(2-\sigma_1-t_1\right)-(1+2\gamma_1[\alpha-Q_1-Q_2])b\delta \\
&\geq \delta\left(1+\frac{{\gamma_1}^2}{1+4\alpha}+\gamma_1\left(\alpha-Q_1\right)(1-b)-b(1+2\gamma_1[\alpha-Q_1-Q_2])\right) \\
&\quad +(t_1-\sigma_1)\left[\frac{{\gamma_1}^2}{1+4\alpha}-\left(1+\gamma_1(\alpha-Q_1)\right)(1-b)\right] \\
&\quad +(t_1-\sigma_1)\left(1+\gamma_1(\alpha-Q_1)\right)(1-b)-\frac{{\gamma_1}^2}{1+4\alpha}\left(2-2\sigma_1\right) \\
&\geq \delta\left(1+\frac{{\gamma_1}^2}{1+4\alpha}+\gamma_1\left(\alpha-Q_1\right)(1-b)-b(1+2\gamma_1[\alpha-Q_1-Q_2])\right) \\
&\quad +(t_1-\sigma_1)\frac{{\gamma_1}^2}{1+4\alpha}-\frac{{\gamma_1}^2}{1+4\alpha}\left(2-2\sigma_1\right) \\
&\geq \delta{\gamma_1}^2\left(1+\alpha-Q_1\right)(1-b)-\frac{{\gamma_1}^2c}{1+4\alpha}\frac{1}{c}\left(2-2\sigma_1\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\|y-z\| - \Gamma^{\sim} &\geq \delta {\gamma_1}^2 \left(1+\alpha - Q_1\right)\left(1-b\right) - \frac{{\gamma_1}^2 c}{1+4\alpha}\delta\left(2-2\sigma_1\right)\\
&\geq \delta {\gamma_1}^2 \left[\left(1+\alpha - Q_1\right)\left(1-b\right) - \frac{2c}{1+4\alpha}\right]\\
&\geq \frac{{\gamma_1}^2 \alpha}{1+4\alpha}\delta\left[\alpha(1-b)-2c\right]\\
&\geq 0.
\end{aligned}$$

Subcase 2.2: Consider $\sigma_1+\sigma_2-t_1-t_2<0$.

Subcase 2.2.1: Here, first we consider $\delta < t_1-\sigma_1+t_2-\sigma_2$.

Since $1-\delta > t_1-\sigma_1+t_2-\sigma_2-\delta>0$, there exists $d>1$ such that $t_1-\sigma_1+t_2-\sigma_2-\delta > \frac{1}{d}$. Now, assume $\alpha > \frac{7+\sqrt{49+16(3+2d)}}{8}$ (so $\alpha>2$) and assume $b<\frac{4{\gamma_1}^2}{41\alpha^2}$ (so $b<\frac{{\gamma_1}^2}{8\alpha^2+4\alpha+1}$). Then, we use another property of the norm and get the following inequalities.

$$\begin{aligned}
\|y-z\| &\geq \|y-z\|_{(3)}\\
&= \left\|\begin{pmatrix}
(t_1-\sigma_1)b+(t_2-\sigma_2)b+(t_3-\sigma_3)+(t_4-\sigma_4)+\dots,\\
(t_2-\sigma_2)b+(t_3-\sigma_3)+(t_4-\sigma_4)+\dots,\\
(t_3-\sigma_3)+(t_4-\sigma_4)+\dots,\\
(t_4-\sigma_4)+(t_5-\sigma_5)+\dots,\\
(t_5-\sigma_5)+(t_6-\sigma_6)+\dots,\dots
\end{pmatrix}\right\|_{(3)}\\
&\geq \gamma_1\left|(t_1-\sigma_1)b+(t_2-\sigma_2)b+(t_3-\sigma_3)+(t_4-\sigma_4)+\dots\right|\\
&\quad +\gamma_1\left|\begin{array}{l}
Q_1\left[(t_1-\sigma_1)b+(t_2-\sigma_2)b\right]+Q_2(t_2-\sigma_2)b\\
+\left[Q_1+Q_2-\alpha\right]\sum_{k=3}^{\infty}(t_k-\sigma_k)\\
+Q_3\sum_{k=3}^{\infty}(t_k-\sigma_k)+Q_4\sum_{k=4}^{\infty}(t_k-\sigma_k)+\dots
\end{array}\right|\\
&\geq \gamma_1(\alpha-Q_1-Q_2-1)|\delta-(t_1-\sigma_1+t_2-\sigma_2)|\\
&\quad +\gamma_1 b(1-Q_1)(t_1-\sigma_1+t_2-\sigma_2)-\gamma_1 Q_2(t_2-\sigma_2)b\\
&\quad -\gamma_1(1-Q_1-Q_2)(2-[\delta+\sigma_1+\sigma_2+t_1+t_2])\\
&\geq \gamma_1(\alpha-Q_1-Q_2-1)|\delta-(t_1-\sigma_1+t_2-\sigma_2)|\\
&\quad -\gamma_1(1-Q_1-Q_2)(2-[\delta+\sigma_1+\sigma_2+t_1]).
\end{aligned}$$

Thus,

$$\begin{aligned}
\|y-z\| - \Gamma^{\sim} &\geq \gamma_1(\alpha-Q_1-Q_2-1)(t_1-\sigma_1+t_2-\sigma_2-\delta)\\
&\quad -\frac{{\gamma_1}^2}{1+4\alpha}(2-[\delta+\sigma_1+\sigma_2+t_1])-b\delta(1+2\gamma_1[\alpha-Q_1-Q_2]).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|y-z\| - \Gamma^{\sim} &\geq \gamma_1(\alpha - Q_1 - Q_2 - 1)(t_1 - \sigma_1 + t_2 - \sigma_2 - \delta) \\
&\quad -\frac{2{\gamma_1}^2}{1+4\alpha} + \frac{1}{1+4\alpha}\left[{\gamma_1}^2 - b(8\alpha^2 + 4\alpha + 1)\right]\delta \\
&\geq \gamma_1(\alpha - Q_1 - Q_2 - 1)(t_1 - \sigma_1 + t_2 - \sigma_2 - \delta) \\
&\quad -\frac{2d{\gamma_1}^2}{1+4\alpha}\frac{1}{d} \\
&\geq \gamma_1(\alpha - Q_1 - Q_2 - 1)(t_1 - \sigma_1 + t_2 - \sigma_2 - \delta) \\
&\quad -\frac{2d{\gamma_1}^2}{1+4\alpha}(t_1 - \sigma_1 + t_2 - \sigma_2 - \delta) \\
&\geq {\gamma_1}^2(\alpha - Q_1 - Q_2 - 1 - \frac{2d}{1+4\alpha})(t_1 - \sigma_1 + t_2 - \sigma_2 - \delta) \\
&\geq (\alpha - 2 - \frac{1}{1+4\alpha} - \frac{2d}{1+4\alpha})(t_1 - \sigma_1 + t_2 - \sigma_2 - \delta) \\
&= \frac{(t_1 - \sigma_1 + t_2 - \sigma_2 - \delta)}{1+4\alpha}\left[4\alpha^2 - 7\alpha - (3+2d)\right] \\
&\geq 0.
\end{aligned}$$

Subcase 2.2.2: Consider $\delta = t_1 - \sigma_1 + t_2 - \sigma_2 > 0$.

Then, assume $2{Q_1}^2 < \alpha \leq 2{Q_1}^2 + 2{Q_2}^2(1-\lambda)$ for $\lambda \in [0,1]$ so the minimum value for $\|h_\lambda - z\|$ is $\Gamma = 2\gamma_1 Q_1 \delta b$. Thus,

$$\begin{aligned}
\|y-z\| &\geq \frac{\|y-z\|_{(1)} + \|y-z\|_{(3)}}{2} \\
&\geq \frac{1}{2}\left(\begin{array}{c}
\left[\begin{array}{c}
\delta b + \gamma_1(Q_1 + {Q_1}^{\frac{3}{2}})|1-\alpha|\delta b \\
+\gamma_1(Q_2 + {Q_2}^{\frac{3}{2}})|(t_2 - \sigma_2)b - \alpha\delta b| \\
+\gamma_1(Q_3 + {Q_3}^{\frac{3}{2}})\alpha\delta b \\
+\gamma_1(Q_4 + {Q_4}^{\frac{3}{2}})\left|\sum_{k=4}^{\infty}(t_k - \sigma_k) - \alpha\delta b\right| \\
+\gamma_1(Q_5 + {Q_5}^{\frac{3}{2}})\left|\sum_{k=5}^{\infty}(t_k - \sigma_k) - \alpha\delta b\right| \\
+\gamma_1(Q_6 + {Q_6}^{\frac{3}{2}})\left|\sum_{k=6}^{\infty}(t_k - \sigma_k) - \alpha\delta b\right| + \cdots
\end{array}\right] \\
+ \\
\left[\begin{array}{c}
\delta b + \gamma_1(Q_1 + {Q_1}^{\frac{3}{2}})\delta b \\
+\gamma_1(Q_2 + {Q_2}^{\frac{3}{2}})|t_2 - \sigma_2|b \\
+\gamma_1(Q_4 + {Q_4}^{\frac{3}{2}})\left|\sum_{k=4}^{\infty}(t_k - \sigma_k)\right| \\
+\gamma_1(Q_5 + {Q_5}^{\frac{3}{2}})\left|\sum_{k=5}^{\infty}(t_k - \sigma_k)\right| \\
+\gamma_1(Q_6 + {Q_6}^{\frac{3}{2}})\left|\sum_{k=6}^{\infty}(t_k - \sigma_k)\right| + \cdots
\end{array}\right]
\end{array}\right).
\end{aligned}$$

Hence,

$$\|y-z\| \geq \delta b + \frac{\alpha\delta b\gamma_1\left(1 + \sum_{k=1}^{\infty}{Q_k}^{\frac{3}{2}}\right)}{2}.$$

Thus,

$$\begin{aligned}\|y-z\|-\Gamma^{\sim} &\geq \delta b+\frac{\alpha\delta b\gamma_1}{2}-2\gamma_1\delta bQ_1=\delta b\left(\frac{\alpha\gamma_1}{2}+1-2\gamma_1Q_1\right)\\ &\geq 0.\end{aligned}$$

Now, consider the final case:

Subcase 2.2.3: $\delta>t_1-\sigma_1+t_2-\sigma_2>0$.

Then, consider the conditions like in the subcase 2.1.1; i.e., since $\delta-(t_1-\sigma_1+t_2-\sigma_2)>0$, there exists $e>1$ such that $\delta-(t_1-\sigma_1+t_2-\sigma_2)>\frac{1}{e}$ and then choose b so that $b<\frac{\gamma_1}{2(4\alpha^2-3\alpha)}$ and $\alpha>\frac{2e}{1-b}$ (so $\alpha>2$ and $2\alpha>\frac{2e}{1-b}-1$); then get

$$\begin{aligned}\|y-z\|-\Gamma^{\sim} &\geq \|y-z\|_{(1)}-\Gamma^{\sim}\\ &\geq |\sigma_1+\sigma_2+\delta-t_1-t_2+(t_1-\sigma_1)b+(t_2-\sigma_2)b|\\ &\quad+\gamma_1\left|(Q_1-\alpha)\left\{(1-b)\left[\sigma_1+\sigma_2+\delta-t_1-t_2\right]+b\delta\right\}\right|\\ &\quad-\gamma_1(1-Q_1)\sum_{j=2}^{\infty}|t_j-\sigma_j|-b\delta(1+2\gamma_1[\alpha-Q_1-Q_2])\\ &\geq (1+\gamma_1[\alpha-Q_1])(1-b)(\delta-(t_1-\sigma_1+t_2-\sigma_2))\\ &\quad-\frac{{\gamma_1}^2}{1+4\alpha}(2-\sigma_1-t_1)\\ &\quad+\left(\frac{{\gamma_1}^2}{1+4\alpha}-2\gamma_1b(\alpha-Q_1)\right)\delta\\ &\geq (1+\gamma_1[\alpha-Q_1])(1-b)(\delta-(t_1-\sigma_1+t_2-\sigma_2))\\ &\quad-\frac{2{\gamma_1}^2e}{1+4\alpha}(\delta-(t_1-\sigma_1+t_2-\sigma_2))\\ &\quad+\frac{\gamma_1}{1+4\alpha}\left(\gamma_1-2b(4\alpha^2-3\alpha)\right)\\ &\geq {\gamma_1}^2\left[(1+\alpha-Q_1)(1-b)-\frac{2e}{1+4\alpha}\right](\delta-(t_1-\sigma_1+t_2-\sigma_2))\\ &\geq \frac{{\gamma_1}^2\alpha}{1+4\alpha}\left[\alpha(1-b)-2e\right](\delta-(t_1-\sigma_1+t_2-\sigma_2))\\ &\geq 0.\end{aligned}$$

In conclusion, from all cases, we see that there exist constant $0<C\leq 1$ and there exist $b\in(0,C)$ and $\alpha\geq\frac{1}{2}$ such that when λ is choosen to be in $\left[-\frac{\sigma_1}{\delta},\ \frac{\sigma_2}{\delta}+1\right]$, for any $y\in E$ and for $z\in W\setminus E$, $\|y-z\|\geq\Gamma$ where

$$\Gamma:=\min_{\lambda\in\left[-\frac{\sigma_1}{\delta},\ \frac{\sigma_2}{\delta}+1\right]}\|h_\lambda-z\|.$$

Then, define

$$\Lambda:=\left\{h_\lambda\ :\ \lambda\in\left[-\frac{\sigma_1}{\delta},\ \frac{\sigma_2}{\delta}+1\right]\right\}.$$

Note that $\Lambda \subseteq E$ is compact as it is the continuous image of compact set $\left[-\frac{\sigma_1}{\delta},\ \frac{\sigma_2}{\delta}+1\right]$ and there exists unique $\lambda_0 \in \Lambda$ such that $\|h_{\lambda_0} - z\|$ is minimizer of Γ. Now, we can see that for $h \in \Gamma$,

$$\begin{aligned}
s\left(Th\right) &= \limsup_m \left\| Th - \frac{1}{m}\sum_{k=1}^{m} x^{(k)} \right\| \\
&\leq \limsup_m \left\| Th - T\left(\frac{1}{m}\sum_{k=1}^{m} x^{(k)}\right)\right\| + \limsup_m \left\| \begin{array}{c} \frac{1}{m}\sum_{k=1}^{m} x^{(k)} \\ - \\ T\left(\frac{1}{m}\sum_{k=1}^{m} x^{(k)}\right) \end{array} \right\| \\
&\quad \text{(since } T \text{ is affine)} \\
&= \limsup_m \left\| Th - T\left(\frac{1}{m}\sum_{k=1}^{m} x^{(k)}\right)\right\| + \limsup_m \left\| \begin{array}{c} \frac{1}{m}\sum_{k=1}^{m} x^{(k)} \\ - \\ \frac{1}{m}\sum_{k=1}^{m} Tx^{(k)} \end{array} \right\| \\
&\leq \limsup_m \left\| h - \frac{1}{m}\sum_{k=1}^{m} x^{(k)} \right\| \\
&= s(h).
\end{aligned}$$

Also, $s(Th) = z + \|z - Th\|$ and $s(h) = z + \|z - h\|$. Hence,

$$\begin{aligned}
\|z - Th\| \leq \|z - h\| &\implies \|z - Th\| = \|z - h\| \\
&\implies Th \in \Lambda.
\end{aligned}$$

Therefore, $T(\Lambda) \subseteq \Lambda$ and since T is continuous, Brouwer's Fixed Point Theorem [1] tells us that T has a fixed point such that $h = h_{\lambda_0}$ is the unique minimizer of $\|y - z\| \ :\ y \in E$ and $Th = h$.

Hence, E has FPP (n.e.) as desired. □

Then, the following corollary is immediate such that its proof is similiar to the proof of our main theorem above.

Corollary 5.3. *Fix $b \in (0,1)$ and define the closed, bounded, convex subset $E = E_b$ of c_0 as in Example 5.1. Define the equivalent norm $\|\cdot\|^{\sim}$ on c_0 as below:*

For $x = (\xi_k)_k \in c_0$ and $\alpha > 0$,

$$\begin{aligned}
\|x\|^{\sim} &= \frac{1}{\gamma_1} \lim_{p\to\infty} \sup_{k\in\mathbb{N}} \gamma_k \left(\sum_{j=k}^{\infty} \frac{\left|\xi_j\right|^p}{j}\right)^{\frac{1}{p}} + \gamma_1 \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k \left|\xi^{\star}{}_k - \alpha \xi^{\star}{}_j\right| \\
&\quad \text{where } \gamma_k \uparrow_k 1,\ \gamma_{k+2} > \gamma_{k+1},\ \forall k \in \mathbb{N}, \\
&\quad x^{\star} := (\xi^{\star}{}_j)_{j\in\mathbb{N}} \text{ is the decreasing rearrangement of } x, \\
&\quad \sum_{k=1}^{\infty} Q_k = 1,\ Q_k \downarrow_k 0\ \ Q_k > Q_{k+1},\ \forall k \in \mathbb{N} \text{ and } Q_1 > \frac{1 - \gamma_1 + 2|\alpha|}{1 + 2|\alpha|}.
\end{aligned}$$

Now, let $\gamma_2 = \gamma_1$. Then, there exist constants $0 < C \leq 1$ and $\alpha \geq \frac{1}{2}$ such that for all $b \in (0, C)$ the set E defined as in the example above has the fixed point property for $\|\cdot\|^{\sim}$-nonexpansive affine mappings.

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