Available online: June 04, 2017

Commun.Fac.Sci.Univ.Ank.Series A1
Volume 67, Number 1, Pages 29–36 (2018)
DOI: 10.1501/Commua1_0000000827
ISSN 1303–5991
http://communications.science.ankara.edu.tr/index.php?series=A1

COMMUNICATIONS
SERIES A1

ON SOME SUBCLASSES OF M-FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS

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Abstract. In this work, we introduce two new subclasses $S_{\Sigma_m}(\alpha, \lambda)$ and $S_{\Sigma_m}(\beta, \lambda)$ of Σ_m consisting of analytic and m-fold symmetric bi-univalent functions in the open unit disc U. Furthermore, for functions in each of the subclasses introduced in this paper, we obtain the coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$.

1. Introduction

Let A denote the class of functions f which are analytic in the open unit disc $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, with in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let S be the subclass of A consisting of the form (1.1) which are also univalent in U. It is well known that every function $f \in S$ has an inverse f^{-1}, satisfying $f^{-1}(f(z)) = z,\ (z \in U)$ and $f\left(f^{-1}(w)\right) = w,\ \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right)$, where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of bi-univalent functions defined in the unit disc U. For a brief history and interesting examples in the class Σ, see [11], (see also [1], [3], [8], [9], [12], [15], [16], [20], [21]).

For each function $f \in S$, the function

$$h(z) = \sqrt[m]{f(z^m)} \qquad (z \in U, \quad m \in \mathbb{N}) \tag{1.3}$$

Received by the editors: January 27, 2017; Accepted: March 08, 2017.
2010 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C50.
Key words and phrases. Analytic functions, m-fold symmetric bi-univalent functions, coefficient bounds.

is univalent and maps the unit disc U into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [7], [10]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (z \in U, \quad m \in \mathbb{N}). \tag{1.4}$$

We denote by S_m the class of m-fold symmetric univalent functions in U, which are normalized by the series expansion (1.4). In fact, the functions in the class S are *one*-fold symmetric.

Analogous to the concept of m-fold symmetric univalent functions, we here introduced the concept of m-fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of f is given as in (1.4) and the series expansion for f^{-1}, which has been recently proven by Srivastava et al. [13], is given as follows:

$$\begin{aligned} g(w) \quad = \quad & w \ - a_{m+1} w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ & - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} \\ & + \cdots \end{aligned} \tag{1.5}$$

where $f^{-1} = g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in U. For $m = 1$, the formula (1.5) coincides with the formula (1.2) of the class Σ. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \quad \left[-\log(1-z^m)\right]^{\frac{1}{m}}, \quad \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)^{\frac{1}{m}}\right].$$

Thus, following Altınkaya and Yalçın [3] constructed the subclasses $S_\Sigma(\lambda, \alpha)$ and $S_\Sigma(\lambda, \beta)$ of bi-univalent functions and obtained estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. Furthermore, in [4], Altınkaya and Yalçın obtained the second Hankel determinant, for the class $S_\Sigma(\lambda, \beta)$.

Recently, certain subclasses of m-fold bi-univalent functions class Σ_m similar to subclasses of introduced and investigated by Altınkaya and Yalçın [2], (see also [13], [14], [17], [18], [19]).

The aim of the this paper is to introduce two new subclasses of the function class Σ_m and derive estimates on the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in these new subclasses of the function class Σ employing the techniques used earlier by Srivastava et al. [11] (see also [6]).

Let P denote the class of functions consisting of p, such that

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

which are regular in the open unit disc U and satisfy $\Re(p(z)) > 0$ for any $z \in U$. Here, $p(z)$ is called Caratheodory function [5].

We have to remember the following lemma so as to derive our basic results:

Lemma 1. *(see [10]) If $p \in P$, then*

$$|p_n| \leq 2 \qquad (n \in \mathbb{N} = \{1, 2, \ldots\}).$$

2. Coefficient bounds for the function class $S_{\Sigma_m}(\alpha, \lambda)$

Definition 1. *A function $f \in \Sigma_m$ is said to be in the class $S_{\Sigma_m}(\alpha, \lambda)$ if the following conditions are satisfied:*

$$\left|\arg\left[\tfrac{1}{2}\left(\tfrac{zf'(z)}{f(z)} + \left(\tfrac{zf'(z)}{f(z)}\right)^{\frac{1}{\lambda}}\right)\right]\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ 0 < \lambda \leq 1, \quad z \in U)$$

and

$$\left|\arg\left[\tfrac{1}{2}\left(\tfrac{wg'(w)}{g(w)} + \left(\tfrac{wg'(w)}{g(w)}\right)^{\frac{1}{\lambda}}\right)\right]\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ 0 < \lambda \leq 1,\ w \in U)$$

where the function $g = f^{-1}$.

Theorem 1. *Let f given by (1.4) be in the class $S_{\Sigma_m}(\alpha, \lambda)$, $0 < \alpha \leq 1$. Then*

$$|a_{m+1}| \leq \frac{4\lambda\alpha}{m\sqrt{(1+\lambda)\left[4\lambda\alpha + (1+\lambda)(1-\alpha)\right] + 2\alpha(1-\lambda)}}$$

and

$$|a_{2m+1}| \leq \frac{2\lambda\alpha}{m\,(1+\lambda)} + \frac{8(m+1)\lambda^2\alpha^2}{m^2(1+\lambda)^2}.$$

Proof. Let $f \in S_{\Sigma_m}(\alpha, \lambda)$. Then

$$\frac{1}{2}\left(\frac{zf'(z)}{f(z)} + \left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\lambda}}\right) = [p(z)]^{\alpha} \tag{2.1}$$

$$\frac{1}{2}\left(\frac{wg'(w)}{g(w)} + \left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\lambda}}\right) = [q(w)]^{\alpha} \tag{2.2}$$

where $g = f^{-1}$, p, q in P and have the forms

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + \cdots$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + \cdots.$$

Now, equating the coefficients in (2.1) and (2.2), we get

$$\frac{m(1+\lambda)}{2\lambda} a_{m+1} = \alpha p_m, \tag{2.3}$$

$$\frac{m(1+\lambda)}{2\lambda}\left(2a_{2m+1} - a_{m+1}^2\right) + \frac{m^2(1-\lambda)}{4\lambda^2} a_{m+1}^2 = \alpha p_{2m} + \tfrac{\alpha(\alpha-1)}{2} p_m^2, \tag{2.4}$$

and

$$-\frac{m(1+\lambda)}{2\lambda}a_{m+1}=\alpha q_m, \tag{2.5}$$

$$\frac{m(1+\lambda)}{2\lambda}\left[(2m+1)a_{m+1}^2-2a_{2m+1}\right]+\frac{m^2(1-\lambda)}{4\lambda^2}a_{m+1}^2=\alpha q_{2m}+\tfrac{\alpha(\alpha-1)}{2}q_m^2. \tag{2.6}$$

Making use of (2.3) and (2.5), we obtain

$$p_m=-q_m. \tag{2.7}$$

and

$$\frac{m^2(1+\lambda)^2}{2\lambda^2}a_{m+1}^2=\alpha^2(p_m^2+q_m^2). \tag{2.8}$$

Also from (2.4), (2.6) and (2.8) we have

$$\left[\tfrac{m^2(1+\lambda)}{\lambda}+\tfrac{m^2(1-\lambda)}{2\lambda^2}\right]a_{m+1}^2=\alpha\left(p_{2m}+q_{2m}\right)+\tfrac{\alpha(\alpha-1)}{2}(p_m^2+q_m^2).$$

$$=\alpha\left(p_{2m}+q_{2m}\right)+\tfrac{\alpha(\alpha-1)}{2}\tfrac{m^2(1+\lambda)^2}{2\lambda^2\alpha^2}a_{m+1}^2.$$

Therefore, we have

$$a_{m+1}^2=\frac{4\lambda^2\alpha^2\left(p_{2m}+q_{2m}\right)}{m^2\left\{(1+\lambda)\left[4\lambda\alpha+(1+\lambda)(1-\alpha)\right]+2\alpha(1-\lambda)\right\}}. \tag{2.9}$$

Applying Lemma 1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}|\leq\frac{4\lambda\alpha}{m\sqrt{(1+\lambda)\left[4\lambda\alpha+(1+\lambda)(1-\alpha)\right]+2\alpha(1-\lambda)}}.$$

Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (2.6) from (2.4), we get

$$\frac{2m(1+\lambda)}{\lambda}a_{2m+1}-\frac{m(m+1)(1+\lambda)}{\lambda}a_{m+1}^2=\alpha\left(p_{2m}-q_{2m}\right)+\tfrac{\alpha(\alpha-1)}{2}(p_m^2-q_m^2).$$

Then, in view of (2.7) and (2.8) , and applying Lemma 1 for the coefficients $p_{2m},\ p_m$ and $q_{2m},\ q_m$, we have

$$|a_{2m+1}|\leq\frac{2\lambda\alpha}{m\left(1+\lambda\right)}+\frac{8(m+1)\lambda^2\alpha^2}{m^2(1+\lambda)^2}.$$

which completes the proof of Theorem 1. □

3. Coefficient bounds for the function class $S_{\Sigma_m}(\beta,\lambda)$

Definition 2. *A function $f\in\Sigma_m$ given by (1.4) is said to be in the class $S_{\Sigma_m}(\beta,\lambda)$ if the following conditions are satisfied:*

$$\Re\left\{\frac{1}{2}\left(\frac{zf'(z)}{f(z)}+\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\lambda}}\right)\right\}>\beta,\qquad(0\leq\beta<1,\quad 0<\lambda\leq1,\quad z\in U) \tag{3.1}$$

and

$$\Re\left\{\frac{1}{2}\left(\frac{wg'(w)}{g(w)}+\left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\lambda}}\right)\right\}>\beta, \qquad (0\le\beta<1, \quad 0<\lambda\le 1, \quad w\in U)\,. \tag{3.2}$$

where the function $g=f^{-1}$.

Theorem 2. *Let* f *given by (1.4) be in the class* $S_{\Sigma_m}(\beta,\lambda)$, $0\le\beta<1$. *Then*

$$|a_{m+1}|\le\frac{2\lambda}{m}\sqrt{\frac{2\,(1-\beta)}{2\lambda^2+\lambda+1}}$$

and

$$|a_{2m+1}|\le\frac{8(m+1)\lambda^2\,(1-\beta)^2}{m^2(1+\lambda)^2}+\frac{2\lambda\,(1-\beta)}{m\,(1+\lambda)}.$$

Proof. Let $f\in S_{\Sigma_m}(\beta,\lambda)$. Then

$$\frac{1}{2}\left(\frac{zf'(z)}{f(z)}+\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\lambda}}\right)=\beta+(1-\beta)p(z) \tag{3.3}$$

$$\frac{1}{2}\left(\frac{wg'(w)}{g(w)}+\left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\lambda}}\right)=\beta+(1-\beta)q(w) \tag{3.4}$$

where $p,q\in P$ and $g=f^{\ -1}$.

It follows from (3.3) and (3.4) that

$$\frac{m(1+\lambda)}{2\lambda}a_{m+1}=(1-\beta)p_m, \tag{3.5}$$

$$\frac{m(1+\lambda)}{2\lambda}\left(2a_{2m+1}-a_{m+1}^2\right)+\frac{m^2(1-\lambda)}{4\lambda^2}a_{m+1}^2=(1-\beta)p_{2m}, \tag{3.6}$$

and

$$-\frac{m(1+\lambda)}{2\lambda}a_{m+1}=(1-\beta)q_m, \tag{3.7}$$

$$\frac{m(1+\lambda)}{2\lambda}\left[(2m+1)a_{m+1}^2-2a_{2m+1}\right]+\frac{m^2(1-\lambda)}{4\lambda^2}a_{m+1}^2=(1-\beta)q_{2m}. \tag{3.8}$$

Then, by making use of (3.5) and (3.7), we get

$$p_m=-q_m. \tag{3.9}$$

and

$$\frac{m^2(1+\lambda)^2}{2\lambda^2}a_{m+1}^2=(1-\beta)^2(p_m^2+q_m^2). \tag{3.10}$$

Adding (3.6) and (3.8), we have

$$\left[\frac{m^2(1+\lambda)}{\lambda}+\frac{m^2\,(1-\lambda)}{2\lambda^2}\right]a_{m+1}^2=(1-\beta)\,(p_{2m}+q_{2m})\,.$$

Therefore, we obtain

$$a_{m+1}^2 = \frac{2\lambda^2(1-\beta)\left(p_{2m}+q_{2m}\right)}{m^2(2\lambda^2+\lambda+1)}.$$

Applying Lemma 1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}| \leq \frac{2\lambda}{m}\sqrt{\frac{2\left(1-\beta\right)}{2\lambda^2+\lambda+1}}.$$

Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (3.8) from (3.6), we obtain

$$\frac{2m(1+\lambda)}{\lambda}a_{2m+1} - \frac{m(m+1)(1+\lambda)}{\lambda}a_{m+1}^2 = (1-\beta)\left(p_{2m}-q_{2m}\right).$$

Then, in view of (3.9) and (3.10) , applying Lemma 1 for the coefficients p_{2m}, p_m and q_{2m}, q_m, we have

$$|a_{2m+1}| \leq \frac{8(m+1)\lambda^2\left(1-\beta\right)^2}{m^2(1+\lambda)^2} + \frac{2\lambda\left(1-\beta\right)}{m\left(1+\lambda\right)}.$$

which completes the proof of Theorem 2. $\square$

If we set $\lambda = 1$ in Theorems 1 and 2, then the classes $S_{\Sigma_m}(\alpha,\lambda)$ and $S_{\Sigma_m}(\beta,\lambda)$ reduce to the classes $S_{\Sigma_m}^{\alpha}$ and $S_{\Sigma_m}^{\beta}$ and thus, we obtain the following corollaries:

Corollary 1. *(see [2]) Let f given by (1.4) be in the class $S_{\Sigma_m}^{\alpha}$ $(0 < \alpha \leq 1)$. Then*

$$|a_{m+1}| \leq \frac{2\alpha}{m\sqrt{\alpha+1}}$$

and

$$|a_{2m+1}| \leq \frac{\alpha}{m} + \frac{2(m+1)\alpha^2}{m^2}.$$

Corollary 2. *(see [2]) Let f given by (1.4) be in the class $S_{\Sigma_m}^{\beta}$ $(0 \leq \beta < 1)$. Then*

$$|a_{m+1}| \leq \frac{\sqrt{2\left(1-\beta\right)}}{m}$$

and

$$|a_{2m+1}| \leq \frac{2(m+1)(1-\beta)^2}{m^2} + \frac{1-\beta}{m}.$$

Remark 1. *For one-fold symmetric bi-univalent functions, if we put $\lambda = 1$ in our Theorems, then we obtain the Corollary 1 and Corollary 2 which were proven earlier by Murugunsundaramoorthy et al. [9].*

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