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ON APPROXIMATION BY NÖRLUND AND RIESZ SUBMETHODS IN VARIABLE EXPONENT LEBESGUE SPACES

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Abstract. In this study the results on the degree of approximation by the Nörlund and the Riesz submethods of the partial sums of their Fourier series of functions where in the variable exponent Lebesgue spaces are given by weakening the monotonicity conditions of sequences in the submethods. Therefore the results given in Güven and İsrafilov (2010) are generalized according to both the monotonicity conditions and both the methods.

1. Background of the Problem and Some Notations

One of the basic problems in the theory of approximation of functions and the theory of Fourier series is to examine the degree of approximation in given function spaces by some certain methods. In this sense, one of the important results encountered belongs to Quade in [1]. He solved a problem related with approximation by trigonometric polynomials on conjecture stated without proof by G. H. Hardy and J. E. Littlewood in 1928. Before giving the results of Quade, we need to some definitions.

Assume that f is a 2π periodic function and $f \in L^1(0, 2\pi)$ where $L^1(0, 2\pi)$ consists of all measurable functions. Moreover, let

$$f \sim \frac{a_o}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(f; x) \tag{1.1}$$

be the Fourier series of a function $f \in L^1$ and $\sigma_n(f)$ denote the n-th term of the $(C, 1)$ transform the partial sums of Fourier series of a 2π periodic function f.

Furthermore, a function f belongs to the $Lip(\alpha, p)$ class if $\omega_p(\delta, f) = O(\delta^\alpha)$, where

$$\omega_p(\delta, f) = \sup_{|t|\leq\delta} ||f(\cdot + t) - f(\cdot)||_p \quad 0 < \alpha \leq 1; \quad p \geq 1,$$

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is the integral modulus of continuity of $f \in L^p$ where $\|.\|_p$ denote the L^p-norm with respect to x defined as

$$\|f\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p \, dx \right\}^{\frac{1}{p}}.$$

Quade, in [1], has shown that if f is a function in the class $Lip(\alpha, p)$, ($0 < \alpha \leq 1$, $p \geq 1$), then for either $p > 1$ and $0 < \alpha \leq 1$ or $p = 1$ and $0 < \alpha < 1$,

$$\|f - \sigma_n\|_p = O(n^{-\alpha}). \tag{1.2}$$

He also showed that if $p = \alpha = 1$, then

$$\|f - \sigma_n\|_1 = O(n^{-1} \log(n+1)).$$

There are several generalizations of (1.2) for $p > 1$ (see [2]-[5] and [8]). In 2002, Chandra gave some attractive results including sharper estimates than some results of Quade by Nörlund and Riesz methods. Therefore, the work of Quade [1] was improved by Chandra [15] for more general trigonometrical polynomials than $\sigma_n(f)$ to yield the same estimate as in (1.2). In 2005, Leindler[17] weakened the conditions of monotonicity given by Chandra according to Nörlund and Riesz methods. We know that Nörlund and Riesz methods generalize the well known Cesáro method which has an important place in this theory. Naturally, there arises the question how we can generalize these approximation methods. There are some possibilities in this way. First it can be generalized by taking into account summability methods. Secondly, it can be weakened the conditions of monotonicity owing to the sequences in Nörlund and Riesz methods. The other one can be generalized with regard to the given function spaces. In this work we shall consider these conditions and move this direction.

1.1. **Nörlund and Riesz Submethods.** Suppose that $\{\lambda(n)\}_{n=1}^{\infty}$ is a strictly increasing sequence of positive integers. The Cesáro submethod C_λ is defined as

$$(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k, \ (n = 1, 2, ...),$$

where (x_k) is a sequence of a real or complex numbers. Therefore, the C_λ-method yields a subsequence of the Cesáro method C_1, and hence it is regular for any λ. Note that C_λ is obtained by deleting a set of rows from Cesáro matrix. The basic properties of C_λ-method can be found in [6] and [12]. By considering this method the following notions was given in [19]: Let (p_n) be a positive sequence of real numbers.

$$N_n^\lambda(f; x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} s_m(f; x),$$

$$R_n^\lambda(f; x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_m s_m(f; x),$$

where

$$s_n(f;x) = \frac{1}{\pi}\int_0^{2\pi} f\left(x+t\right) D_n(t)\ dt,$$

and

$$D_n(t) = \frac{\sin(n+\frac{1}{2})t}{2\sin\left(\frac{t}{2}\right)}.$$

Also,

$$P_{\lambda(n)} = p_0 + p_1 + p_2 + ... + p_{\lambda(n)} \neq 0\ (n \geq 0),$$

and by convention $p_{-1} = P_{-1} = 0$. In case $\lambda(n) = n$, the methods $N_n^{\lambda}(f;x)$ and $R_n^{\lambda}(f;x)$ give us classical known Nörlund and Riesz means. Provided that $p_n = 1$ for all $(n \geq 0)$ both of them yield

$$\sigma_n^{\lambda}(f;x) = \frac{1}{\lambda\left(n\right)+1}\sum_{m=0}^{\lambda(n)} s_m(f;x).$$

In addition to this, if $\lambda(n) = n$ for $\sigma_n^{\lambda}(f;x)$, then it coincides with Cesáro method C_1.

1.2. **Some Sequence Classes.** The monotonicity conditions on the sequence (p_n) in Nörlund and Riesz submethods are quite important in determination of the degree of approach. So let us recall the definitions of some classes of numerical sequences discussed in detail in [13], [17] and [20]. Let $u := (u_n)$ be a nonnegative sequence and $C := (C_n) = \frac{1}{n+1}\sum\limits_{m=0}^{n} u_m$.

A sequence u is called almost monotone decreasing (briefly $u \in AMDS$) (increasing (briefly $u \in AMIS$)), if there exists a constant $K := K(u)$ which only depends on u such that

$$u_n \leq K u_m \qquad (K u_n \geq u_m)$$

for all $n \geq m$. If $C \in AMDS$ $(C \in AMIS)$, then we say that the sequence u is almost monotone decreasing (increasing) mean sequence and denoted by $C \in AMDMS$ $(C \in AMIMS)$. A sequence u tending to zero is called a rest bounded variation sequence $(RBVS)$ (rest bounded variation mean sequence $(RBVMS)$), if it has the property

$$\sum_{m=k}^{\infty} |\Delta u_m| \leq K(u) u_k \quad (\sum_{m=k}^{\infty} |\Delta C_m| \leq K(u) C_k)$$

for all natural numbers k where $\Delta u_m = u_m - u_{m+1}$. Leindler first raised the rest bounded variation condition in [13]. A sequence u is called a head bounded variation sequence ($HBVS$) (head bounded variation mean sequence ($HBVMS$)), if it has the property

$$\sum_{m=0}^{k-1} |\Delta u_m| \leq K(u) u_k \quad (\sum_{m=0}^{k-1} |\Delta C_m| \leq K(u) C_k)$$

for all natural numbers k, or only for all $k \leq N$ if the sequence u has only finite nonzero terms and the last nonzero term u_N. It is clear that the following inclusions are true for the above classes of numerical sequences:

$$RBVS \subset AMDS \ , \quad RBVMS \subset AMDMS$$

and

$$HBVS \subset AMIS \ , \quad HBVMS \subset AMIMS.$$

Moreover, Mohapatra and Szal showed that the following embedding relations are true in [20]:

$$AMDS \subset AMDMS$$

and

$$AMIS \subset AMIMS.$$

Besides, it is obviously that the class of nonnegative and nondecreasing (nonincreasing) sequences is a subset of the class of almost monotone decreasing (increasing) sequences.

1.3. **Generalized Lebesgue Spaces $L^{p(x)}$.** Let $\mathcal{P} := \mathcal{P}(\mathbb{R})$ be the family of all measurable 2π-periodic functions $p : \mathbb{R} \to [1, \infty]$. The space $L^{p(x)} := L^{p(x)}([0, 2\pi])$ is the set of all functions f which is measurable 2π-periodic defined on $[0, 2\pi]$ such that $\varrho_p(\lambda f) < \infty$ for some $\lambda := \lambda(f) > 0$ where

$$\varrho_p(f) := \begin{cases} \int\limits_0^{2\pi} |f(x)|^{p(x)} dx & , 1 \leq p(x) < \infty; \\ \operatorname*{ess\,sup}\limits_{x \in [0,2\pi]} |f(x)| & , p(x) = \infty \end{cases}$$

and $p \in \mathcal{P}$. The generalized Lebesgue space $L^{p(x)}$ is a Banach space with the norm $\| \cdot \|_{p(x)}$ defined by

$$\|f\|_{p(x)} := \inf\{\lambda > 0 : \varrho_p(f/\lambda) \leq 1\}.$$

If $p(x) \equiv q$ is a constant ($1 \leq q < \infty$), then the above norm coincides with the usual L^q norm. Some details and further references for the spaces $L^{p(x)}$ can be found in [11], [14], [9]. Given $p \in \mathcal{P}([0, 2\pi])$ with

$$1 < p_* := \operatorname*{ess\,inf}_{x \in [0,2\pi]} p(x) \leq p^* := \operatorname*{ess\,sup}_{x \in [0,2\pi]} p(x) < \infty. \tag{1.3}$$

Let $\mathcal{P}*([0,2\pi])$ be the set of all the functions $p \in \mathcal{P}([0,2\pi])$ with the condition (1.3). The conjugate function of $p \in \mathcal{P}*([0,2\pi])$ is defined by

$$p'(x) := \begin{cases} \frac{p(x)}{p(x)-1} & , p(x) > 1, \\ \infty & , p(x) = 1 \end{cases}$$

and also $p' \in \mathcal{P}*([0,2\pi])$.

In [9, See, Theorem 2.3], it is known that for all functions $f \in L^{p(x)}$ the norm

$$|||f|||_{p(x)} = \sup_{\varrho_{p'}(g) \leq 1} \int_0^{2\pi} |f(x)g(x)|dx$$

is equivalent to the norm $\|f\|_{p(x)}$ with the inequalities

$$\|f\|_{p(x)} \leq |||f|||_{p(x)} \leq r_p \|f\|_{p(x)}$$

where $r_p = 1 + \dfrac{1}{p_*} - \dfrac{1}{p*}$. Therefore the space $L^{p(x)}$ consists of all measurable $2\pi-$periodic functions f with $|||f|||_{p(x)} < \infty$, as well. The Hardy-Littlewood maximal operator M defined on L^1 for each $f \in L^1$ is denoted by the formula

$$Mf(x) = \sup_I \frac{1}{|I|} \int_I |f(t)|dt$$

where the supremum is taken over all intervals I containing $x \in [0,2\pi]$ and $|I|$ denotes the length of I. Informally, the value of the maximal function of f at x is the largest average value of f on any interval I containing x.

In [16], it is shown that if $p \in \mathcal{P}*([0,2\pi])$ and satisfies the *local continuity condition*(or *Dini-Lipshitz condition*)

$$|p(x) - p(y)| \ln \frac{1}{|x-y|} = O(1) \ ; \ 0 < |x-y| \leq 1/2,$$

then the maximal operator M is bounded on $L^{p(x)}$. The set of all the functions $p \in \mathcal{P}*([0,2\pi])$ satisfying the local continuity condition will be denoted by $\mathcal{P}_{loc}$.

Let $p \in \mathcal{P}_{loc}$ and $0 < \alpha \leq 1$. In [18], the Lipschitz class $Lip(\alpha, p(x))$ is defined as

$$Lip(\alpha, p(x)) = \{f \in L^{p(x)} : \Omega_{p(x)}(f,\delta) = O(\delta^\alpha), \delta > 0\}$$

where

$$\Omega_{p(x)}(f,\delta) = \sup_{|h| \leq \delta} \|T_h(f)\|_{p(x)} \tag{1.4}$$

is the integral modulus of continuity of the function $f \in L^{p(x)}$ which is called moduli of mean smoothness, and here

$$T_h(f)(x) = \frac{1}{h} \int_0^h |f(x+t) - f(x)|dt.$$

The moduli of mean smoothness defined in form (1.4) has been given by N. X. Ky [10] and the existence of $\Omega_{p(x)}(f,\delta)$ is based on that the maximal operator M is bounded on $L^{p(x)}$. This definition yields the basic properties of the modulus of continuity.

2. Auxiliary Results

Lemma 1. [21] *The following inequalities are valid:*

$$A_n^\lambda := \sum_{m=1}^{\lambda(n)} \left|\Delta_m \left\{m^{-1}(P_{\lambda(n)} - P_{\lambda(n)-m})\right\}\right| = O(1) \sum_{m=0}^{\lambda(n)-1} |\Delta p_m| \tag{2.1}$$

and if

$$\sum_{m=1}^{\lambda(n)-1} m|\Delta p_m| = O(P_{\lambda(n)})$$

then

$$A_n^\lambda = O\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right). \tag{2.2}$$

Lemma 2. [21] *Let*

$$(p_n) \in AMDMS$$

or

$$(p_n) \in AMIMS \text{ and satisfy } (\lambda(n)+1)p_{\lambda(n)} = O(P_{\lambda(n)}).$$

Then, for $0 < \alpha < 1$,

$$\sum_{m=0}^{\lambda(n)} (m+1)^{-\alpha} p_{\lambda(n)-m} = O((\lambda(n)+1)^{-\alpha} P_{\lambda(n)}). \tag{2.3}$$

Lemma 3. [21] *Let*

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or

$$(p_n) \in AMDMS \text{ and satisfy } (\lambda(n)+1) = O(P_{\lambda(n)}).$$

Then, for $0 < \alpha < 1$,

$$\sum_{m=0}^{\lambda(n)} (m+1)^{-\alpha} p_m = O((\lambda(n)+1)^{-\alpha} P_{\lambda(n)}). \tag{2.4}$$

Lemma 4. [18]. *Let* $p \in \mathcal{P}_{loc}$ *and* $f \in Lip(\alpha, p(x))$ *for* $0 < \alpha \leq 1$. *Then the estimate*

$$\|f - s_n(f)\|_{p(x)} = O(n^{-\alpha})$$

holds for $n = 1, 2, \ldots$.

Lemma 5. [18]. *Let $p \in \mathcal{P}_{loc}$ and $f \in Lip(1, p(x))$. Then*

$$\|\sigma_n(f) - s_n(f)\|_{p(x)} = O(n^{-1})$$

for $n = 1, 2, \ldots$.

3. Degree of Approximation by Nörlund and Riesz Submethods in $L^{p(x)}$

Taking into Subsection 1.3. we shall extend the results given in [18] both by weakening the monotonicity conditions and by using the C_λ-method of their Fourier series of functions that belonging to the class $L^{p(x)}$ for $p : \mathbb{R} \to [1, \infty)$. Especially, we consider the degree of approximation of $f \in L^{p(x)}$ by trigonometrical polynomials $N_n^\lambda(f; x)$ and $R_n^\lambda(f; x)$ under the perspective of [17, 18, 20]. We see that the results obtained in this studying generalize the results in [15, 17, 18].

Theorem 1. *Let $p \in \mathcal{P}_{loc}$, $f \in Lip(\alpha, p(x))$, $0 < \alpha < 1$ and let (p_n) be a positive sequence. If one of the following conditions satisfies*

(i) $(p_n) \in AMIMS$ with

$$(\lambda(n) + 1)p_{\lambda(n)} = O(P_{\lambda(n)}), \tag{3.1}$$

(ii) $(p_n) \in AMDMS$,

then

$$\left\|f - N_n^\lambda(f)\right\|_{p(x)} = O(\lambda(n)^{-\alpha}).$$

Proof. Due to the definition of $N_n^\lambda(f, x)$, we know that

$$N_n^\lambda(f, x) - f(x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} \left\{s_m(f, x) - f(x)\right\}. \tag{3.2}$$

Taking into account hypothesis and by using Lemma 2 and Lemma 4, we get

$$\begin{aligned}
\left\|N_n^\lambda(f) - f\right\|_{p(x)} &\le \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} \left\|s_m(f) - f\right\|_{p(x)} \\
&= \frac{1}{P_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} p_{\lambda(n)-m} \left\|s_m(f) - f\right\|_{p(x)} + \frac{p_{\lambda(n)}}{P_{\lambda(n)}} \left\|s_0(f) - f\right\|_{p(x)} \\
&= \frac{1}{P_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} p_{\lambda(n)-m} O(m^{-\alpha}) + O(\lambda(n)^{-\alpha}) \\
&= \frac{1}{P_{\lambda(n)}} O(\lambda(n)^{-\alpha} P_{\lambda(n)}) + O(\lambda(n)^{-\alpha}) = O(\lambda(n)^{-\alpha}).
\end{aligned}$$

Thus, the proofs of the cases (i) and (ii) are completed together. □

Remark 1. *Theorem 1 generalizes both cases of Theorem 1 given in* [18] *with respect to both monotonicity condition and Cesáro submethod C_λ. Therefore, the results of Chandra* [15] *and Leindler* [17] *are generalized in case $p(x) \equiv 1$.*

Since $AMDS \subset AMDMS$ and $AMIS \subset AMIMS$, we can derive the following result from Theorem 1.

Corollary 1. *Suppose that $p \in \mathcal{P}_{loc}$, $f \in Lip(\alpha, p(x))$, $0 < \alpha < 1$ and let (p_n) be a positive sequence. If one of the following conditions satisfies*

(i) $(p_n) \in AMIS$ and (3.1) holds,

(ii) $(p_n) \in AMDS$,

then

$$\left\| f - N_n^{\lambda}(f) \right\|_{p(x)} = O(\lambda(n)^{-\alpha}).$$

Remark 2. *This corollary will give us the result of Güven and İsrafilov in case of $\lambda(n) = n$,* [18, Theorem 1]*. Moreover a similar corollary can be also written in accordance with the classes $HBVMS$ and $RBVMS$.*

The next result is related with ones that more general than monotone sequences. We note that if (p_n) is nondecreasing with (3.1), then it is clear that

$$\sum_{k=0}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)}/\lambda(n)).$$

On the other hand, if (p_n) is nonincreasing, then

$$\sum_{k=1}^{\lambda(n)-1} k|\Delta p_k| = O(P_{\lambda(n)}).$$

Theorem 2. *Let $p \in \mathcal{P}_{loc}$, $f \in Lip(1, p(x))$ and let (p_n) be a positive sequence. If one of the following conditions satisfies*

(i) $\sum_{k=0}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)}/\lambda(n))$ with (3.1),

(ii) $\sum_{k=1}^{\lambda(n)-1} k|\Delta p_k| = O(P_{\lambda(n)})$,

then

$$\left\| f - N_n^{\lambda}(f) \right\|_{p(x)} = O(n^{-1}).$$

Proof. Let us prove the case (i). Since

$$N_n^{\lambda}(f, x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} P_{\lambda(n)-m} A_m,$$

we have

$$s_n(f, x) - N_n^{\lambda}(f, x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{n} U_m^{\lambda(n)} A_m - \frac{1}{P_{\lambda(n)}} \sum_{m=n+1}^{\lambda(n)} P_{\lambda(n)-m} A_m.$$

where $U_m^{\lambda(n)} = P_{\lambda(n)} - P_{\lambda(n)-m}$. By Abel's Transformation we have

$$\begin{aligned} s_n(f,x) - N_n^{\lambda}(f,x) &= \frac{1}{P_{\lambda(n)}}\left[\sum_{m=1}^{n-1}\Delta_m\Big(\frac{U_m^{\lambda(n)}}{m}\Big)\sum_{k=0}^{m}kA_k + \frac{U_n^{\lambda(n)}}{n}\sum_{k=0}^{m}kA_k\right] - I \\ &= \frac{1}{P_{\lambda(n)}}\left[\sum_{m=1}^{n}\Delta_m\Big(\frac{U_m^{\lambda(n)}}{m}\Big)\sum_{k=0}^{m}kA_k + \frac{U_{n+1}^{\lambda(n)}}{n+1}\sum_{k=0}^{m}kA_k\right] - I \end{aligned}$$

where $I := \frac{1}{P_{\lambda(n)}}\sum_{m=n+1}^{\lambda(n)}\eta_m A_m$ and $\eta_m := P_{\lambda(n)-m}$. Owing to the method in [21, see p.53], we write

$$\left\|s_n(f) - N_n^{\lambda}(f)\right\|_{p(x)} = O(n^{-1}) \tag{3.3}$$

by considering (2.1) of Lemma 1, Lemma 5 and the condition (i) of Theorem 2. Herefrom, by using 3.3 and Lemma 4 we obtain

$$\left\|f - N_n^{\lambda}(f)\right\|_{p(x)} = O(n^{-1}).$$

for the case (i). Since the case (ii) is proved by similar way, we will omit its proof here. □

Remark 3. *In case $\lambda(n) = n$ in the Theorem 2 this result coincides with the result of* [18, Theorem 2].

We know that if $(p_n) \in RBVS$ with condition $(\lambda(n)+1) = O(P_{\lambda(n)})$, then

$$\sum_{k=1}^{\lambda(n)-1} k|\Delta p_k| = O(P_{\lambda(n)}).$$

Therefore due to Theorem 2-(ii) and above relation, we can write the following result.

Corollary 2. *Let $p \in \mathcal{P}_{loc}$, $f \in Lip(1, p(x))$. If $(p_n) \in RBVS$ and the condition $(\lambda(n)+1) = O(P_{\lambda(n)})$, then*

$$\left\|f - N_n^{\lambda}(f)\right\|_{p(x)} = O(n^{-1}).$$

Theorem 3. *Let $p \in \mathcal{P}_{loc}$, $f \in Lip(\alpha, p(x))$, $0 < \alpha < 1$ and let (p_n) be a positive sequence. If one of the following conditions satisfies*
(i) $(p_n) \in AMDMS$ with $(\lambda(n)+1) = O(P_{\lambda(n)})$,
(ii) $(p_n) \in AMIMS$,
then

$$\left\|f - R_n^{\lambda}(f)\right\|_{p(x)} = O(\lambda(n)^{-\alpha}).$$

Proof. By the definition of Riesz submethod, we have

$$f(x) - R_n^{\lambda}(f;x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_m(f(x) - s_m(f,x)).$$

The expected result is obtained as follows from Lemma 3 and Lemma 4.

$$\|f - R_n^{\lambda}(f)\|_{p(x)} \leq \frac{1}{P_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} p_m \|f - s_m(f)\|_{p(x)} + \frac{1}{P_{\lambda(n)}} p_0 \|f - s_m(f)\|_{p(x)}$$

$$= \frac{O(1)}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_m (m+1)^{-\alpha} + O(\frac{1}{\lambda(n)+1}) = O(\lambda(n)^{-\alpha}).$$

□

Theorem 6. *Let $p \in \mathcal{P}_{loc}$, $f \in Lip(\alpha, p(x))$, $0 < \alpha \leq 1$ and let (p_n) be a positive sequence. If the following condition satisfies*

$$\sum_{m=0}^{\lambda(n)-1} |\Delta(\frac{P_m}{m+1})| = O(\frac{P_{\lambda(n)}}{\lambda(n)+1})$$

then

$$\left\|f - R_n^{\lambda}(f)\right\|_{p(x)} = O(\lambda(n)^{-\alpha}).$$

Proof. We will consider the method which used in [19]. Let us start for $0 < \alpha < 1$. Since

$$f(x) - R_n^{\lambda}(f;x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_m(f(x) - s_m(f,x))$$

we have

$$\|f - R_n^{\lambda}(f)\|_{p(x)} \leq$$

$$= \frac{1}{P_{\lambda(n)}} p_0 \|f - s_0(f)\|_{p(x)} + \frac{1}{P_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} p_m \|f - s_m(f)\|_{p(x)}$$

$$\leq O(\frac{1}{P_{\lambda(n)}}) \sum_{m=1}^{\lambda(n)} m^{-\alpha} p_m \tag{3.4}$$

by Lemma 4. By using Abel's transformation, we get

$$\sum_{m=1}^{\lambda(n)} m^{-\alpha} p_m = \sum_{m=1}^{\lambda(n)-1} \Delta(m^{-\alpha}) P_m + (\lambda(n))^{-\alpha} P_{\lambda(n)}. \tag{3.5}$$

By considering (3.4), we write

$$\sum_{m=1}^{\lambda(n)-1} \Delta(m^{-\alpha})P_m \leq (\sum_{k=1}^{\lambda(n)} k^{-\alpha})[\sum_{m=1}^{\lambda(n)-1} \Delta(\frac{P_m}{m+1}) + \frac{P_{\lambda(n)}}{(\lambda(n)+1)}]$$
$$\leq O[(\lambda(n))^{1-\alpha}\frac{P_{\lambda(n)}}{\lambda(n)+1}] \quad (3.6)$$

Taking into account of (3.4)-(3.6), we obtain

$$\left\|f - R_n^{\lambda}(f)\right\|_{p(x)} = O(\lambda(n)^{-\alpha}).$$

Suppose that $\alpha = 1$. If we apply Abel's transformation to the sum $R_n^{\lambda}(f;x)$, then we write

$$R_n^{\lambda}(f;x) = -\frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)-1} P_m A_{m+1}(f,x) + s_{\lambda(n)}(f;x).$$

Since

$$s_{\lambda(n)}(f;x) = s_n(f;x) + \sum_{m=n+1}^{\lambda(n)} A_m(f;x),$$

we have

$$R_n^{\lambda}(f) - s_n(f) = -\frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)} P_m A_{m+1} + \sum_{m=n+1}^{\lambda(n)+1} A_m.$$

In [19], we know that

$$\|R_n^{\lambda}(f) - s_n(f)\|_{p(x)} \leq \frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)-1} |\Delta(\frac{P_m}{m+1})|\|\sum_{k=0}^{m}(k+1)A_{k+1}\|_{p(x)}$$
$$+ \sum_{m=n+1}^{\lambda(n)} |\Delta(\frac{1}{m})|\|\sum_{k=1}^{m} kA_k\|_{p(x)} + \frac{1}{n+1}\|\sum_{k=1}^{n} kA_k\|_{p(x)} (3.7)$$

Since

$$\|\sum_{k=0}^{n}(k+1)A_{k+1}(f,x)\|_{p(x)} = O(1)$$

by Lemma 5, we have

$$\|R_n^{\lambda}(f) - s_n(f)\|_{p(x)} = O(n^{-1}). \quad (3.8)$$

from (3.4), (3.7) and the condition of Theorem 6. Therefore the expected result is obtained by considering (3.8) and Lemma 4. □

Remark 4. *Theorem 6 generalizes Theorem 3 given in* [18] *with respect to Riesz submethod.*

4. Some Inclusions Related to Nörlund and Riesz Submethods

The purpose of this section is to reveal the importance of the submethods given in Subsection 1.1. Now let us give some inclusions due to these methods. Assume that $E = \{\lambda(n)\}_{n=1}^{\infty}$ and $F = \{\mu(n)\}_{n=1}^{\infty}$ be infinite subsets of $\mathbb{N}$.

Theorem 4. *$N_n^{\lambda} \subseteq N_n^{\mu}$ if and only if $F \setminus E$ is finite where $p_{\lambda(n)}/P_{\lambda(n)} \to 0$ and $p_{\mu(n)}/P_{\mu(n)} \to 0$ as $n \to \infty$.*

Proof. If the set $F \setminus E$ is finite then there exists an integer N such that $\{\mu(n)\}_{n=N}^{\infty} \subset E$. Let $\{k(n)\}$ be a sequence such that for $n \geq N$, $\mu(n) = \lambda_{k(n)}$. Hence $\{k(n)\}$ is an increasing sequence and $k(n) \to \infty$ as $n \to \infty$. Since $p_{\lambda(n)}/P_{\lambda(n)} \to 0$ and $p_{\mu(n)}/P_{\mu(n)} \to 0$ as $n \to \infty$, $N_n^{\lambda} \to \ell$ implies $N_{k(n)}^{\lambda} \to \ell$, and therefore we get $N_n^{\mu} \to \ell$ from definition of Nörlund method. Now, suppose that $N_n^{\lambda} \subseteq N_n^{\mu}$ but the set $F \setminus E$ is infinite. The proof of the theorem will be complete if we can show that there is a sequence $\{s_m\}$ which converges to 0 meaning N_n^{λ}, but not converges meaning N_n^{μ}. Since the set $F \setminus E$ is infinite, there is a strictly increasing sequence $\{\mu_{n(k)}\}$ such that $\mu_{n(k)} \notin E$ for $k = 1, 2, \ldots$. If now

$$c_n = \begin{cases} 0, & n \neq \mu_{n(k)}; \\ (-1)^k, & n = \mu_{n(k)}, \end{cases}$$

and $\{s_m\}$ is the sequence corresponding to $\{c_n\}$ then clearly $s_m \to 0$ meaning N_n^{λ}, but not converges meaning N_n^{μ} as $n \to \infty$ which contradicts the fact that $N_n^{\lambda} \subseteq N_n^{\mu}$. □

Taking into account of Theorem 4 since $E \triangle F = (E \setminus F) \cup (F \setminus E)$, we can write the following result.

Theorem 5. *$N_n^{\lambda} \sim N_n^{\mu}$ if and only if $F \triangle E$ is finite where $p_{\lambda(n)}/P_{\lambda(n)} \to 0$ and $p_{\mu(n)}/P_{\mu(n)} \to 0$ as $n \to \infty$.*

Remark 5. *Especially, we see that $N_n \subseteq N_n^{\mu}$ for any μ. For, the set $F \setminus \{0, 1, 2, \ldots\}$ is empty.*

Remark 6. *The similar results can be also written for the Riesz submethod. In this case we note that $P_{\lambda(n)} \to \infty$ and $P_{\mu(n)} \to \infty$ as $n \to \infty$ instead of $p_{\lambda(n)}/P_{\lambda(n)} \to 0$ and $p_{\mu(n)}/P_{\mu(n)} \to 0$ as $n \to \infty$ in Theorem 4 and Theorem 5, respectively.*

Theorem 6. *Let $\{p_n\}$ be a positive nonincreasing sequence. If*

$$\limsup_{n \to \infty} \frac{\lambda(n+1) - \lambda(n)}{P_{\lambda(n)}} = 0 \tag{4.1}$$

then the N_n^{λ}-method is equivalent to the N_n-method for bounded sequences.

Proof. Let $\{p_n\}$ be a positive nonincreasing sequence, $\{s_n\}$ be a bounded sequence, and N_n be Nörlund transform of $\{s_n\}$. Therefore, for every n and k, we have

$$|N_n - N_{n+k}| = \left| \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} s_m - \frac{1}{P_{n+k}} \sum_{m=0}^{n+k} p_{n+k-m} s_m \right|$$
$$\leq \sum_{m=0}^{n} M \left| \frac{p_{n-m}}{P_n} - \frac{p_{n+k-m}}{P_{n+k}} \right| + M \sum_{m=n+1}^{n+k} \frac{p_{n+k-m}}{P_{n+k}}.$$

Since $\{p_n\}$ is a positive nonincreasing sequence, we get

$$|N_n - N_{n+k}| \leq M \left(1 - \sum_{m=0}^{n} \frac{p_{n+k-m}}{P_{n+k}} \right) + M \sum_{m=n+1}^{n+k} \frac{p_{n+k-m}}{P_{n+k}}$$
$$= 2M \sum_{m=n+1}^{n+k} \frac{p_{n+k-m}}{P_{n+k}} \leq \frac{2Mp_0 k}{P_{n+k}}. \tag{4.2}$$

Assume that $\{s_n\}$ is summable by means of the N_n^λ-method. Let $\varepsilon > 0$. Then there is a k such that $|N_k^\lambda - N_j^\lambda| < \frac{\varepsilon}{2}$ for every $j > k$. On the other hand, we obtain

$$|N_j^\lambda - N_n| < \frac{2Mp_0(n - \lambda(j))}{P_n} < \frac{2Mp_0(\lambda(j+1) - \lambda(j))}{P_{\lambda(j)}} < \frac{\varepsilon}{2}$$

by virtue of (4.1) and (4.2) where $\lambda(j) \leq n < \lambda(j+1)$. Therefore it is concluded that $|N_k^\lambda - N_n| < \varepsilon$ for every $n > \lambda(k)$. Hence $\{s_n\}$ is summable by the N_n-method. We know that $N_n \subseteq N_n^\lambda$ for any λ. So the proof of the theorem is completed. □

Remark 7. *The similar theorem for Riesz method can be written by taking*

$$\limsup_{n\to\infty} \frac{P_{\lambda(n+1)} - P_{\lambda(n)}}{P_{\lambda(n)}} = 0$$

instead of the condition (4.1) in Theorem 6. In this case, we don't need put any monotonicity condition on the sequence $\{p_n\}$.

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