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APPLICATION OF THE $(G^{'}/G)$-EXPANSION METHOD FOR SOME SPACE-TIME FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, the $(G^{'}/G)$-expansion method is presented for finding the exact solutions of the space-time fractional traveling wave solutions for the Joseph-Egri (TRLW) equation and Gardner equation. The fractional derivatives are described by modified Riemann-Liouville sense. Many exact solutions are obtained by the hyperbolic functions, the trigonometric functions and the rational functions. This method is efficient and powerful in performing a solution to the fractional partial differential equations. Also, the method reduces the large amount of calculations.

1. Introduction

In recent years, fractional partial differential equations which are generalizations of classical partial differential equations of integer order have been the focus of many studies [1, 2, 3]. Many powerful methods for obtaining the exact solutions of fractional partial differential equations, such as the fractional the $(G^{'}/G)$-expansion method [4, 5, 6, 7], the fractional first integral method [8, 9], the fractional exp-function method [10, 11, 12], the fractional functional variable method [13] and the fractional sub-equation method [14, 15] have been developed to find exact analytic solutions.

In this paper, the $(G^{'}/G)$-expansion method [16, 17] to solve nonlinear fractional differential equations in the sense of modified Riemann-Liouville derivative by Jumarie is used [18]. The Jumarie's modified Riemann-Liouville derivative of order α is defined by

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$$D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int\limits_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, \ 0<\alpha<1 \\ (f^{(n)}(t))^{\alpha-n} \ , n \leq \alpha < n+1, \ n \geq 1 \end{cases} \tag{1}$$

Some important properties of the fractional modified Riemann-Liouville derivative were given [19] as

$$D_t^{\alpha} x^{\beta} = \frac{\gamma(1+\beta)}{\gamma(1+\beta-\alpha)x^{\beta-\alpha}}, \ \beta > 0 \tag{2}$$

$$D_x^{\alpha}(u(x)v(x)) = v(x)D_x^{\alpha}u(x) + u(x)D_x^{\alpha}v(x) \tag{3}$$

$$D_x^{\alpha}[f(u(x))] = f'_u(u)D_x^{\alpha}u(x) \tag{4}$$

$$D_x^{\alpha}[f(u(x))] = D_u^{\alpha}f(u)(u'^{\alpha} \tag{5}$$

Consider the following general fractional partial differential equations

$$P(u, D_t^{\alpha}u, D_x^{\beta}u, D_t^{2\alpha}u, D_t^{\alpha}D_x^{\beta}u, D_x^{2\beta}u, ...) = 0$$
$$0 < \alpha, \beta < 1 \tag{6}$$

where $u = u(x,t)$ is an unknown function, and P is a polynomial of $u = u(x,t)$ and its partial fractional derivatives, in which the highest order derivatives and the nonlinear terms are involved.

Li and He [20, 21] proposed a fractional complex transform to convert fractional differential equations into ordinary differential equations, so all analytical methods which are devoted to the advanced calculus can be easily applied to the fractional calculus. By using traveling wave variable

$$u(x,t) = U(\xi) \tag{7}$$

$$\xi = \frac{cx^{\beta}}{\Gamma(1+\beta)} - \frac{kx^{\alpha}}{\Gamma(1+\alpha)} \tag{8}$$

where k and c are nonzero arbitrary constants, and Eq. (6) can be written as follows:

$$Q(U, U', U'', U''', ...) = 0. \tag{9}$$

where the prime denotes the derivation with respect to ξ. If the possibility has, then Eq.(9) can be integrated term by term one or more times.

Suppose that the solution of Eq.(9) can be expressed by a polynomial in (G'/G) in the form:

$$U(\xi) = \sum_{i=0}^{m} a_i \Big(\frac{G'}{G}\Big)^i, a_m \neq 0 \tag{10}$$

where $a_i (i = 0, 1, 2, ..., m)$ are constants, while $G(\xi)$ satisfies the following second-order linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \tag{11}$$

with λ and μ are being constants.

The positive integer m can be found by balancing the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq.(9). Substituting Eq.(10) into Eq.(9) and using Eq.(11) and equating each coefficient of the resulting polynomial to zero, a set of algebraic equations for $a_i (i = 0, 1, 2, ..., m)$, λ, μ, k and c is obtained.

Solving the equation system, and substituting $a_i (i = 0, 1, 2, ..., m), \lambda, \mu, k, c$ and the general solutions of Eq.(11) into Eq.(10), a variety of exact solutions of Eq.(6) can be obtained.

2. The space-time fractional Joseph-Egri(TRLW) equation

Consider the following space-time fractional Joseph-Egri (TRLW) equation [22]

$$\begin{gathered} D_t^{\alpha} u + D_x^{\beta} u + \gamma u D_x^{\beta} u + D_x^{\beta} D_t^{2\alpha} u = 0, \ t > 0, \\ 0 < \alpha, \beta \leq 1, \ x > 0 \end{gathered} \tag{12}$$

where γ is a constant.

Substituting Eqs.(7)-(8) into Eq.(12), the following ordinary differential equation can be obtained

$$(c - k)U' + \gamma c U U'^2 U''' = 0 \tag{13}$$

where $U' = \frac{dU}{d\xi}$. By once integrating and setting the constants of integration to zero,

$$(c - k)U + \gamma c \frac{U^2}{2} + ck^2 U'' = 0 \tag{14}$$

is obtained.

For the linear term of highest order U'' with the highest order nonlinear term U^2, balancing the two term in Eq. (14) gives

$$m + 2 = 2m \tag{15}$$

so that

$$m = 2. \tag{16}$$

Assuming that the solutions of Eq.(14) can be expressed by a polynomial in $(G^{'}/G)$ as

$$U(\xi) = a_0 + a_1 \Big(\frac{G^{'}}{G}\Big) + a_2 \Big(\frac{G^{'}}{G}\Big)^2, \ a_2 \neq 0 \tag{17}$$

By using Eq.(11), from Eq.(17), it is derived that

$$\begin{aligned} U^{''}(\xi) = {} & 2a_2\mu^2 + a_1\lambda\mu + (6a_2\lambda\mu + 2a_1\mu + a_1\lambda^2)\Big(\frac{G^{'}}{G}\Big) \\ & +(8a_2\mu + 3a_1\lambda + 4a_2\lambda^2)\Big(\frac{G^{'}}{G}\Big)^2 \\ & +(2a_1 + 10a_2\lambda)\Big(\frac{G^{'}}{G}\Big)^3 + 6a_2\Big(\frac{G^{'}}{G}\Big)^4 \end{aligned} \tag{18}$$

and

$$U^2(\xi) = a_0^2 + 2a_0a_1\Big(\frac{G'}{G}\Big) + (2a_0a_2 + a_1^2)\Big(\frac{G'}{G}\Big)^2$$
$$+2a_1a_2\Big(\frac{G'}{G}\Big)^3 + a_2^2\Big(\frac{G'}{G}\Big)^4 \quad (19)$$

Substituting Eqs.(17)-(19) into Eq.(14), collecting the coefficients of $\left(\frac{G'}{G}\right)^i$ $(i = 0, 1, 2)$ and set it to zero, the following system is obtained:

$$(c-k)a_0 + \frac{\gamma}{2}ca_0^2 + 2ck^2a_2\mu^2 + ck^2a_1\lambda\mu = 0,$$
$$(c-k)a_1 + \gamma ca_0a_1 + 6ck^2a_2\lambda\mu + 2ck^2a_1\mu + ck^2a_1\lambda^2 = 0,$$
$$(c-k)a_2 + \frac{\gamma}{2}ca_1^2 + \gamma ca_0a_2 + 8ck^2a_2\mu$$
$$+3ck^2a_1\lambda + 4ck^2a_2\lambda^2 = 0,$$
$$\gamma ca_1a_2 + 2ck^2a_1 + 10ck^2a_2\lambda = 0,$$
$$\frac{\gamma}{2}ca_2 + 6ck^2 = 0 \quad (20)$$

Solving this system gives

$$a_1 = \frac{-12\lambda c^2}{\gamma\sqrt{-\lambda^2c^2 + 4\mu c^2 + 1}},\ a_2 = \frac{-12c^2}{\gamma\sqrt{-\lambda^2c^2 + 4\mu c^2 + 1}},$$
$$a_0 = \frac{-2\lambda^2c^2 - 4\mu c^2}{\gamma},\ k = \frac{c}{-\lambda^2c^2 + 4\mu c^2 + 1},\ c = c \quad (21)$$

where λ and μ, are arbitrary constants.

By using Eq.(21) expression Eq.(17) can be written as

$$U(\xi) = \frac{-2\lambda^2c^2 - 4\mu c^2}{\gamma} - \frac{12\lambda c^2}{\gamma\sqrt{-\lambda^2c^2 + 4\mu c^2 + 1}}\Big(\frac{G'}{G}\Big)$$
$$-\frac{12c^2}{\gamma\sqrt{-\lambda^2c^2 + 4\mu c^2 + 1}}\Big(\frac{G'}{G}\Big)^2 \quad (22)$$

Substituting general solutions of Eq.(11) into Eq.(22) three types of traveling wave solutions of the space-time fractional Joseph-Egri(TRLW) equation are obtained as follows:

When $\lambda^2 - 4\mu > 0$

$$U_{1,2}(\xi) = \frac{-2c^2(\lambda^2 + 2\mu)}{\gamma} + \frac{3c^2\lambda^2}{\gamma\sqrt{1 - c^2(\lambda^2 - 4\mu)}}$$
$$-\frac{3c^2(\lambda^2 - 4\mu)}{\gamma\sqrt{1 - c^2(\lambda^2 - 4\mu)}}\left(\frac{K_1 sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + K_2 cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{K_1 cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + K_2 sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right) \quad (23)$$

where $\xi = \frac{cx^\beta}{\Gamma(1+\beta)} - \frac{c}{1-c^2(\lambda^2-4\mu)}\frac{t^\alpha}{\Gamma(1+\alpha)}$.

When $\lambda^2 - 4\mu < 0$

$$U_{3,4}(\xi) = \frac{-2c^2(\lambda^2 + 2\mu)}{\gamma} + \frac{3c^2\lambda^2}{\gamma\sqrt{1 + c^2(4\mu - \lambda^2)}}$$
$$-\frac{3c^2(4\mu - \lambda^2)}{\gamma\sqrt{1 + c^2(4\mu - \lambda^2)}}\left(\frac{-K_1 sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + K_2 cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{K_1 cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + K_2 sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\right) \quad (24)$$

where $\xi = \frac{cx^\beta}{\Gamma(1+\beta)} - \frac{c}{1-c^2(\lambda^2-4\mu)}\frac{t^\alpha}{\Gamma(1+\alpha)}$.

When $\lambda^2 - 4\mu = 0$

$$U_{5,6}(\xi) = \frac{-2c^2(\lambda^2 + 2\mu)}{\gamma} - \frac{6c^2\lambda^2}{\gamma\sqrt{1 - c^2(\lambda^2 - 4\mu)}}$$
$$-\frac{12c^2}{\gamma\sqrt{1 - c^2(\lambda^2 - 4\mu)}}\frac{K_2}{K_1 + K_2\xi} \quad (25)$$

where $\xi = \frac{cx^\beta}{\Gamma(1+\beta)} - \frac{c}{1-c^2(\lambda^2-4\mu)}\frac{t^\alpha}{\Gamma(1+\alpha)}$.

3. The space-time fractional Gardner equation

Consider the following space-time fractional Gardner equation [23, 24]

$$D_t^\alpha u = 6uD_x^\beta u + 6\varepsilon^2 u^2 D_x^\beta u + D_x^{3\beta} u,\ t > 0,$$
$$0 < \alpha, \beta \leq 1,\ x > 0 \quad (26)$$

where ε is a constant.

Substituting Eqs.(7)-(8) into Eq.(26) the ordinary differential equation can be obtained as follows:

$$-kU' - 6cUU'^2 cU^2 U'^3 U''' = 0 \quad (27)$$

where $U' = \frac{dU}{d\xi}$. By once integrating and setting the constants of integration to zero,

$$kU + 3cU^2 + 2\varepsilon^2 cU^3 + c^3 U'' + C_0 = 0 \quad (28)$$

is obtained.

For the linear term of highest order U'' with the highest order nonlinear term U^3, balancing the two term in Eq. (28) gives

$$m + 2 = 3m \tag{29}$$

so that

$$m = 1. \tag{30}$$

Assuming that the solutions of Eq. (28) can be expressed by a polynomial in $(G^{'}/G)$ as

$$U(\xi) = a_0 + a_1\Big(\frac{G^{'}}{G}\Big),\ a_1 \neq 0 \tag{31}$$

By using Eq.(11), from Eq.(31), it is derived that

$$U''(\xi) = a_1\lambda\mu + (2a_1\mu + a_1\lambda^2)\Big(\frac{G^{'}}{G}\Big) + 3a_1\lambda\Big(\frac{G^{'}}{G}\Big)^2 + 2a_1\Big(\frac{G^{'}}{G}\Big)^3 \tag{32}$$

and

$$U^2(\xi) = a_0^2 + 2a_0a_1\Big(\frac{G^{'}}{G}\Big) + a_1^2\Big(\frac{G^{'}}{G}\Big)^2 \tag{33}$$

and

$$U^3(\xi) = a_0^3 + 3a_0^2a_1\Big(\frac{G^{'}}{G}\Big) + 3a_0a_1^2\Big(\frac{G^{'}}{G}\Big)^2 + a_1^3\Big(\frac{G^{'}}{G}\Big)^3 \tag{34}$$

Substituting Eqs.(32)-(34) into Eq.(28), collecting the coefficients of $\left(\frac{G^{'}}{G}\right)^i$ $(i = 0, 1)$ and set it to zero, the following system is obtained:

$$\begin{aligned} &ka_0 + 3ca_0^2 + 2\varepsilon^2ca_0^3 + c^3a_1\lambda\mu + C_0 = 0,\\ &ka_1 + 6ca_0a_1 + 6\varepsilon^2ca_0^2a_1 + c^3a_1\lambda^2 + 2c^3a_1\mu = 0,\\ &3ca_1^2 + 6\varepsilon^2ca_0a_1^2 + 3c^3a_1\lambda = 0,\\ &2\varepsilon^2ca_1^3 + 2c^3a_1 = 0. \end{aligned} \tag{35}$$

Solving this system gives

$$\begin{aligned} &a_1 = \mp\frac{ci}{\varepsilon},\ a_0 = \frac{-1 \mp c\varepsilon\lambda i}{2\varepsilon^2},\ k = \frac{c^3}{4\varepsilon^2}(\lambda^2 - 4\mu) + \frac{c}{4\varepsilon^4},\\ &c = c,\ C_0 = \frac{c^3}{2}(\lambda^2 - 4\mu) + \frac{3c}{2\varepsilon^2} \end{aligned} \tag{36}$$

where λ and μ, are arbitrary constants.

By using Eq.(36) expression Eq.(31) can be written as

$$U(\xi) = \frac{-1 \mp c\varepsilon\lambda i}{2\varepsilon^2} \mp \frac{ci}{\varepsilon}\Big(\frac{G^{'}}{G}\Big) \tag{37}$$

Substituting general solutions of Eq.(11) into Eq.(37) three types of traveling wave solutions of the space-time fractional Gardner equation are obtained as follows:

When $\lambda^2 - 4\mu > 0$

$$U_{1,2}(\xi)=\frac{-1}{2\varepsilon^2}$$
$$\mp\frac{ci\sqrt{\lambda^2-4\mu}}{2\varepsilon}\left(\frac{K_1sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+K_2cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{K_1cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+K_2sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right) \tag{38}$$

where $\xi=\frac{cx^\beta}{\Gamma(1+\beta)}-[\frac{c^3}{4\varepsilon^2}(\lambda^2-4\mu)+\frac{c}{4\varepsilon^4}]\frac{t^\alpha}{\Gamma(1+\alpha)}$.

When $\lambda^2-4\mu<0$

$$U_{3,4}(\xi)=\frac{-1}{2\varepsilon^2}$$
$$\mp\frac{ci\sqrt{4\mu-\lambda^2}}{2\varepsilon}\left(\frac{-K_1sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+K_2cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{K_1cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+K_2sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\right) \tag{39}$$

where $\xi=\frac{cx^\beta}{\Gamma(1+\beta)}-[\frac{c^3}{4\varepsilon^2}(\lambda^2-4\mu)+\frac{c}{4\varepsilon^4}]\frac{t^\alpha}{\Gamma(1+\alpha)}$.

When $\lambda^2-4\mu=0$

$$U_{5,6}(\xi)=\frac{-1\mp c\varepsilon\lambda i}{2\varepsilon^2}\mp\frac{ci}{\varepsilon}\frac{K_2}{K_1+K_2\xi} \tag{40}$$

where $\xi=\frac{cx^\beta}{\Gamma(1+\beta)}-[\frac{c^3}{4\varepsilon^2}(\lambda^2-4\mu)+\frac{c}{4\varepsilon^4}]\frac{t^\alpha}{\Gamma(1+\alpha)}$.

4. Conclusion

In this paper, three types of exact analytical solutions including the generalized hyperbolic, trigonometric and rational function solutions for the space-time fractional Joseph-Egri(TRLW) and Gardner equation are presented by using the (G'/G)-expansion method. It can be concluded that this method is very simple, reliable and proposes a variety of exact solutions to space-time fractional partial differential equation.

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