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SEMI-INVARIANT SEMI-RIEMANNIAN SUBMERSIONS

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Abstract. In this paper, we introduce semi-invariant semi-Riemannian submersions from para-Kähler manifolds onto semi-Riemannian manifolds. We give some examples, investigate the geometry of foliations that arise from the definition of a semi-Riemannian submersion and check the harmonicity of such submersions. We also find necessary and sufficient conditions for a semi-invariant semi-Riemannian submersion to be totally geodesic. Moreover, we obtain curvature relations between the base manifold and the total manifold.

1. Introduction

The theory of Riemannian submersion was introduced by O'Neill and Gray in [19] and [13], respectively. Later, Riemannian submersions were considered between almost complex manifolds by Watson in [26] under the name of almost Hermitian submersion. He showed that if the total manifold is a Kähler manifold, then the base manifold is also a Kähler manifold. Since then, Riemannian submersions have been used as an effective tool to describe the structure of a Riemannian manifold equipped with a differentiable structure. Presently, there is an extensive literature on the Riemannian submersions with different conditions imposed on the total space and on the fibres. For instance, Riemannian submersions between almost contact manifolds were studied by Chinea in [5] under the name of almost contact submersions. Riemannian submersions have been also considered for quaternionic Kähler manifolds [14] and para-quaternionic Kähler manifolds [4],[15]. This kind of submersions have been studied with different names by many authors (see [1], [10], [12], [21], [22], [23], [24] and more).

On the other hand, para-complex manifolds, almost para-Hermitian manifolds and para-Kähler manifolds were defined by Libermann [18] in 1952. In fact, such manifolds arose in [25] (see also [6]). Indeed, Rashevskij introduced the properties of para-Kähler manifolds, when he considered a metric of signature (m, m) defined

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from a potential function the so-called scalar field on a $2m-$dimensional locally product manifold called by him stratified space.

2. Preliminaries

In this section, we define almost para-Hermitian manifolds, recall the notion of semi-Riemannian submersions between semi-Riemannian manifolds and give a brief review of basic facts of semi-Riemannian submersions.

An almost para-Hermitian manifold is a manifold M endowed with an almost para-complex structure $P \neq \pm I$ and a semi-Riemannian metric g such that

$$P^2 = I, \quad g(PX, PY) = -g(X, Y) \tag{2.1}$$

for X, Y tangent to M, where I is the identity map. The dimension of M is even and the signature of g is (m,m), where $dimM = 2m$. Consider an almost para-Hermitian manifold (M, P, g) and denote by ∇ the Levi-Civita connection on M with respect to g. Then M is called a para-Kähler manifold if P is parallel with respect to ∇, i.e.,

$$(\nabla_X P)Y = 0 \tag{2.2}$$

for X, Y tangent to M[17].

Let (M, g_1) and (N, g_2) be two connected semi-Riemannian manifolds of index $s(0 \leq s \leq dimM)$ and $s'(0 \leq s' \leq dimN)$ respectively, with $s > s'$. A semi-Riemannian submersion is a smooth map $\pi : M \to N$ which is onto and satisfies the following conditions:
(i) $\pi_{*p} : T_pM \to T_{\pi(p)}N$ is onto for all $p \in M$;
(ii) The fibres $\pi^{-1}(q), q \in N$, are semi-Riemannian submanifolds of M;
(iii) π_* preserves scalar products of vectors normal to fibres.
The vectors tangent to the fibres are called vertical and those normal to the fibres are called horizontal. We denote by $\mathcal{V}$ the vertical distribution, by $\mathcal{H}$ the horizontal distribution and by v and h the vertical and horizontal projection. A horizontal vector field X on M is said to be *basic* if X is $\pi-$related to a vector field X_* on N. It is clear that every vector field X_* on N has a unique horizontal lift X to M and X is *basic*.

We recall that the sections of $\mathcal{V}$, respectively $\mathcal{H}$, are called the vertical vector fields, respectively horizontal vector fields. A semi-Riemannian submersion $\pi : M \to N$ determines two $(1, 2)$ tensor fields T and A on M, by the formulas:

$$T(E, F) = T_E F = h\nabla^1_{vE} vF + v\nabla^1_{vE} hF \tag{2.3}$$

and

$$A(E, F) = A_E F = v\nabla^1_{hE} hF + h\nabla^1_{hE} vF \tag{2.4}$$

for any $E, F \in \Gamma(TM)$, where v and h are the vertical and horizontal projections (see [2],[8]). From (2.3) and (2.4), one can obtain

$$\nabla^1_U W = T_U W + \hat{\nabla}_U W; \tag{2.5}$$

$$\nabla^1_U X = T_U X + h(\nabla^1_U X); \tag{2.6}$$

$$\nabla^1_X U = v(\nabla^1_X U) + A_X U; \tag{2.7}$$

$$\nabla^1_X Y = A_X Y + h(\nabla^1_X Y), \tag{2.8}$$

for any $X, Y \in \Gamma((ker\pi_*)^\perp)$, $U, W \in \Gamma(ker\pi_*)$. Moreover, if X is basic then $h(\nabla^1_U X) = h(\nabla^1_X U) = A_X U$.
We note that for $U, V \in \Gamma(ker\pi_*)$, $T_U V$ coincides with the second fundamental form of the immersion of the fibre submanifolds and for $X, Y \in \Gamma((ker\pi_*)^\perp)$, $A_X Y = \frac{1}{2} v[X, Y]$ reflecting the complete integrability of the horizontal distribution $\mathcal{H}$. It is known that A is alternating on the horizontal distribution: $A_X Y = -A_Y X$, for $X, Y \in \Gamma((ker\pi_*)^\perp)$ and T is symmetric on the vertical distribution: $T_U V = T_V U$, for $U, V \in \Gamma(ker\pi_*)$.

We now recall the following result which will be useful for later.

Lemma 2.1 (see [8],[20]). *If $\pi : M \to N$ is a semi-Riemannian submersion and X, Y basic vector fields on M, $\pi-$related to X_* and Y_* on N, then we have the following properties*

1. $g_1(X, Y) = g_2(X_*, Y_*) \circ \pi$;
2. $h[X, Y]$ *is a basic vector field and* $\pi_* h[X, Y] = [X_*, Y_*] \circ \pi$;
3. $h(\nabla^1_X Y)$ *is a basic vector field $\pi-$related to* $(\nabla^2_{X_*} Y_*)$, *where ∇^1 and ∇^2 are the Levi-Civita connection on M and N;*
4. $[E, U] \in \Gamma(ker\pi_*)$, *for any* $U \in \Gamma(ker\pi_*)$ *and for any basic vector field E.*

Let (M, g_1) and (N, g_2) be (semi-)Riemannian manifolds and $\pi : M \to N$ is a smooth map. Then the second fundamental form of π is given by

$$(\nabla\pi_*)(X, Y) = \nabla^\pi_X \pi_* Y - \pi_*(\nabla^1_X Y) \tag{2.9}$$

for $X, Y \in \Gamma(TM)$, where we denote conveniently by ∇ the Levi-Civita connections of the metrics g and $g^{'}$. Recall that π is called a *totally geodesic* map if $(\nabla\pi_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$[16]. It is known that the second fundamental form is symmetric.

3. Semi-invariant semi-Riemannian submersions

In this section, we define semi-invariant semi-Riemannian submersions from a para-Kähler manifold onto a semi-Riemannian manifold, investigate the integrability of distributions and obtain a necessary and sufficient condition for such submersions to be totally geodesic map.

Definition 3.1. Let (M, g_1, P) be an almost para-Hermitian manifold and (N, g_2) a semi-Riemannian manifold. A semi-Riemannian submersion $\pi : M \to N$ is called a semi-invariant semi-Riemannian submersion if there is a distribution $D_1 \subseteq ker\pi_*$ such that

$$ker\pi_* = D_1 \oplus D_2 \ \ and \ \ PD_1 = D_1, P(D_2) \subseteq (ker\pi_*)^{\perp}$$

where D_2 is orthogonal complementary to D_1 in $ker\pi_*$.

We note that it is known that the distribution $ker\pi_*$ is integrable. Hence, Definition 3.1 implies that the integral manifold (fibre) $\pi^{-1}, q \in B$, of $ker\pi_*$ is a CR-submanifold of M. For CR-submanifolds, see [7].

Note that given a semi-Euclidean space R^{2n}_n with coordinates $(x_1, ..., x_{2n})$ on R^{2n}_n, we can naturally choose an almost para-complex structure P on R^{2n}_n as follows:

$$P(\frac{\partial}{\partial x_{2i}}) = \frac{\partial}{\partial x_{2i-1}}, \ \ P(\frac{\partial}{\partial x_{2i-1}}) = \frac{\partial}{\partial x_{2i}},$$

where $i = 1, ..., n$. Let R^{2n}_n be a semi-Euclidean space of signature (+,-,+,-,...) with respect to the canonical basis $(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_{2n}})$.

Remark 3.1. Let (M, P_1, g_1) and (N, P_2, g_2) be almost para-Hermitian manifolds. A semi-Riemannian submersion $\pi : M \to N$ is called an almost para-Hermitian submersion if π is an almost para-complex map, i.e. $\pi_* \circ P_1 = P_2 \circ \pi_*$.

We now give some examples of a semi-invariant semi-Riemannian submersion.

Example 3.1. Let $\pi : R^4_2 \to R^2_1$ be a map defined $\pi(x_1, x_2, x_3, x_4) = (\frac{x_1+x_3}{\sqrt{2}}, \frac{x_2+x_4}{\sqrt{2}})$. Then it is easy to see that π is an almost para-Hermitian submersion. Every an almost para-Hermitian submersion from an almost para-Hermitian manifold onto an almost para-Hermitian manifold is a semi-invariant semi-Riemannian submersion with $D_2 = \{0\}$.

Example 3.2. Every anti-invariant semi-Riemannian submersion from an almost para-Hermitian manifold onto a semi-Riemannian manifold is a semi-invariant semi-Riemannian submersion with $D_1 = \{0\}$[11].

Example 3.3. Let $\pi : R^6_3 \to R^3_1$ be a map defined $\pi(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, \frac{x_4+x_6}{\sqrt{2}}, \frac{x_3+x_5}{\sqrt{2}})$. Then, by direct calculations

$$ker\pi_* = Span\{V_1 = \frac{\partial}{\partial x_2}, V_2 = -\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_6}, V_3 = -\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}\}$$

and

$$(ker\pi_*)^{\perp} = Span\{X_1 = \frac{\partial}{\partial x_1}, X_2 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_6}, X_3 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}\}.$$

Then it is easy to see that π is a semi-Riemannian submersion. Hence we have $PV_2 = V_3$ and $PV_1 = X_1$. Thus it follows that $D_1 = span\{V_2, V_3\}$ and $D_2 =$

$span\{V_1\}$. Moreover one can see that $\mu = span\{X_2, X_3\}$. As a result, π is a semi-invariant semi-Riemannian submersion.

Let $\pi : (M, g_1, P) \to (N, g_2)$ be a semi-invariant semi-Riemannian submersion from a para-Kähler manifold (M, g_1, P) to a semi-Riemannian manifold (N, g_2). We denote the complementary distribution to PD_2 in $(ker\pi_*)^\perp$ by μ. Then for $V \in \Gamma(ker\pi_*)$, we write

$$PV = \phi V + \omega V, \tag{3.1}$$

where $\phi V \in \Gamma(D_1)$ and $\omega V \in \Gamma(D_2)$. Also for $X \in \Gamma((ker\pi_*)^\perp)$, we have

$$PX = BX + CX, \tag{3.2}$$

where $BX \in \Gamma(D_1)$ and $CX \in \Gamma(\mu)$. Then, by using (2.5), (2.6),(3.1) and (3.2) we get

$$(\nabla_V \phi)W = BT_V W - T_V \omega W, \;\; (\nabla_V \omega)W = CT_V W - T_V \phi W,$$

for $V, W \in \Gamma(ker\pi_*)$, where

$$(\nabla_V \phi)W = \hat{\nabla}_V \phi W - \phi \hat{\nabla}_V W \; and \; (\nabla_V \omega)W = h\nabla^1_V \omega W - \omega \hat{\nabla}_V W.$$

Lemma 3.1. *Let π be a semi-invariant semi-Riemannian submersion from a para-Kähler manifold (M, g_1, P) to a semi-Riemannian manifold (N, g_2). Then we have*

$$g(PT_U V, X) = g(T_U PV, X),$$

for any $U \in \Gamma(ker\pi_*)$, $V \in \Gamma(D_1)$ and $X \in \Gamma(\mu)$.

Proof. Since M is a para-Kähler manifold, then for any $U \in \Gamma(ker\pi_*)$ and $V \in \Gamma(D_1)$ using (2.2) we have

$$P\nabla^1_U V = \nabla^1_U PV.$$

On using (2.5) we get

$$P(T_U V + \hat{\nabla}_U V) = T_U PV + \hat{\nabla}_U PV.$$

Taking inner product with $X \in \Gamma(\mu)$, we get

$$g(PT_U V, X) + g(\hat{\nabla}_U V, X) = g(T_U PV, X) + g(\hat{\nabla}_U PV, X). \tag{3.3}$$

Since μ is invariant under P, then the result follows from (3.3).

Now, we investigate the integrability of the distribution D_1 and D_2. Since fibers of semi-invariant semi-Riemannian submersions from para-Kähler manifolds are CR-submanifolds and T is the second fundamental form of the fibers, we have the following theorem.

Theorem 3.1. *Let π be a semi-invariant semi-Riemannian submersion from a para-Kähler (M, g_1, P) to a semi-Riemannian manifold (N, g_2). Then*

(i) the distribution D_1 is integrable if and only if $g(T_V PW - T_W PV, PU) = 0$ for $V, W \in \Gamma(D_1)$ and $U \in \Gamma(D_2)$,

(ii) the distribution D_2 is integrable.

Proof. (i) Since M is a para-K$\ddot{a}$hler manifold, then for any $V, W \in \Gamma(D_1)$, then (2.2) and (2.5) give

$$\begin{aligned} P[V,W] &= P\nabla^1_V W - P\nabla^1_W V \\ &= \nabla^1_V PW - \nabla^1_W PV \\ &= T_V PW - T_W PV + \hat{\nabla}_V PW - \hat{\nabla}_W PV. \end{aligned}$$

Therefore,

$$T_V PW - T_W PV = P[V,W] - \hat{\nabla}_V PW + \hat{\nabla}_W PV. \tag{3.4}$$

Now if D_1 is integrable then $P[V,W] \in \Gamma(D_1)$ as $[V,W] \in \Gamma(D_1)$. Hence in (3.4) right hand side is vertical while the left hand side is horizontal. On comparing the horizontal and vertical part we get

$$T_V PW = T_W PV,$$

for any $V, W \in \Gamma(D_1)$. In particular, we have

$$g(T_V PW, PU) = g(T_W PV, PU).$$

Conversely, firstly using (3.4), i.e.,

$$g(T_V PW - T_W PV, PU) = 0$$

which shows that

$$T_V PW - T_W PV \in \Gamma(\mu).$$

Now for any $X \in \Gamma(\mu)$, using Lemma 3.1 we have

$$g(T_V PW - T_W PV, X) = g(PT_V W - PT_W V, X) = 0,$$

which implies that $T_V PW - T_W PV = 0$, for any $V, W \in \Gamma(D_1)$. Thus from (3.4), we get

$$P[V,W] = \hat{\nabla}_V PW - \hat{\nabla}_W PV.$$

Since $\hat{\nabla}_V PW - \hat{\nabla}_W PV$ lies in $V, W \in \Gamma(ker\pi_*)$, this implies that $[V,W]$ lies in $\Gamma(D_1)$ and hence $\Gamma(D_1)$ is integrable.

ii) Since M is a para-Kahler manifold, $d\Omega = 0$. For any $X \in \Gamma(D_1)$ and $Y,\ Z \in \Gamma(D_2)$

$$\begin{aligned} 3d\Omega(X,Y,Z) &= X\Omega(Y,Z) - Y\Omega(X,Z) - Z\Omega(X,Y) \\ &- \Omega([X,Y],Z) + \Omega(Y,[X,Z]) + \Omega(X,[Y,Z]) \\ &= Xg_M(Y,JZ) - Yg_M(X,JZ) - Zg_M(X,JY) \\ &- g_M([X,Y],JZ) - g_M(JY,[X,Z]) - g_M(JX,[Y,Z]) \\ &= -g_M(JX,[Y,Z]) \\ &= 0, \end{aligned}$$

which gives the proof (ii). The proof of the following proposition is similar to the proof of Theorem 5.1 in [3].

Proposition 3.1. *Let π be a semi-invariant semi-Riemannian submersion from a para-Kähler (M, g_1, P) to a semi-Riemannian manifold (N, g_2). Then the fibers of π are locally product manifolds if and only if $(\nabla_V \phi)W = 0$ for $V, W \in \Gamma(ker\pi_*)$.*

Now, we obtain necessary and sufficient conditions for a semi-invariant semi-Riemannian submersion to be totally geodesic. We note that a differentiable map π between two semi-Riemannian manifolds is called totally geodesic if $\nabla\pi_* = 0$.

Theorem 3.2. *Let π be a semi-invariant semi-Riemannian submersion from a para-Kähler manifold (M, g_1, P) to a semi-Riemannian manifold (N, g_2). Then π is a totally geodesic map if and only if*

(i) $\hat{\nabla}_X \phi Y + T_X \omega Y$ *and* $\hat{\nabla}_X BZ + T_X CZ$ *belong to* D_1

(ii) $\nabla^1_X \omega Y + T_X \phi Y$ *and* $T_X BZ + h\nabla^1_X CZ$ *belong to* PD_2

for $Z \in \Gamma((ker\pi_*)^\perp)$ *and* $X, Y \in \Gamma(ker\pi_*)$.

Proof. First of all, since π is a semi-Riemannian submersion, we have

$$(\nabla\pi_*)(Z_1, Z_2) = 0, \;\; Z_1, Z_2 \in \Gamma((ker\pi_*)^\perp). \tag{3.5}$$

For $X, Y \in \Gamma(ker\pi_*)$, by using (2.2) we have $(\nabla\pi_*)(X, Y) = -\pi_*(P\nabla^1_X PY)$. Using (3.1) we get $(\nabla\pi_*)(X, Y) = -\pi_*(P\nabla^1_X \phi Y + P\nabla^1_X \omega Y)$. Then from (2.5) and (2.6) we have

$$(\nabla\pi_*)(X, Y) = -\pi_*(P(\hat{\nabla}_X \phi Y + T_X \phi Y + h\nabla^1_X \omega Y + T_X \omega Y)).$$

Using (3.1) and (3.2) in above equation we get

$$\begin{aligned}(\nabla\pi_*)(X, Y) &= -\pi_*(\phi\hat{\nabla}_X \phi Y + \omega\hat{\nabla}_X \phi Y + BT_X \phi Y + CT_X \phi Y \\ &+ Bh\nabla^1_X \omega Y + Ch\nabla^1_X \omega Y + \phi T_X \omega Y + \omega T_X \omega Y).\end{aligned}$$

Since $\phi\hat{\nabla}_X \phi Y + BT_X \phi Y + Bh\nabla^1_X \omega Y + \phi T_X \omega Y \in \Gamma(ker\pi_*)$, we derive

$$(\nabla\pi_*)(X, Y) = -\pi_*(\omega\hat{\nabla}_X \phi Y + CT_X \phi Y + Ch\nabla^1_X \omega Y + \omega T_X \omega Y).$$

Then, since π is a linear isometry between $(ker\pi_*)^\perp$ and TN, $(\nabla\pi_*)(X, Y) = 0$ if only and only if $\omega\hat{\nabla}_X \phi Y + CT_X \phi Y + Ch\nabla^1_X \omega Y + \omega T_X \omega Y = 0$. Thus $(\nabla\pi_*)(X, Y) = 0$ if and only if

$$\omega(\hat{\nabla}_X \phi Y + T_X \omega Y) = 0, \;\; C(T_X \phi Y + h\nabla^1_X \omega Y) = 0. \tag{3.6}$$

In a similar way for $Z \in \Gamma((ker\pi_*)^\perp)$ and $X \in \Gamma(ker\pi_*)$, $(\nabla\pi_*)(X, Z) = 0$ if and only if

$$\omega(\hat{\nabla}_X BZ + T_X CZ) = 0, \;\; C(T_X BZ + h\nabla^1_X CZ) = 0. \tag{3.7}$$

The proof comes from (3.5)-(3.7).

Now, we investigate the geometry of leaves of the distribution $(ker\pi_*)^{\perp}$.

Theorem 3.3. *Let π be a semi-invariant semi-Riemannian submersion from a para-Kähler manifold (M, g_1, P) to a semi-Riemannian manifold (N, g_2). Then the the distribution $(ker\pi_*)^{\perp}$ defines a totally geodesic foliation if and only if*

$$A_{Z_1}BZ_2 + h\nabla^1_{Z_1}CZ_2 \in \Gamma(\mu),\ A_{Z_1}CZ_2 + v\nabla^1_{Z_1}Z_2 \in \Gamma(D_2)$$

for $Z_1, Z_2 \in \Gamma((ker\pi_*)^{\perp})$.

Proof. From (2.1) and (2.2) we obtain $\nabla^1_{Z_1}Z_2 = P\nabla^1_{Z_1}PZ_2$ for $Z_1, Z_2 \in \Gamma((ker\pi_*)^{\perp})$. Using (2.7), (2.8) and (3.2) we have

$$\nabla^1_{Z_1}Z_2 = P(A_{Z_1}BZ_2 + v\nabla^1_{Z_1}BZ_2) + P(A_{Z_1}CZ_2 + h\nabla^1_{Z_1}CZ_2).$$

Then by using (3.1) and (3.2) we obtain

$$\begin{aligned}\nabla^1_{Z_1}Z_2 &= BA_{Z_1}BZ_2 + CA_{Z_1}BZ_2 + \phi v\nabla^1_{Z_1}BZ_2 + \omega v\nabla^1_{Z_1}BZ_2 + \phi A_{Z_1}CZ_2\\ &+ \omega A_{Z_1}CZ_2 + Bh\nabla^1_{Z_1}CZ_2 + Ch\nabla^1_{Z_1}CZ_2.\end{aligned}$$

Hence, we have $\nabla^1_{Z_1}Z_2 \in \Gamma((ker\pi_*)^{\perp})$ if and only if

$$BA_{Z_1}BZ_2 + \phi v\nabla^1_{Z_1}BZ_2 + \phi A_{Z_1}CZ_2 + Bh\nabla^1_{Z_1}CZ_2 = 0.$$

Thus $\nabla^1_{Z_1}Z_2 \in \Gamma((ker\pi_*)^{\perp})$ if and only if

$$B(A_{Z_1}BZ_2 + h\nabla^1_{Z_1}CZ_2) = 0,\ \phi(v\nabla^1_{Z_1}BZ_2 + A_{Z_1}CZ_2) = 0,$$

which completes proof.

Theorem 3.4. *Let π be a semi-invariant semi-Riemannian submersion from a para-Kähler manifold (M, g_1, P) to a semi-Riemannian manifold (N, g_2). Then the the distribution $(ker\pi_*)$ defines a totally geodesic foliation if and only if*

$$T_{X_1}\phi X_2 + h\nabla^1_{X_1}\omega X_2 \in \Gamma(PD_2),\ T_{X_1}\omega X_2 + \hat{\nabla}_{X_1}\phi X_2 \in \Gamma(D_1)$$

for $X_1, X_2 \in \Gamma(ker\pi_*)$.

Proof. From (2.1) and (2.2) we obtain $\nabla^1_{X_1}X_2 = P\nabla^1_{X_1}PX_2$ for $X_1, X_2 \in \Gamma(ker\pi_*)$. Using (2.5), (2.6) and (3.1) we have

$$\nabla^1_{X_1}X_2 = P(T_{X_1}\phi X_2 + \hat{\nabla}_{X_1}\phi X_2) + P(T_{X_1}\omega X_2 + h\nabla^1_{X_1}\omega X_2).$$

Then by using (3.1) and (3.2) we obtain

$$\begin{aligned}\nabla^1_{X_1}X_2 &= BT_{X_1}\phi X_2 + CT_{X_1}\phi X_2 + \phi\hat{\nabla}_{X_1}\phi X_2 + \omega\hat{\nabla}_{X_1}\phi X_2 + \phi T_{X_1}\omega X_2\\ &+ \omega T_{X_1}\omega X_2 + Bh\nabla^1_{X_1}\omega X_2 + Ch\nabla^1_{X_1}\omega X_2.\end{aligned}$$

Hence, we have $\nabla^1_{X_1}X_2 \in \Gamma(ker\pi_*)$ if and only if

$$\omega T_{X_1}\omega X_2 + \omega\hat{\nabla}_{X_1}\phi X_2 + CT_{X_1}\phi X_2 + Ch\nabla^1_{X_1}\omega X_2 = 0.$$

Thus $\nabla^1_{X_1}X_2 \in \Gamma(ker\pi_*)$ if and only if

$$\omega(T_{X_1}\omega X_2 + \hat{\nabla}_{X_1}\phi X_2) = 0,\ C(T_{X_1}\phi X_2 + h\nabla^1_{X_1}\omega X_2) = 0,$$

which completes proof.

From Theorem 3.4, we have the following result.

Corollary 3.1. *Let π be a semi-invariant semi-Riemannian submersion from a para-Kähler manifold (M, g_1, P) to a semi-Riemannian manifold (N, g_2). Then the distribution $ker\pi_*$ defines a totally geodesic foliation if and only if*

$$\begin{aligned} g_2(\nabla\pi_*)(X_1, X_2), \pi_*PZ) &= 0 \\ g_2(\nabla\pi_*)(X_1, \omega X_2), \pi_*W) &= -g_1(T_{X_1}W, \phi X_2) \end{aligned}$$

for $X_1, X_2 \in \Gamma(ker\pi_*)$, $Z \in \Gamma(D_2)$ and $W \in \Gamma(\mu)$.

Proof. For $X_1, X_2 \in \Gamma(ker\pi_*)$, $T_{X_1}\omega X_2 + \hat{\nabla}_{X_1}\phi X_2 \in \Gamma(D_1)$ if and only if $g_1(T_{X_1}\omega X_2 + \hat{\nabla}_{X_1}\phi X_2, Z) = 0$ for $Z \in \Gamma(D_2)$. Skew-symmetric T and (2.5) imply that

$$\begin{aligned} g_1(T_{X_1}\omega X_2 + \hat{\nabla}_{X_1}\phi X_2, Z) &= -g_1(T_{X_1}Z, \omega X_2) + g_1(\nabla^1_{X_1}\phi X_2, Z) \\ &= -g_1(T_{X_1}Z, \omega X_2) + g_1(\nabla^1_{X_1}Z, \phi X_2). \end{aligned}$$

Using (2.5) again we obtain

$$g_1(T_{X_1}\omega X_2 + \hat{\nabla}_{X_1}\phi X_2, Z) = -g_1(T_{X_1}Z, \omega X_2) - g_1(\hat{\nabla}_{X_1}Z, PX_2).$$

Hence we have

$$g_1(T_{X_1}\omega X_2 + \hat{\nabla}_{X_1}\phi X_2, Z) = -g_1(\nabla^1_{X_1}Z, PX_2).$$

Then from (2.2) we derive

$$g_1(T_{X_1}\omega X_2 + \hat{\nabla}_{X_1}\phi X_2, Z) = g_1(\nabla^1_{X_1}PZ, X_2).$$

Thus we have

$$g_1(T_{X_1}\omega X_2 + \hat{\nabla}_{X_1}\phi X_2, Z) = -g_1(\nabla^1_{X_1}X_2, PZ).$$

Then semi-Riemannian submersion π implies that

$$g_1(T_{X_1}\omega X_2 + \hat{\nabla}_{X_1}\phi X_2, Z) = -g_2(\pi_*(\nabla^1_{X_1}X_2), \pi_*(PZ)).$$

Using (2.9) we obtain

$$g_1(T_{X_1}\omega X_2 + \hat{\nabla}_{X_1}\phi X_2, Z) = g_2((\nabla\pi_*)(X_1, X_2), \pi_*(PZ)). \tag{3.8}$$

On the other hand, for $X_1, X_2 \in \Gamma(ker\pi_*)$, $T_{X_1}\phi X_2 + h\nabla^1_{X_1}\omega X_2 \in \Gamma(PD_2)$ if and only if $g_1(T_{X_1}\phi X_2 + h\nabla^1_{X_1}\omega X_2, W) = 0$ for $W \in \Gamma(\mu)$. Since T is skew-symmetric, we have

$$g_1(T_{X_1}\phi X_2 + h\nabla^1_{X_1}\omega X_2, W) = -g_1(T_{X_1}W, \phi X_2) + g_1(h\nabla^1_{X_1}\omega X_2, W).$$

Since π is a semi-Riemannian submersion, we have

$$g_1(T_{X_1}\phi X_2 + h\nabla^1_{X_1}\omega X_2, W) = -g_1(T_{X_1}W, \phi X_2) + g_2(\pi_*(h\nabla^1_{X_1}\omega X_2), \pi_*(W)).$$

Then from (2.9) we arrive at

$$g_1(T_{X_1}\phi X_2 + h\nabla^1_{X_1}\omega X_2, W) = -g_1(T_{X_1}W, \phi X_2) + g_2(\pi_*(-(\nabla\pi_*)(X_1, \omega X_2), \pi_*(W)). \tag{3.9}$$

Thus the proof follow from (3.8),(3.9) and Theorem 3.4.
From Proposition 3.1 and Theorem 3.3 we have the following theorem.

Theorem 3.5. *Let π be a semi-invariant semi-Riemannian submersion a para-Kähler manifold (M, g_1, P) onto a semi-Riemannian manifold (N, g_2). Then M_1 is a locally product manifold $M_{D_1} \times M_{D_2} \times M_{(ker\pi_*)^\perp}$ if and only if $(\nabla\phi) = 0$ on $ker\pi_*$ and*

$$A_{Z_1}BZ_2 + h\nabla^1_{Z_1}CZ_2 \in \Gamma(\mu), \;\; A_{Z_1}CZ_2 + v\nabla^1_{Z_1}Z_2 \in \Gamma(D_2)$$

for $Z_1, Z_2 \in \Gamma((ker\pi_*)^\perp)$, where M_{D_1}, M_{D_2} and $M_(ker\pi_*)^\perp$ are integral manifolds of the distributions D_1, D_2 and $(ker\pi_*)^\perp$.
From Corollary 3.1 and Theorem 3.3 we have the following theorem.

Theorem 3.6. *Let π be a semi-invariant semi-Riemannian submersion a para-Kähler manifold (M, g_1, P) onto a semi-Riemannian manifold (N, g_2). Then M_1 is a locally product manifold $M_{ker\pi_*} \times M_{(ker\pi_*)^\perp}$ if and only if*

$$\begin{aligned} g_2(\nabla\pi_*)(X_1, X_2), \pi_*PZ) &= 0 \\ g_2(\nabla\pi_*)(X_1, \omega X_2), \pi_*W) &= -g_1(T_{X_1}W, \phi X_2) \end{aligned}$$

and

$$A_{Z_1}BZ_2 + h\nabla^1_{Z_1}CZ_2 \in \Gamma(\mu), \;\; A_{Z_1}CZ_2 + v\nabla^1_{Z_1}Z_2 \in \Gamma(D_2)$$

for $X_1, X_2 \in \Gamma(ker\pi_*)$, $Z \in \Gamma(D_2)$, $W \in \Gamma(\mu)$ and $Z_1, Z_2 \in \Gamma((ker\pi_*)^\perp)$, where $M_{ker\pi_*}$ and $M_(ker\pi_*)^\perp$ are integral manifolds of the distributions $ker\pi_*$ and $(ker\pi_*)^\perp$.

Let π be a semi-invariant semi-Riemannian submersion a para-Kähler manifold (M, g_1, P) onto a semi-Riemannian manifold (N, g_2). Then there is a distribution $D_1 \subseteq ker\pi_*$ such that

$$ker\pi_* = D_1 \oplus D_2 \;\; and \;\; PD_1 = D_1, P(D_2) \subseteq (ker\pi_*)^\perp$$

where D_2 is orthogonal complementary to D_1 in $ker\pi_*$.
We choose a local orthonormal frame $\{v_1, ..., v_l\}$ of D_2 and a local orthonormal frame $\{e_1, ..., e_{2k}\}$ of D_1 such that $e_{2i} = Pe_{2i-1}$ for $1 \le i \le k$.
Since $\pi_*(\nabla_{Pe_{2i-1}}Pe_{2i-1}) = \pi_*(\nabla_{e_{2i-1}}e_{2i-1})$, $1 \le i \le k$, we easily have

$$trace(\nabla\pi_*) = 0 \Leftrightarrow \sum_{j=1}^{l} \pi_*(\nabla_{v_j}v_j).$$

Thus, we obtain

Theorem 3.7. *Let π be a semi-invariant semi-Riemannian submersion from a para-Kähler manifold (M, g_1, P) onto a semi-Riemannian manifold (N, g_2). Then π is a harmonic map if and only if trace$(\nabla\pi_*) = 0$ on D_2.*

Corollary 3.2. *Let π be a semi-invariant semi-Riemannian submersion from a para-Kähler manifold (M, g_1, P) onto a semi-Riemannian manifold (N, g_2) such that $ker\pi_* = D_1$. Then π is a harmonic map.*

Let $\pi : (M, g_1) \longrightarrow (N, g_2)$ be a semi-Riemannian submersion. The map π is called a semi-Riemannian submersion with totally umbilical fibers if

$$T_X Y = g_1(X, Y)H \ for \ X, Y \in \Gamma(ker\pi_*),$$

where H is the mean curvature vector field of the fiber.

Proposition 3.2. *Let π be a semi-invariant semi-Riemannian submersion from a para-Kähler (M, g_1, P) to a semi-Riemannian manifold (N, g_2). Then $H \in \Gamma(PD_2)$.*

Proof. For $X, Y \in \Gamma(D_1)$ and $W \in \Gamma(\mu)$ we have

$$\begin{aligned} & T_X PY + \hat{\nabla}_X PY = \nabla^1_X PY = P\nabla^1_X Y \\ = \ & BT_X Y + CT_X Y + \phi\hat{\nabla}_X Y + \omega\hat{\nabla}_X Y \end{aligned}$$

so that

$$g_1(T_X PY, W) = g_1(CT_X Y, W).$$

By the assumption, with some computations we get

$$g_1(X, PY)g_1(H, W) = -g_1(X, Y)g_1(H, PW).$$

Interchanging the role of X and Y, we obtain

$$g_1(Y, PX)g_1(H, W) = -g_1(Y, X)g_1(H, PW).$$

so that combining the above two equations, we have

$$g_1(Y, X)g_1(H, PW) = 0,$$

which means $H \in \Gamma(PD_2)$, since $P\mu = \mu$.

Finally, we are going to obtain curvature relations of semi-invariant semi-Riemannian submersion from a para-Kähler manifold (M, g_1, P) onto a semi-Riemannian manifold (N, g_2).

Let (M,g) be a semi-Riemannian manifold. The sectional curvature K of a 2-plane in T_pM, $p\in M$, spanned by $\{X,Y\}$, is defined by:

$$K(X,Y)=\frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y)-g(X,Y)^2}.$$

It is clear that the above definition makes sense only for non-degenerate planes, i.e. those satisfying $Q(X,Y)=g(X,X)g(Y,Y)-g(X,Y)^2\neq 0$.
As we know, the para-holomorphic sectional curvatures determine the Riemannian curvature tensor in a para-Kähler manifold.
Given a plane D invariant by P in T_pM, $p\in M$, there is an orthonormal basis $\{X,PX\}$ of D. Denote by $K(D)$, $K_*(D)$ and $\hat{K}(D)$ the sectional curvatures of the plane D in M, N and the fiber $\pi^{-1}(\pi(p))$, respectively, where $K_*(D)$ denotes the sectional curvature of the plane $D_* =< \pi_*X,\pi_*PX >$ in N. Using of Corollary 1 in [19], we get the following,

(i) If $D\subset (D_1)_p$, then with some computations we have

$$K(D)=\hat{K}(D)+|T_XX|^2-|T_XPX|^2+\epsilon_X g_1(T_XX,P[PX,X]).$$

(ii) If $D\subset (D_2\oplus PD_2)_p$ with $X\in (D_2)_p$, then we obtain

$$K(D)=-(g_1((\nabla^1_{PX}T)_XX,PX)+|hP\nabla^1_XX|^2-|vP\nabla^1_XX|^2).$$

(iii) If $D\subset (\mu)_p$, then we get

$$K(D)=K_*(D)+3|vP\nabla_XX|^2,$$

where $\epsilon_X=g(X,X)\in\{\pm 1\}$.

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