Available online: June 17, 2017

Commun.Fac.Sci.Univ.Ank.Series A1
Volume 67, Number 1, Pages 102–115 (2018)
DOI: 10.1501/Commua1_0000000834
ISSN 1303–5991
http://communications.science.ankara.edu.tr/index.php?series=A1

STRONGLY $*$-CLEAN PROPERTIES AND RINGS OF FUNCTIONS

HUANYIN CHEN AND ABDULLAH HARMANCI

ABSTRACT. A $*$-ring R is called a strongly $*$-clean ring if every element of R is the sum of a unit and a projection that commute with each other. In this paper, we explore strong $*$-cleanness of rings of continuous functions over spectrum spaces. We prove that a $*$-ring R is strongly $*$-clean if and only if R is an abelian exchange ring and $C(X)$ $\big(C^*(X)\big)$ is $*$-clean, if and only if R is an abelian exchange ring and the classical ring of quotients $q(C(X))$ of $C(X)$ is $*$-clean, where X is a spectrum space of R.

1. Introduction

Let R be an associative ring with unity. A ring R is called *clean* if every element of a ring R is the sum of an idempotent and a unit in R. If, in addition, these elements are commute, then the ring is called *strongly clean*. Cleanness of a ring is widely worked since 1977 in many aspects. In 2002, Azarpanah [1], and in 2003, McGovern [11] consider this notion in topological aspects. Let $C(X)$ denote the ring of real valued continuous functions over a topological space X. Azarpanah and McGovern independently prove that if X is a completely regular Hausdorff space, then $C(X)$ is clean if and only if X is strongly zero dimensional, if and only if $C^*(X)$ is clean where $C^*(X)$ is the subring of $C(X)$ consisting of all bounded functions in $C(X)$ [1]. On the other hand, in the first section of [12], commutative clean rings are studied by using all maximal ideals and all prime ideals of the ring.

An *involution* of a ring R is an operation $* : R \to R$ such that $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. A ring R with involution $*$ is called a $*$-ring, which has its roots in rings of operators, that is , $*$-algebras of operators on a Hilbert space. An element p in a $*$-ring R is called a *projection* if $p^2 = p = p^*$. Recently Vas [14] consider cleanness for any $*$-ring. A $*$-ring R is called $*$*-clean* if each of its elements is the sum of a unit and a projection, and R a *strongly* $*$*-clean* if each of its elements is the sum of a unit and a projection that commute with each

Received by the editors: December 14, 2016; Accepted: March 15, 2017.
2010 *Mathematics Subject Classification.* Primary 16W10; Secondary 16E50, 16U99.
Key words and phrases. Strongly $*$-clean ring; ring of continuous functions; topological $*$-space.

other. Also Li and Zhou [8] deal with these notions and answer some questions in [14].

In this paper, we are concern with the topological properties of strongly $*$-clean rings. Let $Max(R)$ and $Spec(R)$ be the sets of all maximal ideals and all prime ideals of the ring R, respectively. Let $J\text{-}spec(R) = \{P \in Spec(R) \mid J(R) \subseteq P\}$. These sets form topological spaces under Zariski topology. We call such topological spaces the *spectrum space* of R. For a $*$-ring R, we endow the ring $C(X)$ of continuous functions on X with involution $*$, where X is a spectrum space of R. By the help of this, the relationship between strongly $*$-clean rings and the corresponding rings of continuous functions are developed. We then look at the special case of rings of bounded functions. We shall prove that a $*$-ring R is strongly $*$-clean if and only if R is an abelian exchange ring and $C(X)\big(C^*(X)\big)$ is $*$-clean, where X is a spectrum space of R. Along the way, we provide topological characterization of a strongly $*$-clean ring in terms of the classical ring of quotients over its spectrum spaces.

Throughout this paper all rings are associative with unity. We write $J(R)$, $P(R)$ and $U(R)$ for the Jacobson radical, the prime radical and the set of all invertible elements of a ring R, respectively. Let $C(X)$ denote the ring of real valued continuous functions over a topological space X. Let S and T be two sets. We use $S \sqcup T$ to denote the set $S \cup T$ with $S \cap T = \emptyset$

2. $*$-Spaces of Prime Ideals

As is well known, $Spec(R)$ is a topological space with Zariski topology. Let I be an ideal of R, and let $E_S(I) = \{P \in Spec(R) \mid I \nsubseteq P\}$. Set $V_S(I) = Spec(R) - E_S(I)$, and $V_S(a) := V_S(RaR)$ for any $a \in R$. Then $V_S(I)$ is a closed set of $Spec(R)$. We say that X is a *$*$-space* provided that $C(X)$ is a $*$-ring.

Lemma 1. *Let R be a $*$-ring. Then $Spec(R)$ is a $*$-space.*

Proof. Let P be a prime ideal of R. Set $P^* = \{a \in R \mid a^* \in P\}$. It is easy to check that P^* is an ideal of R. If $aRb \in P^*$, then $b^*Ra^* \subseteq P$. As P is prime, we see that $b^* \in P$ or $a^* \in P$. Thus, $a \in P^*$ or $b \in P^*$. This implies that P^* is a prime ideal of R. Construct a map $* : C\big(Spec(R)\big) \to C\big(Spec(R)\big)$ given by $f \mapsto f^*$, where $f^*(P) = f(P^*)$ for any $P \in Spec(R)$. Clearly, f^* is continuous for any $f \in C(X)$, thus this map is well defined. It is easy to verify that $*$ is a ring morphism. If $f^* = 0$, then for any $P \in Spec(R)$, $f^*(P^*) = 0$, and so $f(P) = 0$. Thus, $f = 0$. That is, $*$ is injective. For any $g \in C\big(Spec(R)\big)$, we see that $f^* = g$ where $f = g^*$. Therefore $*$ is an involution as $C\big(Spec(R)\big)$ is commutative. □

A ring R is called *abelian* if every idempotent in R is central. A ring R is an *exchange ring* provided that for any $a \in R$, there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$. For general theory of exchange rings, we refer the reader to [13].

Lemma 2. *Let R be a $*$-ring, let $a \in R$, and let $e \in R$ be a projection. If R is an abelian exchange ring, then the following are equivalent:*

(1) *$a - e \in U(R)$, i.e. a is $*$-clean.*
(2) *$V_S(a-1) \subseteq V_S(e) \subseteq Spec(R) - V_S(a)$.*

Proof. (1) $\Rightarrow$ (2) Set $u := a - e \in U(R)$. Then $1 - a = 1 - e - u$. For any $P \in V_S(a-1)$, we have $P \notin V_S(1-e)$; otherwise, $u = (1-e)+(a-1) \in P$. As R is abelian, $Spec(R) = V_S(e) \bigsqcup V_S(1-e)$, and so $P \in V_S(e)$. Thus, $V_S(a-1) \subseteq V_S(e)$. If $P \in Spec(R)$ and $P \notin Spec(R) - V_S(a)$, then $P \in V_S(a)$. This implies that $P \notin V_S(e)$; otherwise, $u = a - e \in P$. As a result, $V_S(a-1) \subseteq V_S(e) \subseteq Spec(R) - V_S(a)$.

(2) $\Rightarrow$ (1) Assume that $V_S(a-1) \subseteq V_S(e) \subseteq Spec(R) - V_S(a)$. Let $u = a - e$. Assume that $RuR \neq R$. Then there exists a maximal ideal M of R such that $RuR \subseteq M \subsetneqq R$. Clearly, $e \in M$ or $1 - e \in M$. Thus, $M \in V_S(e)$ or $M \in V_S(1-e)$. If $M \in V_S(e)$, then $a = e + u \in M$, whence, $M \in V_S(a)$. This gives a contradiction. If $M \in V_S(1-e)$, then $a - 1 = (e-1) + u \in M$, whence, $M \in V_S(a-1)$. This implies that $M \in V_S(e)$, a contradiction. Thus $RuR = R$. Since R is an exchange ring, analogously to [3, Proposition 17.1.9] that there exists an idempotent $f \in R$ such that $RfR = R$, where $f \in uR$. Since R is abelian, we derive $f = 1$, and so $u \in U(R)$. Therefore $a - e \in R$ is invertible, hence the result holds. □

Let X be a topological space. As is well known, a subset U of X is a clopen subset of X if and only if there exists an idempotent $e \in C(X)$ such that $e(x) = 1$ for any $x \in U$ and $e(x) = 0$ for any $x \in X - U$. We say that a subset U of a $*$-space X is *$*$-clopen* provided that there exists a projection $e \in C(X)$ such that $e(x) = 1$ for any $x \in U$ and $e(x) = 0$ for any $x \in X - U$. A $*$-space X is *strongly $*$-zero-dimensional* provided that for any disjoint closed subsets A and B there exists a $*$-clopen subset U of X such that $A \subseteq U$ and $B \subseteq X - U$.

Theorem 1. *Let R be a $*$-ring. Then R is strongly $*$-clean if and only if*

(1) *R is an abelian exchange ring;*
(2) *$Spec(R)$ is strongly $*$-zero-dimensional.*

Proof. Assume that R is strongly $*$-clean. In view of [8, Lemma 2.1], R is an abelian exchange ring. Let A and B be disjoint closed sets of $Spec(R)$. Then $A \bigcap B = \emptyset$. Clearly, there exist two ideals I and J such that $A = V_S(I)$ and $B = V_S(J)$; hence, $V_S(I) \bigcap V_S(J) = \emptyset$. If $I + J \neq R$, then there exists a maximal ideal P of R such that $I + J \subseteq P \subsetneqq R$. Hence, $P \in V_S(I+J) = V_S(I) \bigcap V_S(J)$, a contradiction. This implies that $I + J = R$. Write $a + b = 1$ where $a \in I$ and $b \in J$. By hypothesis, there exists a projection $e \in R$ such that

$$V_S(a-1) \subseteq V_S(1-e) \subseteq Spec(R) - V_S(a).$$

It is easy to check that

$$\begin{aligned} &B = V_S(J) \subseteq V_S(b) \\ &= V_S(a-1) \subseteq V_S(1-e) \subseteq Spec(R) - V_S(a) \\ &\subseteq Spec(R) - V_S(I) = Spec(R) - A. \end{aligned}$$

Clearly, $B \subseteq V_S(1-e)$. As $V_S(1-e) \subseteq Spec(R) - A$, we see that $A \subseteq V_S(e)$. Obviously, $Spec(R) = V_S(e) \bigsqcup V_S(1-e)$. Define $f : Spec(R) \to \mathbb{R}$ given by $f(P) = 1$ for any $P \in V_S(e)$ and $f(P) = 0$ for any $P \in V_S(1-e)$. Then $f \in C\big(Spec(R)\big)$. Clearly, $f^2 = f$. For any $P \in V_S(e)$, we have $e \in P$, and so $e \in P^*$. This implies that $P^* \in V_S(e)$. Thus, $f^*(P) = f(P^*) = f(P) = 1$. Likewise, $f^*(P) = f(P) = 0$ for any $P \in V_S(1-e)$. Therefore $f = f^*$. This shows that $V_S(e)$ is a $*$-clopen set. Therefore $Spec(R)$ is strongly $*$-zero-dimensional.

Conversely assume that (1) and (2) hold. For any $a \in R$, we see that $V_S(a) \bigcap V_S(1-a) = \emptyset$, and so there exists a $*$-clopen U such that $V_S(a-1) \subseteq U \subseteq Spec(R) - V_S(a)$. Thus, we have a projection $f \in C\big(Spec(R)\big)$ such that $f(P) = 1$ for any $P \in U$ and $f(P) = 0$ for any $P \in Spec(R) - U$. As U is clopen and the prime radical $P(R)$ is nil, analogously to [3, Lemma 17.1.10], we can find an idempotent $e \in R$ such that $U = V_S(e)$.

Now we claim that e is a projection. For any $P \in V_S(e)$, we see that $f(P) = 1$, and so $f^*(P) = f(P^*) = f(P) = 1$. This implies that $P^* \in U = V_S(e)$, and so $e \in P^*$. Hence, $e^* \in P$, and then $P \in V_S(e^*)$. As a result, $V_S(e) \subseteq V_S(e^*)$. For any $P \in V_S(1-e)$, we see that $f(P) = 0$, and so $f(P^*) = f^*(P) = f(P) = 0$, and so $P^* \in V_S(1-e)$. This implies that $1-e \in P^*$, and so $1-e^* \in P$. Hence, $P \subseteq V_S(1-e^*)$. This shows that $V_S(1-e) \subseteq V_S(1-e^*)$. As $Spec(R) = V_S(e) \bigsqcup V_S(1-e) = V_S(e^*) \bigsqcup V_S(1-e^*)$, we get $V_S(e) = V_S(e^*)$ and $V_S(1-e) = V_S(1-e^*)$. For any $P \in Spec(R)$, if $P \in V_S(e)$, then $P \in V_S(e^*)$, and so $e, e^* \in P$. Thus, $e - e^* \in P$. If $P \in V_S(1-e)$, then $P \in V_S(1-e^*)$, and so $1-e, 1-e^* \in P$. This implies that $e - e^* = (1-e^*) - (1-e) \in P$. Therefore $e - e^* \in P(R)$. As $P(R)$ is nil, we see that $(e-e^*)^n = 0$ for some $n \in \mathbb{N}$. As $e - e^* = (e-e^*)^3$, we see that $e = e^*$. That is, $e \in R$ is a projection. In view of Lemma 2, we complete the proof. □

Recall that two subsets A and B of X is said to be *completely separated* if there exists $f \in C(X)$ such that $0 \leq f \leq 1$, $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. Let X be a topological space, and let A be a subset of X. Then A is a *zero set* in X provided that there exists an element $f \in C(X)$ such that $A = \{x \in X \mid f(x) = 0\}$, and denote A by $Z(f)$. Every zero set is a closed set, but the converse does not always hold.

Lemma 3. *Let X be a $*$-space. Then X is strongly $*$-zero-dimensional if and only if*

(1) *$C(X)$ is $*$-clean;*
(2) *Any two disjoint closed sets of X are completely separated.*

Proof. Suppose that X is strongly $*$-zero-dimensional. Then any disjoint closed sets of X are completely separated. Let $f \in C(X)$. Let $A = f^{-1}(0)$ and $B = f^{-1}(1)$. Since every zero set of X is closed, we see that A and B are both disjoint closed sets of X. By hypothesis, there exists a $*$-clopen set U of X such that $A \subseteq U$ and $B \subseteq X - U$. Let $e \in C(X)$ be a projection such that $e(x) = 1$ for any $x \in U$ and $e(x) = 0$ for any $x \in X - U$. Let $u = f - e$. For any $x \in U$, $e(x) = 1$. If $f(x) = 1$, then $x \in B$, and so $x \in X - U$, a contradiction. Thus, $f(x) \neq 1$. This implies that $u(x) \neq 0$ for any $x \in U$. If $x \in X - U$, then $e(x) = 0$. If $f(x) = 0$, then $x \in A \subseteq U$, a contradiction, and so $f(x) \neq 0$. This implies that $u(x) \neq 0$ for any $x \in X - U$. Therefore $u(x) \neq 0$ for any $x \in X$. Hence $u^{-1}(x) := \frac{1}{u(x)}$ for any $x \in X$. That is, $u \in C(X)$ is invertible. Therefore $f = e + u \in C(X)$ is $*$-clean.

Conversely, assume that (1) and (2) hold. Let A and B be disjoint closed sets. Then A and B are completely separated. In light of [5, Theorem 1.15], A and B are contained in disjoint zero sets. Thus, we can find some $f_1, f_2 \in C(X)$ such that $A \subseteq Z(f_1), B \subseteq Z(f_2)$ and $Z(f_1) \bigcap Z(f_2) = \emptyset$. This shows that $|f_1| + |f_2| > 0$. Choose $h = \frac{|f_1|}{|f_1|+|f_2|} \in C(X)$. Since $C(X)$ is $*$-clean, there exist a projection $e \in C(X)$ and a unit $u \in C(X)$ such that $h = e + u$. For any $x \in X$, $e(x) \cdot e(x) = e(x)$, and so $e(x) = 0$ or $e(x) = 1$. Set $U = \{x \in X \mid e(x) = 0\}$ and $V = \{x \in X \mid e(x) = 1\}$. Then $X = U \bigsqcup V$. As U and V are closed, and so V is clopen. Further, V is $*$-clopen. As $u \in C(X)$ is a unit, we see that $u(x) \neq 0$ for all $x \in X$. For any $x \in A$, we see that $f_1(x) = 0$, and so $h(x) = 0$. Thus, $e(x) \neq 0$ as $u(x) \neq 0$, and then $x \in V$. That is, $A \subseteq V$. For any $x \in B$, $f_2(x) = 0$, and so $h(x) = 1$. This implies that $e(x) = 0$; hence, $x \in X - V$. Thus, $B \subseteq X - V$. Therefore X is strongly $*$-zero-dimensional. □

Theorem 2. *Let R be a $*$-ring. Then R is strongly $*$-clean if and only if*

(1) *R is an abelian exchange ring;*
(2) *$C\big(Spec(R)\big)$ is $*$-clean.*

Proof. If R is strongly $*$-clean, then (1) and (2) follows from Theorem 1 and Lemma 3.

Conversely, assume that (1) and (2) hold. Then R is strongly clean. In view of [3, Lemma 17.1.12], $Spec(R)$ is strongly zero dimensional. Thus, for any disjoint closed sets A and B of $Spec(R)$, there exists a clopen U such that $A \subseteq U$ and $B \subseteq Spec(R) - U$. It follows from Urysohn's Lemma, there exists a continuous function $f : Spec(R) \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. Thus, A and B are completely separated. By virtue of Lemma 3, $Spec(R)$ is strongly $*$-zero-dimensional. Therefore we complete the proof from Theorem 1. □

The condition "$C\big(Spec(R)\big)$ is $*$-clean" in Theorem 2 is necessary, as the following shows.

Example 1. *Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then the map $* : R \to R, (a, b)^* = (b, a)$ is an involution. Obviously, R is an abelian exchange ring. Further, R is a commutative $*$-ring. But R is not strongly $*$-clean, as the idempotent $e = (1, 0) \in R$ is not a projection (see* [8, Theorem 2.2]*).*

3. Extensions to $*$-Subspaces

Let I be an ideal of a $*$-ring R, and let $E_M(I) = \{P \in Max(R) \mid I \nsubseteq P\}$. Set $V_M(I) = Max(R) - E_M(I)$. Then $Max(R)$ is a topological space with closed sets $V_M(I)$. Denote $M^* = \{a \in R \mid a^* \in M\}$ for a maximal ideal M. Clearly, $M \in Max(R)$ if and only if $M^* \in Max(R)$. Construct a map $* : C\big(Max(R)\big) \to C\big(Max(R)\big)$ given by $f \mapsto f^*$, where $f^*(M) = f(M^*)$ for any $M \in Max(R)$. As in the proof of Lemma 1, $*$ is an anti-automorphism of $C\big(Max(R)\big)$. Therefore $Max(R)$ is a $*$-space.

Lemma 4. *Let R be a $*$-ring. Then R is strongly $*$-clean if and only if*

(1) *R is an abelian exchange ring;*
(2) *$Max(R)$ is strongly $*$-zero-dimensional.*

Proof. Suppose that R is strongly $*$-clean. Then it is an abelian exchange ring. As in the proof of Lemma 1, $a - e \in U(R)$ and only if $V_M(a-1) \subseteq V_M(e) \subseteq Max(R) - V_M(a)$, where $e \in R$ is a projection. Let A and B be disjoint closed sets of $Max(R)$. Analogously to the discussion in Theorem 1, there exists a projection $e \in R$ such that $A \subseteq V_M(e)$ and $B \subseteq V_M(1-e)$. Define $f : Max(R) \to \mathbb{R}$ given by $f(M) = 1$ for any $M \in V_M(e)$ and $f(M) = 0$ for any $M \in V_M(1-e)$. Then $f \in C\big(Max(R)\big)$. Similar to the consideration in Theorem 1, $V_S(e)$ is a $*$-clopen set. Therefore $Max(R)$ is strongly $*$-zero-dimensional.

Conversely, assume that (1) and (2) hold. Then R is clean. In view of [3, Theorem 17.1.13], R is a pm ring, where a ring is a pm ring provided that each prime ideal is contained in exactly one maximal ideal. Thus, there exists a map $\varphi : Spec(R) \to Max(R)$, $\varphi(P) = M$, where M is the unique maximal ideal such that $P \subseteq M$. It is easy to check that $\varphi\big(V_S(I)\big) = V_M(I)$. This shows that φ is continuous. For any disjoint closed sets $A, B \subseteq Spec(R)$, there exist two ideals I and J of R such that $A = V_S(I)$ and $B = V_S(J)$. Hence, $\varphi(A)$ and $\varphi(B)$ are both closed. As $V_S(I) \bigcap V_S(J) = \emptyset$, we see that $V_S(I+J) = \emptyset$; hence, $I + J = R$. Thus, we infer that $V_M(I) \bigcap V_M(J) = V_M(I+J) = V_M(R) = \emptyset$. This shows that $\varphi(A)$ and $\varphi(B)$ are disjoint closed sets of $Max(R)$. By hypothesis, $Max(R)$ is strongly $*$-zero-dimensional, there exist disjoint $*$-clopen sets $U, V \subseteq Max(R)$ such that $V_M(I) \subseteq U, V_M(J) \subseteq V$. Clearly, $A \subseteq \varphi^{\leftarrow}\varphi(A) \subseteq \varphi^{\leftarrow}(U)$ and $B \subseteq \varphi^{\leftarrow}\varphi(B) \subseteq \varphi^{\leftarrow}(V)$. Clearly, $\varphi^{\leftarrow}(U)$ and $\varphi^{\leftarrow}(V)$ are clopen. For any $P \in \varphi^{\leftarrow}(U) \bigcap \varphi^{\leftarrow}(V)$, there exists a unique $M \in Max(R)$ such that $P \subseteq M$. Hence, $M \in U \bigcap V$, a contradiction. This shows that $\varphi^{\leftarrow}(U) \bigcap \varphi^{\leftarrow}(V) = \emptyset$.

As U is a $*$-clopen set of $Max(R)$, there exists a projection $e \in C\big(Max(R)\big)$ such that $e(x) = 1$ for any $x \in U$ and $e(x) = 0$ for any $x \in Max(R) - U$. Construct a function $f : Spec(R) \to \mathbb{R}$ given by $P \mapsto e\varphi(P)$ for any $P \in Spec(R)$. Then $f \in C\big(Spec(R)\big)$ is a projection. Further, we see that $f(y) = e\varphi(y) = 1$ for any $y \in \varphi^{\leftarrow}(U)$ and $f(y) = e\varphi(y) = 0$ for any $y \in Spec(R) - \varphi^{\leftarrow}(U)$. This implies that $\varphi^{\leftarrow}(U)$ is $*$-clopen. Likewise, $\varphi^{\leftarrow}(V)$ is $*$-clopen. Therefore $Spec(R)$ is strongly $*$-zero-dimensional, and thus completing the proof by Theorem 1. □

Theorem 3. *Let R be a $*$-ring. Then R is strongly $*$-clean if and only if*

(1) *R is an abelian exchange ring;*
(2) *$C\big(Max(R)\big)$ is $*$-clean.*

Proof. Suppose that R is strongly $*$-clean. In view of Lemma 4, R is an abelian exchange ring and $Max(R)$ is strongly $*$-zero-dimensional. According to Lemma 3, $C\big(Max(R)\big)$ is strongly $*$-clean.

Conversely, as R is an abelian exchange ring, it is clean. In view of [3, Theorem 17.1.13], $Max(R)$ is strongly zero-dimensional. As in the proof of Theorem 2, by Urysohn's Lemma, any two disjoint closed sets of $Max(R)$ are completely separated. According to Lemma 3, $Max(R)$ is strongly $*$-zero-dimensional. This completes the proof by Lemma 4. $\square$

The following observation is crucial.

Example 2. *Let* $R = \{\frac{m}{n} \in \mathbb{Q} \mid m, n \in \mathbb{Z}, (n, 6) = 1\}$. *We choose the involution as the identity. Then* R *is a commutative ring. Clearly,* $Max(R) = \{2R, 3R\}$. *As* $Max(R)$ *is a finite set, it follows from* [5, Remark 2.3] *that* $C\big(Max(R)\big)$ *is* $*$*-clean. But* R *is not strongly* $*$*-clean. In fact,* R *is not an exchange ring.*

Clearly, the Jacobson radical $J(R)$ is semiprime, and so $J(R)$ is the intersection of some prime ideals. Thus, $J(R) = \bigcap\limits_{P\in J\text{-}spec(R)} P$. Let I be an ideal of R, and let $F(I) = \{P \in J\text{-}spec(R) \mid I \not\subseteq P\}$. Then $F(R) = J\text{-}spec(R), F(0) = \emptyset, F(I) \cap F(J) = F\big(IJ\big)$ and $\bigcup\limits_i F(I_i) = F\big(\sum\limits_i I_i\big)$. So J-*spec*(R) is a topological subspace of *Spec*(R), where $\{F(I) \mid I \trianglelefteq R\}$ is the collection of its open sets. Let $W(I) = J\text{-}spec(R) - F(I)$. Then $W(I) = \{P \in J\text{-}spec(R) \mid I \subseteq P\}$ is the collection of its closed sets. Let R be a $*$-ring. As in the proof of Lemma 1, J-*spec*(R) is a $*$-space. The next aim is to investigate strong $*$-cleanness of $*$-rings by such $*$-subspaces. The following observation will clear our path.

Lemma 5. *Let* R *be a* $*$*-ring. Then* R *is strongly* $*$*-clean if and only if*

(1) R *is an abelian exchange ring;*
(2) $R/J(R)$ *is strongly* $*$*-clean.*

Proof. One direction is obvious. Conversely, assume that (1) and (2) hold. For any $a \in R$, there exists a projection $\overline{f} = f + J(R) \in R/J(R)$ and a unit $\overline{u} \in R/J(R)$ such that $\overline{a} = \overline{f} + \overline{u}$. As $f - f^2 \in J(R)$, by hypothesis, there exists an idempotent $e \in R$ such that $f - e \in J(R)$. Since every unit lifts modulo $J(R)$, we may assume that $u \in U(R)$. Thus, $a = e + u + r$ for some $r \in J(R)$. Set $v = u + r$. Then $a = e + v$ with $e = e^2 \in R, v \in U(R)$. As R is abelian, $ae = ea$ and $ae^* = e^*a$. Further, $e - e^* \equiv f - f^* \in J(R)$.

Let $p = 1 + (e^* - e)^*(e^* - e)$. As $ae = ea, ae^* = e^*a$, we see that $ap = pa$. Clearly, $p \in U(R)$. Write $q = p^{-1}$. Then $p^* = p$, and so $q^* = q$. Further, $ep = e(1 - e - e^* + ee^* + e^*e) = ee^*e = (1 - e - e^* + ee^* + e^*e)e = pe$. Thus, we see that $eq = qe$ and $e^*q = qe^*$. Set $g = ee^*q$. Then $g^2 = ee^*qee^*q = qee^*ee^*q = qpee^*q = ee^*q = g$. In addition, $g^* = q^*ee^* = ee^*q = g$, i.e., $g \in R$ is a projection. As $aq = qa$, we see that $ag = ga$. One easy check that $eg = g$ and $ge = ee^*qe = ee^*eq = epq = e$. This implies that $e-g=e-ee^*q=e(ep-ee^*)q=e(ee^*e-ee^*)q=ee^*(e-e^*)q\in J(R)$. Therefore $a=e+v=g+(e-g)+v$. Clearly, $(e-g)+v\in U(R)$. Let $w=(e-g)+v$. Then $a=g+w$, $g^2=g=g^*$, $w\in U(R)$ and $ag=ga$. Therefore R is strongly $*$-clean. □

Theorem 4. *Let R be a $*$-ring. Then R is strongly $*$-clean if and only if*

(1) *R is an abelian exchange ring;*
(2) *$C\big(J\text{-}spec(R)\big)$ is $*$-clean.*

Proof. Construct a map $\varphi : J\text{-}spec(R)\to Spec\big(R/J(R)\big)$ given by $P\mapsto \overline{P}$ for any $P\in J\text{-}spec(R)$. Then φ is a continuous map. If $\varphi(P)=\varphi(Q)$, then $\overline{P}=\overline{Q}$. For any $p\in P$, write $p+J(R)=q+J(R)$ for some $q\in Q$. This implies that $p\in q+J(R)\subseteq Q$, and so $P\subseteq Q$. Likewise, $Q\subseteq P$. Hence, $P=Q$, and so φ is injective. For any $\overline{P}\in Spec\big(R/J(R)\big)$, then $P\in J\text{-}spec(R)$, and then φ is surjective. That is, φ is bijective. Further, one can easily check that φ is a homeomorphism. Construct a map $\phi : C\big(J\text{-}spec(R)\big)\to C\big(Spec(R/J(R))\big)$ given by $f\mapsto f\varphi^{-1}$ for any $f\in C\big(J\text{-}spec(R)\big)$. In addition, $\varphi(f^*)=\big(\varphi(f)\big)^*$. Therefore $C\big(J\text{-}spec(R)\big)$ and $C\big(Spec(R/J(R))\big)$ are $*$-isomorphic.

If R is strongly $*$-clean, then R is an abelian exchange ring. In view of Lemma 5, $R/J(R)$ is strongly $*$-clean. It follows from Theorem 2, $C\big(Spec(R/J(R))\big)$ is strongly $*$-clean, and therefore so is $C\big(J\text{-}spec(R)\big)$. Conversely, assume that (1) and (2) hold. Then $R/J(R)$ is an abelian exchange ring and $C\big(Spec(R/J(R))\big)$ is strongly $*$-clean. In light of Theorem 2, $R/J(R)$ is strongly $*$-clean. Therefore R is strongly $*$-clean by Lemma 5. □

Corollary 1. *Let R be a $*$-ring. Then R is strongly $*$-clean if and only if*

(1) *R is an abelian exchange ring;*
(2) *$J\text{-}spec(R)$ is strongly $*$-zero-dimensional.*

Proof. Suppose that R is strongly $*$-clean. Then R is an abelian exchange ring. It follows by Theorem 4 that $C\big(J\text{-}spec(R)\big)$ is $*$-clean. Analogously to the proof of Theorem 2, any two disjoint closed sets of $J\text{-}spec(R)$ are completely separated. Therefore $J\text{-}spec(R)$ is strongly $*$-zero-dimensional from Lemma 3.

Conversely, assume that (1) and (2) hold. In view of Lemma 3, $C\big(J\text{-}spec(R)\big)$ is $*$-clean. Hence the result follows by Theorem 4. □

Combining Theorems 2, 3 and 4, we come now to the following main result.

Theorem 5. *Let R be a $*$-ring, and let X be a spectrum space of R. Then R is strongly $*$-clean if and only if*

(1) *R is an abelian exchange ring;*
(2) *$C(X)$ is $*$-clean.*

4. The Ring of Bounded Continuous Functions

Let X be a topological space. $C^*(X)$ denote the subring of $C(X)$ of all bounded functions. In the following lemma we follow the technique of [1, Lemma 2.1].

Lemma 6. *Let X be a $*$-space. Then $f \in C(X)$ is $*$-clean if and only if there exists a $*$-clopen set U in X such that $f^{-1}(1) \subseteq U \subseteq X - Z(f)$.*

Proof. Let $f \in C(X)$ be $*$-clean. Then there exists a projection $e \in C(X)$ such that $f - e \in U(C(X))$. Set $U = Z(e)$. Clearly, $X = Z(e) \sqcup Z(1-e), e\big(Z(e)\big) = \{0\}$ and $e\big(Z(1-e)\big) = \{1\}$. Thus, U is a $*$-clopen set. One easily checks that $f^{-1}(1) \subseteq U \subseteq X - Z(f)$. Conversely, assume that $f^{-1}(1) \subseteq U \subseteq X - Z(f)$ for a $*$-clopen set U. Then $U = Z(e)$ for some projection e. Construct $u : X \to \mathbb{R}$ given by $u(x) = f(x)$ for any $x \in Z(e)$ and $u(x) = f(x) - 1$ for any $x \in X - Z(e)$. Then $u = f - e$. If $x \in Z(e)$, then $x \notin Z(f)$, and so $f(x) \neq 0$. Hence, $u(x) \neq 0$. If $x \in X - Z(e)$, then $x \notin f^{-1}(1)$, and so $f(x) \neq 1$. This implies that $u(x) \neq 0$. Consequently, $u \in U\big(C(X)\big)$, as required. □

Lemma 7. *Let R be a $*$-ring, and let X be a spectrum space of R. Then $C(X)$ is $*$-clean if and only if so is $C^*(X)$.*

Proof. For any $f \in C^*\big(X\big)$, we define $f^* : X \to \mathbb{R}$ given by $f^*(P) = f(P^*)$ for any $P \in X$. One easily checks that $f^* \in C^*\big(X\big)$. This induces an involution $* : C^*\big(X\big) \to C^*\big(X\big)$ given by $f \mapsto f^*$. Therefore $C^*\big(X\big)$ is a $*$-ring.

Suppose that $C(X)$ is $*$-clean. Let $f \in C^*(X)$. Choose $A = \{x \in X \mid f(x) \geq \frac{2}{3}\}$ and $B = \{x \in X \mid f(x) \leq \frac{1}{3}\}$. Construct a function $g \in C(X)$ such that $g(x) = 1, x \in A; g(x) = 0, x \in B$ and $g(x) = \frac{1}{2}$, otherwise. Then $g \in C(X)$ is $*$-clean. In view of lemma 6, there exists a $*$-clopen set U in X such that $g^{-1}(1) \subseteq U \subseteq X - Z(g)$. Write $U = Z(e)$ for a projection $e \in C(X)$. Construct $u : X \to \mathbb{R}$ given by $u(x) = f(x)$ for any $x \in Z(e)$ and $u(x) = f(x) - 1$ for any $x \in X - Z(e)$. Then $u = f - e$. If $x \in Z(e)$, then $x \notin Z(g)$, and so $g(x) \neq 0$. Thus, $x \notin B$, and so $f(x) \neq 0$. This shows that $u(x) \neq 0$. If $x \in X - Z(e)$, then $x \notin g^{-1}(1)$, and so $g(x) \neq 1$. Hence, $x \notin A$. This shows that $f(x) \neq 1$. This implies that $u(x) \neq 0$. In addition, $u \in C^*(X)$. Therefore $u \in U\big(C^*(X)\big)$, and thus $f \in C^*(X)$ is $*$-clean, as desired.

We now assume $C^*(X)$ is $*$-clean. Let $f \in C(X)$. Set $h(x) = \{ \begin{array}{ll} -1, & \text{if } f(x) < -1; \\ f(x), & \text{if } f(x) \geq -1. \end{array}$

Choose $g(x) = \{ \begin{array}{ll} h(x), & \text{if } h(x) < 1; \\ 1, & \text{if } h(x) \geq 1. \end{array}$ Then $g \in C^*(X)$. By hypothesis, g is $*$-clean. This implies that $g \in C(X)$ is $*$-clean. In view of Lemma 6, there exists a $*$-clopen set U in X such that $g^{-1}(1) \subseteq U \subseteq X - Z(g)$. It is easy to check that $f^{-1}(1) \subseteq g^{-1}(1)$ and $X - Z(g) \subseteq X - Z(f)$. Therefore $f^{-1}(1) \subseteq U \subseteq X - Z(f)$. This completes the proof by Lemma 6. □

Theorem 6. *Let R be a $*$-ring, and let X be a spectrum space of R. Then R is strongly $*$-clean if and only if*

(1) *R is an abelian exchange ring;*
(2) *$C^*(X)$ is $*$-clean.*

Proof. In view of Lemma 7, $C(X)$ is strongly $*$-clean if and only if so is $C^*(X)$. Therefore we complete the proof by Theorem 5. □

The Stone-Cěch compactification βX of a topological space X is the largest compact Hausdorff space "generated" by X, in the sense that any map from X to a compact Hausdorff space factors through βX (in a unique way). That is, βX is a compact Hausdorff space together with a continuous map from X and has the following universal property: any continuous map f : $X \to K$, where K is a compact Hausdorff space, lifts uniquely to a continuous map $\beta f : \beta X \to K$.

Corollary 2. *Let R be a $*$-ring, and let X be a spectrum space of R. Then R is strongly $*$-clean if and only if*

(1) *R is an abelian exchange ring;*
(2) *The Stone-Cěch compactification βX of X is strongly $*$-zero dimensional.*

Proof. Suppose that R is strongly $*$-clean. Then R is an abelian exchange ring. In view of [5, Remark 6.6], $C\big(\beta X\big) \cong C^*(X)$. Thus, $C\big(\beta X\big)$ is $*$-clean by Theorem 6. Hence, βX is a $*$-space. Clearly, $C\big(\beta X\big)$ is a commutative clean ring. According to [1, Theorem 2.5], βX is strongly zero dimensional. This shows that any two disjoint closed sets of βX are completely separated. Therefore βX of X is strongly $*$-zero dimensional by Lemma 3.

Conversely, assume that (1) and (2) hold. In light of Lemma 3, $C\big(\beta X\big)$ is $*$-clean. By virtue of [5, Remark 6.6], $C^*(X)$ is $*$-clean. Accordingly, R is strongly $*$-clean from Theorem 6. □

Corollary 3. *Let R be a $*$-ring, and let X be a spectrum space of R. Then R is strongly $*$-clean if and only if*

(1) *R is an abelian exchange ring;*
(2) *$Max\big(C^*(X)\big)$ is strongly $*$-zero dimensional.*

Proof. By virtue of [5, 14.8] or [10, p. 463], the prime ideals containing a given ideal forms a chain in $C^*(X)$, and so $C^*(X)$ is a pm-ring. In view of [3, Corollary 17.1.14], $C^*(X)$ is $*$-clean. This completes the proof by Theorem 6. □

5. Strong $*$-Cleanness of $q(X)$

Let R be a commutative $*$-ring with an identity, and let $q(R)$ be the classical ring of quotients of R. We say that $x \in R$ is *self-adjoint* provided that $x^* = x$. Construct a ring morphism $* : q(R) \to q(R), \frac{r}{s} \mapsto \frac{r^*}{s^*}$. Then $*$ is also an involution of $q(R)$. Thus, $q(R)$ is a $*$-ring.

Let $N_D(R)$ denote the set of all nonzero divisors of R, and let $N_D(X) := N_D\big(C(X)\big)$ for a topological space X.

Lemma 8. *Let R be a commutative $*$-ring. If $e \in q(R)$ is self-adjoint, then there exist self-adjoint $a, b \in R$ such that $e = \frac{a}{b}$.*

Proof. Write $e = \frac{c}{d}$. As $e \in q(R)$ is self-adjoint, $e^* = \left(\frac{c}{d}\right)^* = \frac{c^*}{d^*} = \frac{c}{d}$. Thus, $c^*d = d^*c$. Clearly, $d, d^* \in N_D(R)$; hence, $e = \frac{cd^*}{dd^*}$. Set $a = cd^*$ and $b = dd^*$. Then $a^* = dc^* = a$ and $b^* = b$. That is, $a, b \in R$ are self-adjoint. In addition, $e = \frac{a}{b}$, as required. □

Lemma 9. *Let R be a commutative $*$-ring. Then the following are equivalent:*

(1) *$q(R)$ is $*$-clean.*
(2) *For any $a, b \in R$ with $a+b \in N_D(R)$, there exist self-adjoint $x \in aR, y \in bR$ such that $x + y \in N_D(R)$ and $xy = 0$.*
(3) *For any $a, b \in R$ with $a + b \in N_D(R)$, there exist $x \in aR, y \in bR$ such that $x + y \in N_D(R), xy = 0$ and x^*y is self-adjoint.*

Proof. (1) ⇒ (2) Suppose that $a+b \in N_D(R)$ with $a, b \in R$. Then there exists some $\alpha \in q(R)$ such that $a\alpha + b\alpha = 1$. Since $q(R)$ is $*$-clean, we can find a projection $e \in q(R)$ such that $e \in b\alpha q(R) \subseteq bq(R)$ and $1 - e \in a\alpha q(R) \subseteq aq(R)$. Write $e = \frac{bs}{t} = \frac{bst^*}{tt^*}$. Set $w = bst^*$ and $u = tt^*$. Then $e = \frac{w}{u}$, where $w, u \in R$ are self-adjoint and $w \in bR$. Analogously, $1 - e = \frac{z}{t}$, where $z, t \in R$ are self-adjoint and $z \in aR$. Obviously, $\frac{w}{u} + \frac{z}{t} = 1$, and so $wt + zu = ut$. Choose $x = wt$ and $y = zu$. Then $x + y = ut \in N_D(R)$. Further, $xy = (wz)(ut) = 0$ and $x, y \in R$ are self-adjoint.

(2) ⇒ (3) is trivial.

(3) ⇒ (1) Suppose that $\frac{a}{s} + \frac{b}{s} = 1$ in $q(R)$. Then $a + b = s \in N_D(R)$. By hypothesis, there exist $x \in aR, y \in bR$ such that $x + y \in N_D(R), xy = 0$ and x^*y is self-adjoint. Let $e = \frac{x}{x+y}$. Then $e(1 - e) = \frac{xy}{x+y} = 0$, and so $e = e^2 \in q(R)$ is an idempotent. Since $x^*y \in R$ is self-adjoint, we see that $(x^*y)^* = x^*y = xy^*$, and so $e^* = \frac{x^*}{x^*+y^*} = \frac{x}{x+y} = e$; hence, $e \in q(R)$ is a projection. Moreover, $e = \frac{x}{x+y} \in \left(\frac{a}{s}\right)q(R)$ and $1 - e = \frac{y}{x+y} \in \left(\frac{b}{s}\right)q(R)$. Therefore $q(R)$ is strongly $*$-clean. □

Let X be a $*$-space. Then $C(X)$ is a $*$-ring. We denote $q\big(C(X)\big)$ by $q(X)$, and so $q(X)$ is a $*$-ring. We say that U is a *$*$-zero set* of X provided that there exists a self-adjoint $f \in C(X)$ such that $A = Z(f)$. Let A be a subset of X. We say that A is *nowhere dense* if every open set of X contains an open subset that is disjoint from A. This is equivalent to saying that the closure of A contains no open set of A which is not empty. Clearly, every subset of a nowhere dense set is nowhere dense. We say that A is *dense* in X if $X - A$ is nowhere dense.

Recall that a topological space X is *completely regular* if for every point and a closed set not containing the point, there is a continuous function that has value 0 at the given point and value 1 at each point in the closed set. Almost every topological space studied is completely regular. For instance, metric spaces, Tychonoff spaces (e.g.,topological manifolds, CW complexes, Niemytzki planes), topological groups, etc. The following result is known, we include a simple proof for the self-contained.

Lemma 10. *Let X be a completely regular space, and let $f \in C(X)$. Then the following are equivalent:*

1. $f \in N_D(X)$.
2. $Z(f)$ *is nowhere dense.*

Proof. (2) ⇒ (1) Assume that $f\phi = 0$ for a $\phi \in C(X)$. Assume that $Z(\phi) \neq X$. By hypothesis, there exists an open subset B of $X - Z(\phi)$ such that $Z(f)\bigcap B = \emptyset$. Thus, we can find $x \in B$ such that $x \notin Z(f)$. This implies that $f(x), \phi(x) \neq 0$. This yields that $f\phi \neq 0$, a contradiction. Thus, $Z(\phi) = X$, and so $\phi = 0$. This means that $f \in N_D(X)$.

(1) ⇒ (2) Let C be an open set of X, and let $B = C\bigcap\big(X - Z(f)\big)$. If $B \neq \emptyset$, then B is an open subset of C. In addition, $Z(f)\bigcap B = \emptyset$. If $B = \emptyset$, then we have $C \subseteq Z(f)$, and so $f(C) = 0$. Choose $a \in C$. Since X is a completely regular space, we can find some $g \in C(X)$ such that $g(x) = 0$ for any $x \in X - C$ and $g(a) = 1$. This implies that $fg = 0$. By hypothesis, $g = 0$, a contradiction. Therefore we complete the proof. □

Theorem 7. *Let X be a completely regular ∗-space. Then the following are equivalent:*

1. $q(X)$ *is ∗-clean.*
2. *For any zero sets A and B of X such that $A\bigcap B$ is nowhere dense, there exist ∗-zero sets U, V such that $A \subseteq U, B \subseteq V$ such that $U\bigcap V$ is nowhere dense and $U\bigcup V = X$.*

Proof. (1) ⇒ (2) For any zero sets A and B of X such that $A\bigcap B$ is nowhere dense, we can write $A = Z(f)$ and $B = Z(g)$. Since $Z(f^2 + g^2) = Z(f)\bigcap Z(g) = U\bigcap V$ is nowhere dense, it follows from Lemma 10 that $f^2 + g^2 \in N_D(X)$. In view of Lemma 9, there exist self-adjoint $h \in f^2C(X), k \in g^2C(X)$ such that $h+k \in N_D(X)$ and $hk = 0$. Let $U = Z(h)$ and $V = Z(k)$. Then $A \subseteq U, B \subseteq V$. In addition, $U\bigcup V = Z(h)\bigcup Z(k) = Z(hk) = Z(0) = X$. Further, $U\bigcap V = Z(h)\bigcap Z(k) = Z(h^2 + k^2)$. As $h^2 + k^2 = (h + k)^2$, we see that $U\bigcap V = Z\big((h + k)^2\big) = Z(h + k)$ is nowhere dense from Lemma 10. Since $h, k \in q(X)$ are self-adjoint, U and V are both ∗-zero sets, as required.

(2) ⇒ (1) Let $f, g \in C(X)$ such that $f + g \in N_D(X)$. Let $A = Z(f)$ and $B = Z(g)$. Then $A\bigcap B = Z(f)\bigcap Z(g) \subseteq Z(f + g)$; hence, $A\bigcap B$ is nowhere dense from Lemma 10. By hypothesis, there exist ∗-zero sets U, V such that $A \subseteq U, B \subseteq V$ such that $U\bigcap V$ is nowhere dense and $U\bigcup V = X$. Thus, we can find self-adjoint $h, k \in C(X)$ such that $U = Z(h)$ and $V = Z(k)$. Set $\varphi = fh \in fC(X)$ and $\psi = gk \in gC(X)$. Then $Z(\varphi) = Z(fh) = Z(f)\bigcup Z(h) = Z(h)$. Likewise, $Z(\psi) = Z(k)$. Thus, $Z(\varphi^2 + \psi^2) = Z(\varphi)\bigcap Z(\psi) = Z(h)\bigcap Z(k)$ is nowhere dense, and so

$\varphi^2 + \psi^2 \in N_D(X)$ from Lemma 10. As $Z(\varphi^2\psi^2) = Z(\varphi) \bigcup Z(\psi) = Z(h) \bigcup Z(k) = X$, we see that $\varphi^2\psi^2 = 0$. In addition, it follows from $Z(hk) = Z(h) \bigcup Z(k) = X$ that $hk = 0$. Therefore $\big(\varphi^2\big)^*\psi^2 = (fg)^2 hk(hk) = 0$. According to Lemma 9, $q(X)$ is $*$-clean. □

Lemma 11. *Let X be a completely regular $*$-space. Then $C(X)$ is $*$-clean if and only if*

(1) *X is strongly zero-dimensional;*
(2) *$q(X)$ is $*$-clean.*

Proof. Suppose that $C(X)$ is $*$-clean. Then $q(X)$ is $*$-clean. By [1, Theorem 2.5], X is strongly zero-dimensional, as desired.

Conversely, assume that (1) and (2) hold. Let A and B be disjoint closed sets. Since X is strongly zero-dimensional, there exists a clopen set U such that $A \subseteq U$ and $B \subseteq V$. Thus, there exists an $e \in C(X)$ such that $e(x) = 1$ for any $x \in U$ and $e(x) = 0$ for any $x \in X - U$. Clearly, $e = e^2 \in C(X)$. By hypothesis, we have a projection $f \in q(X)$ and a unit $u \in q(X)$ such that $e = f + u$. In view of Lemma 8, write $f = \frac{a}{b}$ with self-adjoint $a, b \in R$. Since $e, f \in q(X)$ are idempotents, we see that $(e - f)^3 = e - f$, and so $u^2 = 1$. That is, $(e - f)^2 = 1$. This implies that $e(1 - 2f) = 1 - f$, and so $e = (1 - 2f)(1 - f)$. This means that $\frac{e}{1} = \frac{(b-2a)(b-a)}{b^2}$, and so $eb^2 = (b - 2a)(b - a)$. Since $a, b \in R$ are self-adjoint, we see that $e^*b^2 = \big((b - 2a)(b - a)\big)^* = (b - 2a)(b - a) = eb^2$. But $b \in N_D(R)$, and so $e = e^* = e^2$. Thus, U is a $*$-clopen; hence that X is strongly $*$-zero-dimensional. According to Lemma 3, we complete the proof. □

Theorem 8. *Let R be a $*$-ring, and let X be a spectrum space of R. Then R is a strongly $*$-clean ring if and only if*

(1) *R is an abelian exchange ring;*
(2) *$q(X)$ is $*$-clean.*

Proof. Since every locally compact Hausdorff space is completely regular, we see that the spectrum space X of R is always completely regular.

If R is a strongly $*$-clean ring, then R is an abelian exchange ring. By virtue of Theorem 5, $C(X)$ is $*$-clean. In light of Lemma 11, $q(X)$ is $*$-clean.

Conversely, assume that (1) and (2) hold. Then X is strongly zero-dimensional. According to Lemma 11, $C(X)$ is $*$-clean. Therefore R is strongly $*$-clean by Theorem 5. □

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Current address: Huanyin Chen: Department of Mathematics, Hangzhou Normal University, Hangzhou, 310036, China.

E-mail address: `huanyinchen@aliyun.com`

Current address: Abdullah Harmanci: Hacettepe University, Department of Mathematics, 06800 Beytepe Ankara, Turkey.

E-mail address: `harmanci@hacettepe.edu.tr`