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LOCALIZED RADIAL SOLUTION TO A SUPERLINEAR DIRICHLET PROBLEM IN ANNULAR DOMAIN

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ABSTRACT. In this paper, we are interested to the existence of radially symmetric solutions of $\Delta u(x) + f(u) = 0$ with prescribed number of zeros on annular domain in $\mathbb{R}^N$, when f grows superlinearity at infinity. Our approach is based on a shooting method and using fairly straightforward tools of the theory of ordinary differential which is convenient to count the number of nodes.

1. INTRODUCTION

In this paper, we shall consider classical radial of superlinear boundary-value problem

$$\begin{aligned} \Delta u(x) + f(u) = 0 \quad \text{if } x \in \Omega \\ u = 0 \quad \text{if } x \in \partial\Omega \end{aligned} \tag{1.1}$$

where $|x|$ denotes the standard norm of x in $\mathbb{R}^N$, $N \geq 3$ and Ω is the annulus of $\mathbb{R}^N$ defined by

$$\Omega = \mathbf{C}(0, R, T) = [x \in \mathbb{R}^N : R < |x| < T]$$

where R and T are two real numbers such that $0 < R < T$, $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear function.

We will assume henceforth that the following hypothesis:

(H1) f is locally Lipschitzian,

(H2) f is superlinear, i.e.,

$$\lim_{|u| \to \infty} \frac{f(u)}{u} = +\infty,$$

(H3) $u \to f(u)$ is increasing for $|u|$ large.

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Note: From (H2) and L'Hopital's Rule it follows that

$$\lim_{|u|\to\infty} \frac{F(u)}{u^2} = +\infty, \tag{1.2}$$

where $F(u) = \int_0^u f(s)\mathrm{d}s$.

It is well known on the ball domain case, the superlinear problem (1.1) has been widely studied. Most of these results are based on variational and phase-plane analysis methods. However, these arguments are quite difficult and provide no specific information of qualitative properties. Thereafter another approach proposed by Pudipeddi [2, 4] gives an easy proof by using Bessel's functions and revealing qualitative properties of radial solutions with (H1)–(H3) hypothesis and adding the additional conditions:

(H4) The function $u \to N\,F(u) - \dfrac{N-2}{2}\,u\,f(u)$ is bounded above.

(H5) There exists a $0 < k \le 1$, such that

$$\lim_{u\to\infty} \Big(\frac{u}{f(u)}\Big)^{N/2}\Big(NF(ku) - \frac{N-2}{2}uf(u)\Big) = +\infty.$$

Recently, there has been an interest in studying this problem on annular domain. We cite in our work [1], and we show that the superlinear nonhomogeneous Dirichlet problem has infinitely many radially symmetric solutions with prescribed number of zeros with (H1)–(H3) and (H4) hypothesis. Here we use the same method as in [1] without adding (H4) to prove the existence of radial solutions (1.1) which is convenient to count the number of zeros. We note that for example the function $f(u) = 8u^7 - 4u^3$, grows superlinearity at infinity but (H4) is not satisfied.

Our paper is organized as follows: In Section 2 we begin by establishing some preliminary results concerning the existence of radial solutions and by analyzing the energy we show that the energy function converges uniformly to infinity without using the Pohozaev-type identity. In Section 3 we obtain the localization of zeros of the solution and lastly in section 4 we shall prove the main theorem 1.1.

Theorem 1.1. *If (H1)–(H3) are satisfied then* (1.1) *has infinitely many radially symmetric solutions u with $u'(R) \neq 0$. For $k \in \mathbb{N}^*$ sufficiently large there exist two radially symmetric solutions u_k and w_k of problem* (1.1) *which have exactly* $(k-1)$ *zeros on* (R,T) *such that $w_k'(R) < 0 < u_k'(R)$.*

2. Preliminaries

The existence of radially symmetric solution $u(x) = u(r)$ with $r = |x|$ of (1.1) is equivalent to the existence of a solution u of the nonlinear ordinary differential equation

$$u''(r) + \frac{N-1}{r}u'(r) + f(u) = 0 \quad \text{if } R < r < T, \tag{2.1}$$

$$u(R) = u(T) = 0. \tag{2.2}$$

To solve (2.1)-(2.2), we apply the shooting method, by considering the initial value problem

$$\begin{aligned} u''(r) + \frac{N-1}{r}u'(r) + f(u) &= 0 \quad \text{if } R < r < T, \\ u(R) &= 0 \quad \text{and} \quad u'(R) = d \end{aligned} \tag{2.3}$$

with d an arbitrary nonzero real number. Denote $u(r,d)$ as the solution of (2.3) which depends on parameter d. By varying d, we shall attempt to choose the parameter appropriately to have (2.2) and if k is a sufficiently large nonnegative integer then $u(r,d)$ has exactly $(k-1)$ zeros on (R,T).

Let $d > 0$. From (H1) and since the initial value problem is not singular on the domain then the existence and uniqueness of the local solution denoted $u(\cdot,d)$ of (2.3) on $[R, R+\varepsilon]$ for some $\varepsilon > 0$ to obtain by the standard existence-uniqueness theorem for ordinary differential equations. For the existence on $[R,T]$ we define the energy function of a solution $u(\cdot,d) = u$ of (2.3) as

$$E(r,d) = E(r) = \frac{u'^2(r)}{2} + F(u(r)). \tag{2.4}$$

Then, we see from (1.2) that $F(u) > 0$ for u large enough so there exists a $J > 0$ such that

$$F(u) > -J \quad \forall u \in \mathbb{R}. \tag{2.5}$$

Therefore,

$$E'(r) = -\frac{N-1}{r}u'^2 \le 0.$$

So, E is nonincreasing and by (2.4) and (2.5) we see that

$$|u'| \le \sqrt{d^2 + 2J}.$$

It follows that $|u'|$ is uniformly bounded wherever it is defined and hence u and u' are defined on all of $[R,T]$. Thus (2.3) has a unique solution $u(r,d)$ defined on interval $[R,T]$.

Remark 2.1. *The solution $u(r,d)$ of* (2.3) *depends continuously on d in the sense that if the sequence (d_n) converges to d, then the sequence of functions $u(\cdot,d_n)$ converges uniformly to $u(\cdot,d)$ on any bounded interval. A similar property is also true for $u'(\cdot,d_n)$.*

As $u'(R,d) = d > 0$ and by continuity, then there exists $r > R$ such that $u' > 0$ on (R,r). Denote $r_0(d)$ as the largest $r \in (R,T)$ such that $u' > 0$ on (R,r).

Lemma 2.2. *Assume (H1) and (H2) hold. Then*

(1) $\lim_{d\to+\infty} r_0(d) = R$,
(2) $\lim_{d\to+\infty} u(r_0(d),d) = +\infty$.

Proof. Multiplying (2.1) by $r^{N-1}u$ and by integrating on (R, r) with the initial conditions gives

$$u'(r) = \frac{1}{r^{N-1}} : \left(d\, R^{N-1} - \int_R^r t^{N-1}\, f(u)\, \mathrm{d}t \right). \tag{2.6}$$

Integrating this, we obtain

$$u(r) = \frac{d\, R^{N-1}}{N-2}\Big(\frac{1}{R^{N-2}} - \frac{1}{r^{N-2}}\Big) - \int_R^r \frac{1}{t^{N-1}}\Big(\int_R^t s^{N-1} f(u)\, \mathrm{d}s\Big)\, \mathrm{d}t. \tag{2.7}$$

For (1), we argue by contradiction. Suppose that there exists $\varepsilon > 0$ such that for all $\gamma > 0$ there exists $d > \gamma$ for which

$$R + \varepsilon \le r_0(d).$$

Denote $R_0 = R + \varepsilon$. Then, there exists a sequence $d_n \to +\infty$ such that

$$\begin{aligned} r_0(d_n) &\ \ge\ R_0, \\ u(r, d_n) &\ >\ 0, \quad u'(r, d_n) \ge 0 \quad \forall r \in (R, R_0), \forall n \in \mathbb{N}. \end{aligned} \tag{2.8}$$

We set $\overline{r} = (R + R_0)/2$ and $u(\overline{r}, d_n) = u_n(\overline{r})$. We now show that the sequence $(u_n(\overline{r}))$ is unbounded. Again by contradiction, we suppose that there exists $M > 0$ such that for all $n \in \mathbb{N}$, $0 < u_n(\overline{r}) \le M$. By (2.7) and u_n is increasing on $[R, R_0]$ we obtain

$$\begin{aligned} \frac{d_n R^{N-1}}{N-2}\Big(\frac{1}{R^{N-2}} - \frac{1}{\overline{r}^{N-2}}\Big) &= u_n(\overline{r}) + \int_R^{\overline{r}} \frac{1}{t^{N-1}}\Big(\int_R^t s^{N-1} f(u)\, \mathrm{d}s\Big)\, \mathrm{d}t \\ &\le M + \frac{T^2}{N} \sup_{0 \le \zeta \le M} |f(\zeta)| < \infty \end{aligned}$$

which is a contradiction to $d_n \to +\infty$. Hence, the sequence $(u_n(\overline{r}))$ is unbounded and passing to subsequence we can suppose that

$$\lim_{n \to +\infty} u_n(\overline{r}) = +\infty.$$

Now, for all $n \in \mathbb{N}$, we denote

$$M_n = \inf_{\overline{r} \le r \le R_0} \frac{f(u_n)}{u_n}.$$

Since, $0 < u_n(\overline{r}) \le u_n(r)$ for all $r \in [\overline{r}, R_0]$ we see that

$$M_n \ge \inf_{u_n(\overline{r}) \le u \le u_n(R_0)} \frac{f(u)}{u}.$$

On the other hand, from (H2) and $\lim_{n \to +\infty} u_n(\overline{r}) = +\infty$ we have $\lim_{n \to +\infty} M_n = +\infty$. Thus, there exists $n_0 \in \mathbb{N}$ such that $M_{n_0} > \mu_2$ where $\mu_2 > 0$ is the second eigenvalue of $-[\frac{d^2}{dr^2} + \frac{N-1}{r}\frac{d}{dr}]$ in $(\overline{r}, R_0)$ with Dirichlet boundary conditions. It is known that the first eigenfunction of this operator can be chosen to be positive. Then, since the second eigenfunction is orthogonal to the first eigenfunction then necessarily the second Φ_2 eigenfunction must be zero somewhere on $(\overline{r}, R_0)$. Then by Sturm comparison theorem since $\mu_2 < M_{n_0}$ it follows that u_{n_0} has at least one zero in $(\overline{r}, R_0)$. This is a contradiction with (2.8), and finally we deduce that $\lim_{d\to+\infty} r_0(d) = R$.

For (2), since $\lim_{d\to+\infty} r_0(d) = R$ then for $d > 0$ sufficiently large we have $R < r_0(d) < T$. On the other hand, u has a local maximum at $r_0(d)$, then there exists $r^* \in (r_0(d), T)$ such that u is decreasing and nonnegative on $(r_0(d), r^*)$. Now, we will show that

$$\lim_{d\to+\infty} u(r_0(d), d) = +\infty.$$

Suppose that there exists a sequence $d_n \to +\infty$ such that $(u(r_0(d_n), d_n))$ is bounded by M. From (2.6) we obtain that for all $n \in \mathbb{N}$ and for all $r \in (r_0(d_n), r^*)$

$$r^{N-1}u'(r) = d_n\, R^{N-1} - \int R^r t^{N-1} f(u)\,\mathrm{d}t \le 0,$$

$$\begin{aligned} d_n R^{N-1} &\le \int_R^r t^{N-1} f(u)\,\mathrm{d}t \quad (0 \le u \le M) \\ &\le \frac{T^N}{N} \sup_{0\le\zeta\le M} |f(\zeta)| < \infty. \end{aligned}$$

It follows that (d_n) is bounded which is a contradiction to $d_n \to +\infty$. □

Lemma 2.3. *Assume (H1)–(H2) hold. Then*

$$\lim_{d\to\infty} E(r, d) = +\infty$$

uniformly for $r \in [R, T]$.

Proof. We see that the energy $E(r, d)$ is decreasing in $r \in [R, T]$ and

$$E'(r, d) = -\frac{N-1}{r} u'^2.$$

Using (2.4) and (2.5) we have

$$E'(r, d) \ge -\frac{2(N-1)}{r}(E(r, d) + J).$$

Integrating this on $[R, T]$ gives

$$ln(E(T, d) + J) - ln(E(R, d) + J) \ge -2(N-1)\, ln(\frac{T}{R}).$$

We deduce that

$$E(T, d) + J \ge \Big(\frac{d^2}{2} + J\Big)\Big(\frac{T}{R}\Big)^{-2(N-1)}.$$

Therefore,

$$E(r, d) \ge E(T, d) \ge C_1 d^2 + C_2,\ \forall r \in [R, T],$$

with $C_1 = \frac{1}{2}\Big(\frac{T}{R}\Big)^{-2(N-1)} > 0$ and $C_2 = (2C_1 - 1)J$. Finally, we deduce that $\lim_{d\to\infty} E(r, d) = +\infty$ uniformly for $r \in [R, T]$. □

Lemma 2.4. *Assume (H1)–(H2) hold. If d is sufficiently large, then*

1. *all the zeros of $u(r,d)$ are simple on $[R,T]$,*
2. *$u(r,d)$ has a finite number of zeros on $[R,T]$.*

Proof. (1) From Lemma 2.3, for d sufficiently large we have $E(T,d) > 0$. If t_0 is a zero of $u(r,d)$, then $E(t_0,d) = \frac{u'^2(t_0,d)}{2} \geq E(T,d) > 0$; thus $u'(t_0,d) \neq 0$. Then, t_0 is a simple zero of $u(r,d)$.

For (2), we argue by contradiction. Suppose if d is sufficiently large there exists $R < t_1 < \dots . < t_n < t_{n+1} \leq T$ and $u(t_n) = 0$ for all $n \in \mathbb{N}$. Using the mean value theorem, there exists $z_n \in (t_n, t_{n+1})$ such that $u'(z_n,d) = 0$ for all $n \in \mathbb{N}$. So, (t_n) converges to $t \leq T$, and by continuity of u and u' we deduce that $u(t,d) = u'(t,d) = 0$. This is a contradiction to (1). Thus, for d sufficiently large u has a finite number of zeros on $[R,T]$. □

3. On the number of zeros of solutions to (2.3)

In this section, we show that the solution $u(r,d)$ has a large number of zeros for d sufficiently large. Also, assuming (H1)–(H3) hold, it is obvious that the first zero of $u(r,d)$ is $z_0(d) = R$. We know from (1.2) that $F(u) \to +\infty$ as $u \to \hat{A} \pm \infty$. Therefore, since $\lim_{d\to\infty} E(T,d) = +\infty$ and by (H2), the mapping $u \mapsto F(u)$ is increasing for large u and decreasing when u is a large negative number, then for d sufficiently large the equation $F(u) = \frac{1}{2}E(T,d)$ has exactly two solutions, which we denote $h_1(d)$ and $h_2(d)$ such that

$$h_2(d) < 0 < h_1(d).$$

From (1.2) and Lemma 2.3, we see that

$$\lim_{d\to+\infty} h_1(d) = +\infty. \tag{3.1}$$

Also, $\lim_{d\to+\infty} h_2(d) = -\infty$.

On the other hand, by (H2), for d large enough, $u''(r_0(d)) = -f(u(r_0(d)) < 0$. As $u'(r_0(d)) = 0$ so u is decreasing on $(r_0(d), r)$ for r close enough to $r_0(d)$. Hence, (see [1]) for d sufficiently large there is a smallest $r \in (r_0(d), T)$ denoted $r_1(d)$ such that

$$u(r_1(d)) = h_1(d), \quad h_1(d) < u \leq u(r_0(d)) \quad \text{on } [r_0(d), r_1(d)). \tag{3.2}$$

Lemma 3.1. *If (H1)–(H3) are satisfied, then*

1. $\lim_{d\to+\infty} r_1(d) = R$,
2. *For d sufficiently large, $u(r,d)$ has a first zero $z_1(d)$ in the interval (R,T), and $\lim_{d\to+\infty} z_1(d) = R$.*

Proof. For (1), let

$$C(d) = \frac{1}{2} \min_{r\in[r_0(d),r_1(d)]} \frac{f(u)}{u} = \frac{1}{2} \min_{r\in[h_1(d),u(r_0(d))]} \frac{f(s)}{s}.$$

It follows from (3.1) and (H2) that

$$\lim_{d\to+\infty} C(d) = +\infty. \tag{3.3}$$

We now compare the problem

$$u''(r) + \frac{N-1}{r}u'(r) + \frac{f(u)}{u}u = 0 \tag{3.4}$$

with

$$v''(r) + \frac{N-1}{r}v'(r) + C(d)v = 0 \tag{3.5}$$

with the initial conditions

$$u(r_0(d)) = v(r_0(d)) \quad \text{and} \quad u'(r_0(d)) = v'(r_0(d)) = 0. \tag{3.6}$$

Then, by (3.3) we see that for d sufficiently large and all $r \in [r_0(d), r_1(d)]$, we have

$$\frac{f(u)}{u} \geq 2C(d) > C(d). \tag{3.7}$$

Claim: For d sufficiently large, $u < v$ on $(r_0(d), r_1(d)]$.

Indeed, multiplying (3.4) by $r^{N-1}v$ and (3.5) by $r^{N-1}u$ and subtracting give

$$\big(r^{N-1}(u'v-uv')\big)'^{N-1}uv\Big(\frac{f(u)}{u} - C(d)\Big) = 0.$$

Integrating this on $(r_0(d), r)$ and using the initial conditions give

$$r^{N-1}(u'v-uv') = -\int_{r_0(d)}^{r} t^{N-1}uv\Big(\frac{f(u)}{u} - C(d)\Big)\,\mathrm{d}t. \tag{3.8}$$

From (3.1), (3.3) and (3.7), we see that for d sufficiently large,

$$\frac{f(u)}{u} - C(d) \geq C(d) > 0. \tag{3.9}$$

For d sufficiently large, let $\mathscr{F} = \{r \in (r_0(d), r_1(d)) : u < v \text{ on } (r_0(d), r)\}$. Then

$$\begin{aligned} u''(r_0(d)) &= -f(u(r_0(d))) \\ &= u(r_0(d))\Big(-\frac{f(u(r_0(d)))}{u(r_0(d))} + C(d)\Big) - C(d)u(r_0(d)). \end{aligned}$$

From (H2) and Lemma 2.2, it follows that for d sufficiently large

$$u(r_0(d)) > 0 \quad \text{and} \quad -\frac{f(u(r_0(d)))}{u(r_0(d))} + C(d) < 0.$$

Then, for d sufficiently large, we have

$$u''(r_0(d)) < -C(d)u(r_0(d)) = v''(r_0(d)).$$

By continuity, there exists $\varepsilon > 0$ such that $(u-v)''(r) < 0$ on $(r_0(d), r_0(d)+\varepsilon)$. Using the initial conditions (3.6) we deduce that $u < v$ on $(r_0(d), r_0(d)+\varepsilon)$. Thus,

$\mathscr{F} \neq \emptyset$. We denote $\overline{r} = \sup \mathscr{F}$. Now, we will show that $\overline{r} = r_1(d)$. Otherwise, suppose that

$$u < v \quad \text{on } (r_0(d), \overline{r}) \quad \text{and} \quad u(\overline{r}) = v(\overline{r}).$$

Since $0 < h_1(d) < u < v$ on $(r_0(d), \overline{r})$ and by (3.9) we see that for d sufficiently large

$$r^{N-1} uv \Big(\frac{f(u)}{u} - C(d) \Big) > 0.$$

Therefore, by (3.8) $u'(r)v(r) - u(r)v'(r) < 0$ on $(r_0(d), \overline{r}]$. Thus, $u'(\overline{r}) < v'(\overline{r})$. On the other hand, as $u(r) < v(r)$ for $r < \overline{r}$ we have

$$\frac{u(r) - u(\overline{r})}{r - \overline{r}} > \frac{v(r) - v(\overline{r})}{r - \overline{r}}.$$

Hence $u'(\overline{r}) \geq v'(\overline{r})$. This is a contradiction. It follows that $\overline{r} = r_1(d)$ which completes the proof of the claim.

Now, we set

$$z(r) = \left(r/\sqrt{C(d)} \right)^{\frac{N-2}{2}} v \left(r/\sqrt{C(d)} \right).$$

It is easy to verify that $z(r)$ is a solution of Bessel's equation of order $\nu = \frac{N-2}{2} > 0.$, i.e.,

$$z'' + \frac{z'}{r} + \big(1 - \frac{\nu^2}{r^2}\big) z = 0.$$

Then, there exists a constant $K > 0$ such that every interval of length K has at least one zero of $z(r)$ (see [3]). It follows that every interval of length $K/\sqrt{C(d)}$ contains at least one zero of $v(r)$. Hence, by claim for d sufficiently large, we have

$$r_0(d) < r_1(d) < r_0(d) + \frac{K}{\sqrt{C(d)}}.$$

Now (1) of this lemma is a consequence of Lemma 2.2 and (3.3).

For (2), suppose not, which means $u > 0$ on $(R, T]$ and consider $r > r_1(d)$. Then $0 < u < u(r_1(d))$. Also as $F(h_1(d)) = \frac{1}{2} E(T, d)$ for large d, thus

$$2F(h_1(d)) \leq \frac{u'^2}{2} + F(u) \leq \frac{u'^2}{2} + F(h_1(d)).$$

Therefore

$$-u' = |u'| \geq \sqrt{2F(h_1(d))} \quad \text{for } r_1(d) \leq r \leq T.$$

Integrating on $(r_1(d), r)$ and by (3.2) we obtain

$$h_1(d) - u(r) = u(r_1(d)) - u(r) \geq \sqrt{2F(h_1(d))}(r - r_1(d)),$$

so that

$$h_1(d) - \sqrt{2F(h_1(d))}(r - r_1(d)) \geq u(r) > 0,$$

thus

$$r - r_1(d) \leq \frac{h_1(d)}{\sqrt{2F(h_1(d))}} \tag{3.10}$$

for large d.

Taking $r = T$ and taking the limit as $d \to \infty$ in (3.10) as well as using (1.2), (3.1) and $r_1(d) \to R$ we see that

$$0 < T - R \leq \frac{h_1(d)}{\sqrt{2F(h_1(d))}} \to 0$$

as $d \to \infty$. This is impossible since $T > R$. Thus, u has a first zero $z_1(d)$. Then using a similar argument on $[r_1(d), z_1(d)]$ and letting $r = r_1(d)$ in (3.10) we obtain $\lim_{d\to+\infty} z_1(d) = R$. The proof is complete. □

Lemma 3.2. *Let (H1)–(H3) be satisfied. Then for d sufficiently large the solution $u(r, d)$ attains a local minimum at $r_3(d) \in (r_2(d), T)$ and moreover $\lim_{d\to\infty} r_3(d) = R$.*

Proof. We begin to establish the following claim.

Claim: For d sufficiently large, $u(r, d)$ attains the value $h_2(d)$ on $(z_1(d), T)$.

Otherwise, suppose that $u(r) > h_2(d)$ on $(z_1(d), T)$. By Lemma 2.4 and u is decreasing on $(r_0(d), z_1(d)$, we see that $u'(z_1(d)) < 0$ then $u' < 0$ on a maximal interval $(z_1(d), r^*)$. Thus $F(u) < F(h_2(d))$ on $[z_1(d), r^*[$. Hence

$$2F(h_2(d)) \leq E(r, d) < \frac{u'^2}{2} + F(h_2(d)).$$

Therefore

$$0 < \sqrt{2F(h_2(d))} \leq |u'| = -u' \quad \forall r \in [z_1(d), r^*].$$

In particular $u'^*) < 0$. This implies $r^* = T$. Now integrating this inequality on $(z_1(d)), r)$ we obtain

$$h_2(d) < u(r) \leq -\sqrt{2F(h_2(d))}(r - z_1(d)) \quad \forall r \in [z_1(d), T]. \tag{3.11}$$

Taking $r = T$, we have

$$T - z_1(d) \leq \frac{-h_2(d)}{\sqrt{2F(h_2(d))}}.$$

Since $\lim_{d\to\infty} h_2(d) = -\infty$, by (1.2) we deduce that $\lim_{d\to\infty} \frac{-h_2(d)}{\sqrt{2F(h_2(d))}} = 0$. As $\lim_{d\to\infty} z_1(d) = R$ then $T = R$. This is a contradiction. End of proof of the claim.

We denote by $r_2(d)$ the smallest $r \in (z_1(d), T)$ such that $u(r_2(d)) = h_2(d)$ and $h_2(d) < u(r, d)$ on $[z_1(d), r_2(d)]$. By (3.11) taking $r = r_2(d)$ we see that

$$\lim_{d\to\infty} r_2(d) = R. \tag{3.12}$$

Now, suppose by contradiction that u is decreasing on $(r_2(d), T)$. Then $u < h_2(d) < 0$ on $(r_2(d), T)$. We set

$$C(d) = \frac{1}{2} \min_{u \leq h_2(d)} \frac{f(u)}{u}.$$

By (H2), we see that
$$\lim_{d\to+\infty} C(d) = +\infty. \tag{3.13}$$
Now, we compare the problem
$$u''(r) + \frac{N-1}{r}u'(r) + \frac{f(u)}{u}u = 0 \tag{3.14}$$
with
$$v''(r) + \frac{N-1}{r}v'(r) + C(d)v = 0 \tag{3.15}$$
and with the initial conditions
$$v(r_2(d)) = u(r_2(d)) = h_2(d) \text{ and } v'(r_2(d)) = u'(r_2(d)). \tag{3.16}$$
As in the proof of Lemma 3.1, we see that $u > v$ on $(r_2(d), T)$ for d large enough. We see that
$$z(r) = \left(r/\sqrt{C(d)}\right)^{\frac{N-2}{2}} v\left(r/\sqrt{C(d)}\right)$$
is a solution of the Bessel's equation of order $\nu = \frac{N-2}{2}$. Then, there exists $K > 0$ such every interval of length K has at least one zero of $z(r)$. We deduce that for large d, v must have a zero on $(r_2(d), T)$ and since $u > v$ we see that u gets positive which contradicts that u is decreasing on $(r_2(d), T)$. It follows that u has a local minimum at $r_3(d) \in (r_2(d), T)$. Also , for d sufficiently large we have
$$r_2(d) < r_3(d) \le r_2(d) + \frac{K}{\sqrt{C(d)}}.$$
It follows from (3.13) and (3.12) as $d \to \infty$ that $r_3(d) \to R$. This completes the proof. □

As $F(u(r_3(d))) = E(r_3(d)) \to \infty$ as $d \to \infty$ (by Lemma 2.3), in a similar way we can show that for d large enough, $u(r, d)$ has a second zero $z_2(d)$ with $r_3(d) < z_2(d) < T$ and moreover $\lim_{d\to+\infty} z_2(d) = R$. Proceeding in the same way, we can show that for d sufficiently large, $u(r, d)$ has a second local maximum at $r_4(d) \in (z_2(d), T)$ with $\lim_{d\to+\infty} u(r_4(d)) = +\infty$ and therefore, there exists $z_3(d)$ the third zero of $u(r, d)$ on (R, T) with $\lim_{d\to+\infty} z_3(d) = R$.

Remark 3.3. *Continuing in the same way, we can obtain as many zeros of $u(r, d)$ as desired on (R, T) for d large enough.*

4. Proof of theorem 1.1

For $d > 0$, let us denote by $N_d := \text{Card}\{\text{ zeros of } u(r, d) \text{ on } (R, T)\}$. For $k \ge 1$ defined by set
$$S_k = \{d > 0 : N_d = k - 1\}.$$
By Lemma 2.3 and remark 3.3, we see that for d sufficiently large, S_k is not empty for some k and $E(T, d) > 0$ and we denote $k_0 = \min\{k \in \mathbb{N}^* : S_k \neq \emptyset\}$. It follows that S_{k_0} is not empty and is bounded above. Let $d_{k_0} = \sup S_{k_0}$.

Lemma 4.1. *$u(r, d_{k_0})$ has exactly $k_0 - 1$ zeros on (R, T).i.e., $N_{d_{k_0}} = k_0 - 1$.*

Proof. By definition of k_0 we have $N_{d_{k_0}} \geq k_0 - 1$. Suppose now that $N_{d_{k_0}} \geq k_0$. Then for d close to d_{k_0} and $d \leq d_{k_0}$ by remark 2.1 with respect to initial conditions and by Lemma 2.4 we see that $N_d \geq k_0$. However, if $d \in S_{k_0}$ and is close to d_{k_0} and $d < d_{k_0}$ then $N_d = k_0 - 1$. This is a contradiction to the definition of d_{k_0}. Hence $N_{d_{k_0}} = k_0 - 1$. $\square$

Lemma 4.2. *$u(T, d_{k_0}) = 0$.*

Proof. We argue by contradiction and assume that $u(T, d_{k_0}) \neq 0$, then by remark 2.1 with respect to initial conditions and by Lemma 2.4, we deduce that if d is close to d_{k_0} then $N_d = N_{d_{k_0}}$ Now, for d close to d_{k_0} and $d > d_{k_0}$ then $d \notin S_{k_0}$ therefore, $N_d \neq k_0 - 1$. This is a contradiction with Lemma 4.1. Hence $u(T, d_{k_0}) = 0$. $\square$

We denote $S_{k_0+1} = \{d > d_{k_0} : N_d = k_0\}$.

Lemma 4.3. *$S_{k_0+1} \neq \emptyset$.*

Proof. We want to show the following result first.

Claim: If d close to d_{k_0} and $d > d_{k_0}$ then $N_d \leq k_0$.

Suppose by contradiction that there exists a sequence $q_n \to d_{k_0}$ such that $N_{q_n} \geq k_0 + 1$. For all $1 \leq i \leq k_0$ let us denote z_i^n the ith zero of $u(r, q_n)$ on (R, T) such that

$$R < z_1^n < z_2^n < \cdots < z_{k_0}^n < z_{k_0+1}^n < T.$$

For every $1 \leq i \leq k_0 + 1$ the sequence (z_i^n) is bounded and converges to z_i thus, we see that

$$R < z_1 < z_2 < \cdots < z_{k_0} < z_{k_0+1} < T.$$

It follows that $N_{d_{k_0}} \geq k_0$, which contradicts Lemma 4.1. Thus the claim is proven.

Finally, if $d > d_{k_0}$ then $N_d \leq k_0$ and $N_d \neq k_0 - 1$ thus, $N_d = k_0$ and $S_{k_0+1} \neq \emptyset$ which completes the proof. $\square$

By remark 3.3, it follows that S_{k_0+1} is not empty and bounded above, thus we denote $d_{k_0+1} = \sup S_{k_0+1}$. We show in a similar way as Lemmas 4.1 and 4.2 that $N_{d_{k_0+1}} = k_0$ and $u(T, d_{k_0+1}) = 0$. Proceeding inductively we can show, for all $k \geq k_0$ there exists a solution $u_k(r) = u(r, d_k)$ of (2.1)-(2.2) which has exactly $(k-1)$ zeros on (R, T) with $u_k'(R) = d_k > 0$.

Now, in the case $d < 0$ we consider the problem

$$\begin{gathered} u''(r) + \frac{N-1}{r} u'(r) + f(u) = 0 \quad \text{if } R < r < T \\ u(R) = 0, \quad u'(R) = d < 0. \end{gathered} \tag{4.1}$$

We denote $v(r) = -u(r)$ on $[R, T]$ and $f_1(s) = -f(-s)$ on $\mathbb{R}$ then the problem (4.1) is equivalent to

$$\begin{gathered} v''(r) + \frac{N-1}{r} v'(r) + f_1(v) = 0, \quad \text{if } R < r < T \\ v(R) = 0, \quad v'(R) = -d > 0. \end{gathered} \tag{4.2}$$

It is clear that the assumptions (H1), (H2) and (H3) are satisfied.

Next, according to the case $d > 0$ we deduce that, for k sufficiently large, (2.1)-(2.2) has a solution v_k which has exactly $(k-1)$ zeros on (R, T) with $v_k'(R) > 0$. Finally, for k sufficiently large, (2.1)-(2.2) has a solution $w_k = -v_k$ which has $(k-1)$ zeros on (R, T) and $w_k'(R) < 0$. End of proof of the main Theorem 1.1.

5. Conclusion

By this work, we managed to establish the existence of infinitely many localized radial solution to superlinear Dirichlet problem (1.1) on annular domain in $\mathbb{R}^N$, when f grows superlinearity at infinity, the proof presented here seems more natural and more easier.

We use a shooting method and we show that the energy converges to infinity which leads to reveal some properties of zeros of solutions. Finally, by approximating solutions of (1.1) with an appropriate linear Bessel's equation, we deduce that there are localized solutions with any prescribed number of zeros.

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