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A NOTE ON TOPOLOGIES GENERATED BY m-STRUCTURES AND ω-TOPOLOGIES

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Abstract. Let τ^α (resp. $SO(X,\tau)$) be the family of all α-open (resp. semi-open) sets in a topological space (X,τ). The topology τ^α is constructed in [10] as follows: $\tau^\alpha = \mathcal{T}(SO(X)) = \{U \subset X : U \cap S \in SO(X,\tau)$ for every $S \in SO(X,\tau)\}$. By the same method, we construct topologies $\mathcal{T}(m_X)$ and $\mathcal{T}(\omega m_X)$ for m-structrues m_X and ωm_X defined in [11], respectively, and show that $\omega\mathcal{T}(m_X) \subset \mathcal{T}(\omega m_X)$. Furthermore, in [2], a topology $\mathcal{M}_*$ is constructed by using an M-space $(X,\mathcal{M})$ with an ideal $\mathcal{I}$. In this note, we define ωM-open sets on $(X,\mathcal{M})$ and show that the family $\omega\mathcal{M}$ of all ωM-open sets is a topology for X and $\omega(\mathcal{M}_*) = (\omega\mathcal{M})_* = (\omega\mathcal{M})^*$.

1. Introduction

In 1982, Hdeib [6] introduced and investigated the notions of ω-closed sets and ω-closed mappings. Al-Zoubi and Al-Nashef [3] investigated several properties of the topology of ω-open sets. Recently, Noiri and Popa [11] have introduced the notion of ωm-open sets in an m-space and, by utilizing ωm-open sets, obtained several properties of m-Lindelöf spaces. Let τ^α (resp. $SO(X,\tau)$) be the family of all α-open (resp. semi-open) sets of a topological space (X,τ). Then $SO(X,\tau)$ is not a topology but the topology τ^α is constructed in [10] as follows: $\{U \subset X : U \cap S \in SO(X,\tau)$ for every $S \in SO(X,\tau)\} = \tau^\alpha$. In this note, by the same method we construct some topologies from an m-structure and the family of ωm-open sets and investigate their relations.

On the other hand, Al-Omari and Noiri [2] constructed the topology $\mathcal{M}_*$ from an M-space $(X,\mathcal{M})$ with an ideal $\mathcal{I}$. In this note, we define the notion of ωM-open sets in $(X,\mathcal{M})$ and show that the family $\omega\mathcal{M}$ of all ωM-open sets is a topology for X and also $\omega(\mathcal{M}_*) = (\omega\mathcal{M})_*$.

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2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. We recall some definitions and theorems used in this note.

Definition 1. *Let (X, τ) be a topological space. A subset A of X is said to be ω-open* [6] *if for each $x \in A$ there exists $U \in \tau$ containing x such that $U \setminus A$ is a countable set.*

The family of all ω-open sets in (X, τ) is denoted by $\omega\tau$.

Lemma 1. *(Al-Zoubi and Al-Nashef* [3]*). Let (X, τ) be a topological space. Then $\omega\tau$ is a topology and it is strictly finer than τ.*

Definition 2. *Let X be a nonempty set and $P(X)$ the power set of X. A subfamily m_X of $P(X)$ is called an m-structure on X* [11] *if m_X satisfies the following properties:*
(1) $\emptyset \in m_X$ and $X \in m_X$,
(2) The arbitrary union of the sets belonging to m_X belongs to m_X.

By (X, m_X), we denote a set X with an m-structure m_X and call it an *m-space*. Each member of m_X is said to be *m_X-open* and the complement of an m_X-open set is said to be *m_X-closed*.

Definition 3. *Let (X, m_X) be an m-space. A subset A of X is said to be ωm_X-open* [11] *if for each $x \in A$, there exists $U_x \in m_X$ containing x such that $U_x \setminus A$ is a countable set. The complement of an ωm_X-open set is said to be ωm_X-closed.*

The family of all ωm_X-open sets in (X, m_X) is denoted by ωm_X.

Remark 1. *Let (X, τ) be a topological space and m_X an m-structure on X. If $\tau \subset m_X$, then the following relations hold. We can observe that the implications in the diagram below are not reversible.*

$$\begin{array}{ccc} \text{open} & \longrightarrow & m_X\text{-open} \\ \downarrow & & \downarrow \\ \omega\text{-open} & \longrightarrow & \omega m_X\text{-open} \end{array}$$

Lemma 2. *(Noiri and Popa* [11]*). Let (X, m_X) be an m-space and A a subset of X. Then the following properties hold:*

(1) *A is ωm_X-open if and only if for each $x \in A$, there exists $U_x \in m_X$ containing x and a countable subset C_x of X such that $U_x \setminus C_x \subset A$,*
(2) *The family ωm_X is an m-structure on X and ωm_X is a topology if m_X is a topology,*
(3) *$m_X \subset \omega m_X$ and $\omega(\omega m_X) = \omega m_X$.*

Definition 4. *Let (X, τ) be a topological space. A subset A of X is said to be*

(1) *α-open* [10] *if* $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,
(2) *semi-open* [8] *if* $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,
(3) *preopen* [9] *if* $A \subset \mathrm{Int}(\mathrm{Cl}(A))$,
(4) *b-open* [4] *if* $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$,
(5) *β-open* [1] *if* $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$.

The family of all α-open (resp. semi-open, preopen, b-open, β-open) sets in (X, τ) is denoted by τ^α (resp. SO(X), PO(X), BO(X), $\beta(X)$).

Definition 5. *For an m-space (X, m_X), we define $T(m_X)$ as follows:*
$T(m_X) = \{U \subset X : U \cap M_X \in m_X$ *for every* $M_X \in m_X\}$.

Remark 2. *Let (X, τ) be a topological space and $m_X = PO(X)$ (resp. $BO(X)$, $\beta(X)$). Then $T(m_X)$ is denoted by T_γ (resp. T_b, T_δ) and the following properties are known:*

(1) $T(SO(X)) = \tau^\alpha$ [10], (2) $\tau^\alpha \subset T_\gamma = T_\delta$ [5], and (3) $T_\gamma = T_b$ [4].

3. Topologies generated by m_X and ωm_X

Theorem 1. *Let (X, m_X) be an m-space. Then $\mathcal{T}(m_X)$ is a topology for X such that $\mathcal{T}(m_X) \subset m_X$.*

Proof. (1) It is obvious that $\emptyset, X \in \mathcal{T}(m_X)$.

(2) Let $V_\alpha \in \mathcal{T}(m_X)$ for each $\alpha \in \Delta$. Let A be an arbitrary element of m_X. For each $\alpha \in \Delta$, $V_\alpha \cap A \in m_X$ and $\{\cup V_\alpha : \alpha \in \Delta\} \cap A = \cup\{V_\alpha \cap A : \alpha \in \Delta\} \in m_X$ by Definition 2 (2). Therefore, we have $\cup_{\alpha \in \Delta} V_\alpha \in \mathcal{T}(m_X)$.

(3) Let $V_1, V_2 \in \mathcal{T}(m_X)$. For any $A \in m_X$, $V_2 \cap A \in m_X$ and $(V_1 \cap V_2) \cap A = V_1 \cap (V_2 \cap A) \in m_X$. Therefore, we obtain $V_1 \cap V_2 \in \mathcal{T}(m_X)$.

Furthermore, for any $V \in \mathcal{T}(m_X)$, $V = V \cap X \in m_X$ and hence $\mathcal{T}(m_X) \subset m_X$. □

The following corollary is results established by Nåstad [10], Ganster and Andrijeviá [5] and Andrijeviá [4].

Corollary 1. *Let (X, τ) be a topological space. Then the families SO(X), PO(X), BO(X), $\beta(X)$ are m-structures on X. Therefore, τ^α, $\mathcal{T}_\gamma$, $\mathcal{T}_b$ and $\mathcal{T}_\delta$ are topologies for X.*

Theorem 2. *Let (X, m_X) be an m-space. Then $\omega\mathcal{T}(m_X) \subset \mathcal{T}(\omega m_X)$.*

Proof. Suppose that $A \in \omega\mathcal{T}(m_X)$. To obtain that $A \in \mathcal{T}(\omega m_X)$, we show that $A \cap B \in \omega m_X$ for every $B \in \omega m_X$. For each $x \in A \cap B, x \in A \in \omega\mathcal{T}(m_X)$ and by Lemma 2, there exist $U_x \in \mathcal{T}(m_X)$ containing x and a countable set C_x such that $U_x \setminus C_x \subset A$. On the other hand, since $x \in B \in \omega m_X$, there exist $V_x \in m_X$ containing x and a countable set D_x such that $V_x \setminus D_x \subset B$. Now, we have

$A \cap B \supset (U_x \setminus C_x) \cap (V_x \setminus D_x) = U_x \cap (X \setminus C_x)) \cap (V_x \cap (X \setminus D_x)) = (U_x \cap V_x) \cap [(X \setminus C_x) \cap (X \setminus D_x)] = (U_x \cap V_x) \cap [X \setminus (C_x \cup D_x)] = (U_x \cap V_x) \setminus (C_x \cup D_x)$.

Since C_x and D_x are countable, $C_x \cup D_x$ is a countable set. Since $U_x \in \mathcal{T}(m_X)$ and $V_x \in m_X$, $U_x \cap V_x \in m_X$ and $x \in U_x \cap V_x$. Therefore, by Lemma 2, $A \cap B \in \omega m_X$. This shows that $A \in \mathcal{T}(\omega m_X)$. Therefore, $\omega\mathcal{T}(m_X) \subset \mathcal{T}(\omega m_X)$. □

Remark 3. *By Lemma 2 and Theorems 1 and 2, we obtain the following diagram:*

$$\begin{array}{ccc} m_X & \longrightarrow & \omega m_X \\ & & \uparrow \\ \uparrow & & \mathcal{T}(\omega m_X) \\ & & \uparrow \\ \mathcal{T}(m_X) & \longrightarrow & \omega\mathcal{T}(m_X) \end{array}$$

QUESTION: Is the converse implication of Theorem 2 true ?

Corollary 2. *Let (X, τ) be a topological space. Then $\omega\tau^\alpha \subset \mathcal{T}(\omega SO(X))$.*

4. Topologies generated by M-spaces with ideals

Definition 6. *Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily $\mathcal{M}$ of $\mathcal{P}(X)$ is called an* M-structure *on X* [2] *if $\mathcal{M}$ satisfies the following properties:*

(1) *$\mathcal{M}$ contains $\emptyset$ and X,*
(2) *$\mathcal{M}$ is closed under the finite intersection.*

By $(X, \mathcal{M})$, we denote a set X with an M-structure $\mathcal{M}$ and call it an *M-space*. Each member of $\mathcal{M}$ is said to be *M-open* and the complement of an M-open set is said to be M-closed.

Definition 7. *Let (X, M) be an M-space. A subset A of X is said to be ωM-open if for each $x \in A$, there exists $U_x \in M$ containing x such that $U_x \setminus A$ is a countable set. The complement of an ωM-open set is said to be ωM-closed.*

The family of all ωM-open sets in $(X, \mathcal{M})$ is denoted by $\omega\mathcal{M}$.

Lemma 3. *Let $(X, \mathcal{M})$ be an M-space and A a subset of X. Then A is ωM-open if and only if for each $x \in A$, there exists $U_x \in \mathcal{M}$ containing x and a countable subset C_x of X such that $U_x \setminus C_x \subset A$.*

Proof. Necessity. Let A be ωM-open and $x \in A$. Then there exists $U_x \in \mathcal{M}$ containing x such that $U_x \setminus A$ is a countable set. Let $C_x = U_x \setminus A$. Then we have $U_x \setminus C_x \subset A$.

Let $x \in A$. Then there exists $U_x \in \mathcal{M}$ containing x and a countable set C_x such that $U_x \setminus C_x \subset A$. Therefore, $U_x \setminus A \subset C_x$ and $U_x \setminus A$ is a countable set. Hence A is ωM-open. □

Theorem 3. *For an M-space $(X, \mathcal{M})$, the following properties hold:*

(1) *The family* $\omega\mathcal{M}$ *is a topology for* X,
(2) $\mathcal{M} \subset \omega\mathcal{M}$ *and* $\omega(\omega\mathcal{M}) = \omega\mathcal{M}$.

Proof. (1) (i) It is obvious that $\emptyset, X \in \omega\mathcal{M}$.

(ii) Let $A, B \in \omega\mathcal{M}$ and $x \in A \cap B$. Then, by Lemma 3, there exist $U, V \in \mathcal{M}$ and countable sets C, D such that $x \in U$ and $U \setminus C \subset A$ and $x \in V$ and $V \setminus D \subset B$. Therefore, $x \in U \cap V \in \mathcal{M}$, $C \cup D$ is countable and we have

$(U \cap V) \setminus (C \cup D) = (U \cap V) \cap [(X \setminus C) \cap (X \setminus D)] = [U \cap (X \setminus C)] \cap [V \cap (X \setminus D)] = (U \setminus C) \cap (V \setminus D) \subset A \cap B$.

This shows that $A \cap B \in \omega\mathcal{M}$.

(iii) Let $\{A_\alpha : \alpha \in \Delta\}$ be any subfamily of $\omega\mathcal{M}$. Then for each $x \in \cup_{\alpha\in\Lambda} A_\alpha$, there exists $\alpha(x) \in \Lambda$ such that $x \in A_{\alpha(x)}$. Since $A_{\alpha(x)} \in \omega\mathcal{M}$, there exists $U_x \in \mathcal{M}$ containing x such that $U_x \setminus A_{\alpha(x)}$ is a countable set. Since $U_x \setminus (\cup_{\alpha\in\Lambda} A_\alpha) \subset U_x \setminus A_{\alpha(x)}$, $U_x \setminus (\cup_{\alpha\in\Lambda} A_\alpha)$ is a countable set. Therefore, $\cup_{\alpha\in\Lambda} A_\alpha \in \omega\mathcal{M}$. This shows that $\omega\mathcal{M}$ is a topology.

(2) Since every M-open set is ωM-open, $\mathcal{M} \subset \omega\mathcal{M}$. Therefore, by (1) we have $\omega\mathcal{M} \subset \omega(\omega\mathcal{M})$. Let $A \in \omega(\omega\mathcal{M})$. By Lemma 3, for each $x \in A$, there exists $U_x \in \omega\mathcal{M}$ containing x and a countable set C_x such that $U_x \setminus C_x \subset A$. Furthermore, by Lemma 3, there exists $V_x \in \mathcal{M}$ containing x and a countable set D_x such that $V_x \setminus D_x \subset U_x$. Therefore, we have $V_x \setminus (C_x \cup D_x) = (V_x \setminus D_x) \setminus C_x \subset U_x \setminus C_x \subset A$. Since $C_x \cup D_x$ is a countable set, we obtain that $A \in \omega\mathcal{M}$. Therefore, $\omega(\omega\mathcal{M}) \subset \omega M$ and hence $\omega(\omega\mathcal{M}) = \omega M$. □

A subfamily $\mathcal{I}$ of $\mathcal{P}(X)$ is called an *ideal* [7] if it satisfies the following properties:
(1) $A \in \mathcal{I}$ and $B \subset A$ imply that $B \in \mathcal{I}$;
(2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ imply that $A \cup B \in \mathcal{I}$.

An M-space $(X, \mathcal{M})$ with an ideal $\mathcal{I}$ is called an ideal M-space and is denoted by $(X, \mathcal{M}, \mathcal{I})$ [2]. In [2], for a subset A of X the $\mathcal{M}$-local function of A is defined as follows:

$$A_*(\mathcal{I}, \mathcal{M}) = \{x \in X : A \cap U \notin \mathcal{I} \text{ for every } U \in \mathcal{M}(x)\},$$

where $\mathcal{M}(x) = \{U \in \mathcal{M} : x \in U\}$. In case there exists no confusion $A_*(\mathcal{I}, \mathcal{M})$ is briefly denoted by A_*. By Theorem 4.2 of [2], it is shown that $Cl_*(A) = A \cup A_*$ is a Kuratowski closure operator. The topology generated by Cl_* is denoted by $\mathcal{M}_*$, that is, $\mathcal{M}_* = \{U \subset X : Cl_*(X \setminus U) = X \setminus U\}$.

Lemma 4. *(Al-Omari and Noiri* [2]*) Let* $(X, \mathcal{M}, \mathcal{I})$ *be an ideal* M*-space. Then* $\beta(\mathcal{M}, \mathcal{I}) = \{V \setminus I : V \in \mathcal{M}, I \in \mathcal{I}\}$ *is a basis for* $\mathcal{M}_*$.

Theorem 4. *For any ideal* M*-space* $(X, \mathcal{M}, \mathcal{I})$, $\omega(\mathcal{M}_*) = (\omega\mathcal{M})_*$.

Proof. First, we show that $\omega(\mathcal{M}_*) \supset (\omega\mathcal{M})_*$. Let $A \in (\omega\mathcal{M})_*$ and $x \in A$. Then, by Lemma 4, there exist $V \in \omega\mathcal{M}$ and $I \in \mathcal{I}$ such that $x \in V \setminus I \subset A$. Since $x \in V \in \omega\mathcal{M}$, there exist $G_x \in \mathcal{M}$ and a countable set C_x such that $x \in G_x \setminus C_x \subset V$. Therefore, $x \in G_x \setminus I \in \mathcal{M}_*$ and $(G_x \setminus I) \setminus C_x = (G_x \setminus C_x) \setminus I \subset V \setminus I \subset A$. This shows that $A \in \omega(\mathcal{M}_*)$. Therefore, $\omega(\mathcal{M}_*) \supset (\omega\mathcal{M})_*$.

Next, we show that $\omega(\mathcal{M}_*) \subset (\omega\mathcal{M})_*$. Let $A \in \omega(\mathcal{M}_*)$ and $x \in A$. Then there exist $V_x \in \mathcal{M}_*$ and a countable set C_x such that $x \in V_x \setminus C_x \subset A$. Since $V_x \in \mathcal{M}_*$, by Lemma 4, there exist $G_x \in \mathcal{M}$ and $I \in \mathcal{I}$ such that $x \in G_x \setminus I \subset V_x$. Then $x \in G_x \setminus C_x \in \omega\mathcal{M}$ and $(G_x \setminus C_x) \setminus I = (G_x \setminus I) \setminus C_x \subset V_x \setminus C_x \subset A$. This shows that $A \in (\omega\mathcal{M})_*$. Therefore, $\omega(\mathcal{M}_*) \subset (\omega\mathcal{M})_*$. Consequently, we obtain that $\omega(\mathcal{M}_*) = (\omega\mathcal{M})_*$. □

In an ideal topological space $(X, \tau, \mathcal{I})$, the topology generated by the local function is denoted by τ^*. It is known in [7] that $\tau^* = \tau^\alpha$ if $\mathcal{I}$ is the nowhere dense ideal. Thus, we have the following corollaries:

Corollary 3. *For any ideal M-space $(X, \mathcal{M}, \mathcal{I})$, $\omega(\mathcal{M}_*) = (\omega\mathcal{M})_* = (\omega\mathcal{M})^*$.*

Corollary 4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $\omega(\tau^*) = (\omega\tau)^*$.*

Corollary 5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\mathcal{I}$ is the nowhere dense ideal, then $\omega\tau^\alpha = (\omega\tau)^\alpha$.*

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