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VECTOR MATRIX REPRESENTATION OF OCTONIONS AND THEIR GEOMETRY

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Abstract. In this paper we investigate octonions and their special vector matrix representation. We give some geometrical definitions and properties related with them. Furthermore, we use the vector matrix representation to show its advantageous sides.

1. Introduction

Hamilton discovered non commutative four dimensional quaternion algebra in 1843. In the same year J. T. Graves discovered non commutative, non associative octonion algebra. After the discovery of quaternion algebra and octonion algebra, Hamilton noticed and mentioned the associativity property which octonion algebra doesn't satisfy [3].

Octonion algebra is an important subject for particle physics, quantum mechanics and many areas of mathematics. Some of our references include usage areas of octonion algebra [1, 3, 5, 6, 7, 8].

To construct octonion algebra $\mathbb{O}$, one can use Cayley-Dickson process by quaternion algebra $\mathbb{H}$. It is well known that Cayley-Dickson process is a generalization of technique which can be used to construct $\mathbb{C}$ from $\mathbb{R}$, $\mathbb{H}$ from $\mathbb{C}$ and $\mathbb{O}$ from $\mathbb{H}$ and so on [3, 4, 10].

It should be noted that the set of real quaternions is as follows;

$$\mathbb{H} = \{\alpha = a_0 + a_1 i + a_2 j + a_3 k \mid a_0,\ a_1,\ a_2,\ a_3 \in \mathbb{R}\}. \tag{1.1}$$

$\mathbb{H}$ is an algebra over $\mathbb{R}$ and the set $\{1,\ i,\ j,\ k\}$ is a base for this algebra $\mathbb{H}$ [10]. In here the equations $i^2 = j^2 = k^2 = -1,$ and $ijk = -1$ are satisfied.

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By the aid of Cayley-Dickson process any real octonion k can be written as follows [3, 8]. In octonion algebra any elements k and l

$$k = \sum_{n=0}^{7} a_n e_n = \alpha + \beta e \text{ and } l = \sum_{n=0}^{7} b_n e_n = \gamma + \delta e \tag{1.2}$$

can be written. Where $\alpha, \beta, \gamma, \delta \in \mathbb{H}$ and $e^2 = -1$. According to the Cayley-Dickson process the addition and multiplication operations can be defined as

$$k + l = (\alpha + \beta e) + (\gamma + \delta e) = (\alpha + \gamma) + (\beta + \delta)e \tag{1.3}$$

and

$$kl = (\alpha + \beta e)(\gamma + \delta e) = (\alpha\gamma - \delta\overline{\beta}) + (\overline{\alpha}\delta + \gamma\beta)e, \tag{1.4}$$

where $\overline{\alpha}$ denotes the conjugate of the quaternion α. The conjugate of any quaternion α and octonion k are defined as follows.

$$\overline{\alpha} = a_0 - a_1 i - a_2 j - a_3 k \tag{1.5}$$

and

$$\overline{k} = \overline{\alpha + \beta e} = (a_0 - a_1 i - a_2 j - a_3 k) - (a_4 + a_5 i + a_6 j + a_7 k)e, \tag{1.6}$$

respectively. Then, with the help of the definition of conjugate the norms of quaternion α and octonion k can be written as follows.

$$Nr(\alpha) = \overline{\alpha}\alpha = \alpha\overline{\alpha} = a_0^2 + a_1^2 + a_2^2 + a_3^2 \tag{1.7}$$

and

$$Nr(k) = \overline{(\alpha + \beta e)}(\alpha + \beta e) = (\alpha + \beta e)\overline{(\alpha + \beta e)} \tag{1.8}$$

$$Nr(k) = a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2, \tag{1.9}$$

respectively.

It should be noted that the quaternion and octonion algebras satisfy composition algebra property; that is

$$Nr(\alpha\beta) = Nr(\alpha)Nr(\beta), \tag{1.10}$$

$$Nr(kl) = Nr(k)Nr(l), \tag{1.11}$$

where $\alpha, \beta \in \mathbb{H}$ and $k, l \in \mathbb{O}$. The octonion algebra $\mathbb{O}$ is a nonassociative but alternative and an eight dimensional division algebra. We know that the canonical base of algebra $\mathbb{O}$ is

$$e_0 = 1,\ \ e_1 = i,\ \ e_2 = j,\ \ e_3 = k,\ \ e_4 = e,\ \ e_5 = ie,\ \ e_6 = je,\ \ e_7 = ke. \tag{1.12}$$

By the aid of the equation (1.12) we give another basic octonion algebra construction method which is explicitly giving octonion algebra's multiplication table as follows.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	$-e_5$	e_4	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	$-e_6$	e_7	e_4	$-e_5$
e_3	e_2	$-e_1$	-1	$-e_7$	$-e_6$	e_5	e_4
e_4	e_5	e_6	e_7	-1	$-e_1$	e_2	e_3
e_5	$-e_4$	$-e_7$	e_6	e_1	-1	$-e_3$	e_2
e_6	e_7	$-e_4$	$-e_5$	e_2	e_3	-1	$-e_1$
e_7	$-e_6$	e_5	$-e_4$	e_3	$-e_2$	e_1	-1

Table 1. Octonion multiplication table

2. Geometry of Octonion Algebra

In this section, we will give some fundamental geometrical definitions related with octonion algebra.

Let $k = \sum_{i=0}^{7} a_i e_i$ be any octonion. The vectorial part of k is $Vec(k) = \sum_{i=1}^{7} a_i e_i$ and scalar part is $Sc(k) = a_0 e_0$. So, the polar form of any octonion k

$$k = \sqrt{Nr(k)}\left(\frac{Sc(k)}{\sqrt{Nr(k)}} + \frac{Vec(k)}{\sqrt{Nr(k)}}\right) = \sqrt{Nr(k)}(cos\theta + \hat{k}sin\theta) \tag{2.1}$$

can be written. Where $\hat{k}$ is defined as $\frac{Vec(k)}{Nr(Vec(k))}$ [5]. De Moivre formula is

$$k^n = cos(n\theta) + \hat{k}sin(n\theta). \tag{2.2}$$

where k is an unit octonion. We note that definitions (2.1) and (2.4) depend on angles are valid as long as they satisfy the following equation.

$$Sc^2(k) + Nr(Vec(k)) = Nr(k). \tag{2.3}$$

An angle λ which is between two octonions k and l is defined as;

$$cos\lambda = \frac{Sc(k\bar{l})}{\sqrt{Nr(k)}\sqrt{Nr(l)}}. \tag{2.4}$$

We can deduce that k and l are perpendicular if $Sc(k\bar{l})$ equals to zero. On the other hand k and l are parallel if $Vec(k\bar{l})$ equals to zero.

Now, we define a map as follows.

$$\phi_x : \mathbb{R}^8 \to \mathbb{R}^8, \quad \phi_x(k) = x(kx^{-1}). \tag{2.5}$$

This function can be interpreted geometrically as describing a rotation of the vector part of k about the vector part of x through an angle 2ϕ. Now, we define a new octonion $\hat{p}$, such that any vector $\hat{p}$ is to be a unit pure octonion and normal to $\hat{x}$. Also, an angle between $\hat{x}$ and $\hat{k}$ can be calculated by equation (2.4). If $\hat{p}$ is a unit pure octonion then $\hat{p}^{-1} = -\hat{p}$ and the angle of this octonion is $\frac{\pi}{2}$. Thus, $\hat{p}(k\hat{p}^{-1}) = -\hat{p}(k\hat{p})$ describes a rotation of $Vec(k)$ through π about $\hat{p}$. Similarly, $\hat{p}(k\hat{p})$ describes a reflection map in the plane [5].

Let X be an octonion whose vector part lies in the plane which is normal to $\hat{p}$. Then, we have

$$Vec(\hat{p}(X\hat{p})) = \hat{p}(Vec(X)\hat{p}) = Vec(X). \tag{2.6}$$

The equation (2.6) means that the elements in the plane normal to $\hat{p}$ are unchanged by the transformation $\hat{p}(X\hat{p})$ [5]. Moreover, in agreement with definitions the direction of any vector parallel to $\hat{p}$ is reversed. For $r \in R$

$$Vec(\hat{p}((r\hat{p})\hat{p})) = -r\hat{p}. \tag{2.7}$$

Now, we mention firstly vector matrix representation of real octonions. Then, we combine vector matrix representation of real octonions with its geometry.

3. Vector Matrix Representation of Octonion Algebra

It is well known that any finite dimensional associative algebra is algebraically isomorphic to a subalgebra of a total matrix algebra. In other words, one can find matrix representation for any associative algebra, but octonion algebra is not associative. Therefore, octonion algebra cannot be isomorphic to associative matrix algebra [11]. To overcome this problem, Zorn defined vector matrix representation for split octonion algebra [2]. After the definition of vector matrix representation this representation is altered for real octonion algebra [7].

Vector matrix representation of any octonion can be denoted as vector matrices according to following vector matrix representations [9].

$$e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \;\; e_4 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \;\; e_j = \begin{bmatrix} 0 & -u_j \\ u_j & 0 \end{bmatrix}, \;\; e_{4+j} \begin{bmatrix} 0 & iu_j \\ iu_j & 0 \end{bmatrix} \tag{3.1}$$

where

$$u_1 = [1,0,0], \;\; u_2 = [0,1,0], \;\; u_3 = [0,0,1] \;\text{ and } j = 1,2,3. \tag{3.2}$$

In order to prevent confusion we will use different notations for both octonions and their representations. For the octonion $k = \sum_{i=0}^{7} a_i e_i$ we can use the equations (3.1) and (3.2). So, we get vector matrix representation. Vector matrix representation for octonion k will be denoted as O and it can be defined as follows,

$$O = \left[\begin{array}{cc} a_0 + ia_4 & -a_1 + ia_5, -a_2 + ia_6, -a_3 + ia_7 \\ a_1 + ia_5, a_2 + ia_6, a_3 + ia_7 & a_0 - ia_4 \end{array} \right]. \tag{3.3}$$

For simplicity, we denote matrix representation of k as follows:

$$O = \left[\begin{array}{cc} z_1 & V_1 \\ V_2 & z_2 \end{array} \right]. \tag{3.4}$$

where

$$z_1 = a_0 + ia_4, \;\; z_2 = \overline{z_1}, \;\; V_1 = -a_1 + ia_5, -a_2 + ia_6, -a_3 + ia_7 \tag{3.5}$$

and

$$V_2 = -\overline{V_1}. \tag{3.6}$$

Multiplication of two vector matrices O and P can be defined as follows [9]:

$$OP = \begin{bmatrix} z_1 & V_1 \\ V_2 & z_2 \end{bmatrix} \begin{bmatrix} z_3 & V_3 \\ V_4 & z_4 \end{bmatrix}, \tag{3.7}$$

$$OP = \begin{bmatrix} z_1 z_3 + V_1 \bullet V_4 & z_1 V_3 + z_4 V_1 - V_2 \times V_4 \\ V_2 z_3 + z_2 V_4 + V_1 \times V_3 & z_2 z_4 + V_2 \bullet V_3 \end{bmatrix}. \tag{3.8}$$

Determinant of O can be calculated as follows:

$$Det(O) = z_1 z_2 + V_1 \bullet V_2. \tag{3.9}$$

Here the operation $\bullet$ is dot product and the operation $\times$ is cross product.

During calculations we see that the vector matrix representation of octonions and multiplication associated with this representation has some advantages; if one is not familiar with octonions, then he or she can use vector matrix representation. Because, this representation contains complex numbers and modified matrix multiplication. Another advantage of this representation is to calculate multiplication of the octonions without knowing the octonion multiplication table.

4. Geometry and Vector Matrix Representation of Octonions

In this section, we will give some examples to combine vector matrix representations of octonions with its geometry. For this purpose, we choose any k and l octonions as follows:

$$k = 3 + 2e_1 - 1e_2 + 1e_4 + 3e_6 - 1e_7, \;\; l = -1e_1 + 2e_3 + 6e_4 - 3e_7. \tag{4.1}$$

Let O and P be vector matrix representations of k and l, respectively. In this case, we calculate O and P as follows:

$$O = \begin{bmatrix} 3+i & -2, 1+3i, -i \\ 2, -1+3i, -i & 3-i \end{bmatrix}, \;\; P = \begin{bmatrix} 6i & 1, 0, -2-3i \\ -1, 0, 2-3i & -6i \end{bmatrix}. \tag{4.2}$$

Now, let us calculate angles of these octonions. From equation (2.1) angle θ of k can be calculated as

$$cos(\theta) = \frac{Sc(k)}{\sqrt{Nr(K)}} = \frac{3}{5}. \tag{4.3}$$

And from equation (2.1) angle σ of l can be calculated

$$cos(\sigma) = \frac{Sc(l)}{\sqrt{Nr(l)}} = 0. \tag{4.4}$$

In these equations one can use the value $Det(O)$ instead of $Nr(k)$.

Now, let us calculate the angle between k and l octonions by equation (2.4),

$$cos(\lambda) = \frac{Sc(k\bar{l})}{\sqrt{Nr(k)}\sqrt{Nr(l)}} = \frac{7}{25\sqrt{2}} \tag{4.5}$$

where λ is the angle between k and l octonions. Multiplication $k\bar{l}$ can be obtained by the Table 1 as

$$k\bar{l} = 7e_0 - 4e_1 + 22e_2 - 8e_3 - 16e_4 - 4e_5 + 13e_6 + 14e_7.$$

or by using vector matrix representation as follows.

For $k\bar{l}$ we use the vector matrix representation. That is,

$$O\bar{P} = \begin{bmatrix} 3+i & -2, 1+3i, -i \\ 2, -1+3i, -i & 3-i \end{bmatrix} \begin{bmatrix} -6i & -1, 0, 2+3i \\ 1, 0, -2+3i & 6i \end{bmatrix}, \quad (4.6)$$

$$O\bar{P} = \begin{bmatrix} 7-16i & 4-4i, -22+13i, 8+14i \\ -4-4i, 22+13i, -8+14i & 7+16i \end{bmatrix}. \quad (4.7)$$

To calculate rotated octonion k' we use the equation (2.5). Necessary calculations can be done either using Table 1 or the vector matrix representation.

According to the equation (2.5) we have the vector matrix representation for k'. That is, we have

$$O' = P(OP^{-1}). \quad (4.8)$$

Also, the vector matrix representations of P, O and P^{-1} are as follows:

$$P = \begin{bmatrix} 6i & 1, 0, -2-3i \\ -1, 0, 2-3i & -6i \end{bmatrix},$$

$$O = \begin{bmatrix} 3+i & -2, 1+3i, -i \\ 2, -1+3i, -i & 3-i \end{bmatrix}$$

and

$$P^{-1} = \frac{1}{5\sqrt{2}} \begin{bmatrix} -6i & -1, 0, 2+3i \\ 1, 0, -2+3i & 6i \end{bmatrix}.$$

Therefore, the vector matrix O' can be obtained as follows:

$$O' = \frac{1}{5\sqrt{2}} \begin{bmatrix} 150+34i & 114, -50-150i, -28+8i \\ -114, 50-150i, 28+8i & 150-34i \end{bmatrix}.$$

Moreover, with this representation we can write

$$k' = \frac{1}{5\sqrt{2}}(150 - 114e_1 + 50e_2 + 28e_3 + 34e_4 - 150e_6 + 8e_7).$$

Consequently, with the help of vector matrix representation, we can calculate the octonion multiplication without using the octonion multiplication table.

5. Conclusion

In this study, we investigated octonions and their geometry with the help of vector matrix representation. It should be noted that the vector matrix representation can be used to calculate octonion addition and multiplication over the field $\mathbb{C}$.

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