



STOCHASTIC STABILITY IN TERMS OF AN ASSOCIATED TRANSFER FUNCTION MATRIX FOR SOME HYBRID SYSTEMS WITH MARKOVIAN SWITCHING

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ABSTRACT. This work is devoted to stability for a dynamical system with Markovian switched controller and a finite number of linear differential sub-systems. We suppose sufficient conditions on a state space representation of an associated transfer function matrix which guarantees stochastic stability. The given examples are detailed to illustrate the main result.

1. INTRODUCTION

Much attention has been drawn by switched dynamical systems due to their extensive applications in many fields such as economic systems, financial markets, communication networks..., (For more information see [4]). The important problem is to ensure stability because the typical problem of switched systems is as follow: it can be that all sub-systems of the jump system are stable but the switched system can be unstable [20].

We refer the reader to the book by Sun and Ge ([17]) for examples, and a recent systematic manner presentation of the stability theory of general switched systems under different switching mechanisms. The other new stability issues, concepts and methods for Markovian switching systems, were introduced by Benaïm et al. in [2] where the same system was studied supposing also that the matrices of the subsystems are Hurwitz (all eigenvalues with negative real parts) without supposing that the subsystems are in companion form. The authors have dealt with the stability problem, but only for planar randomly switched systems and they have proved that under some conditions on the Hurwitz matrices, the norm of the continuous component may go to zero or to infinity depending on the existence of few or many jumps.

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Zhu et al. ([23]) have studied stability of random switching systems of differential equations. They have presented sufficient conditions in terms of a Lyapunov function and they also established verifiable conditions for stability and instability of systems arising in approximation. They have used a logarithm transformation to derive necessary and sufficient conditions for systems that are linear in the continuous state component. They have concluded their paper with several examples, and have noticed a somewhat different behavior from the well-known Hartman-Grobman theorem. A deterministic counterpart in \mathbb{R}^2 was detailed by Balde et al.[1] in the case of arbitrary switching. Also, an implemented method of simulation as Scilab and Matlab toolboxes was presented in [5] and [6] for the deterministic switching case.

In this paper, we consider a hybrid system modulated by a random-switching process which is equivalent to finite number of ordinary differential equations coupled by a switching or jump component. The hybrid system with Markovian switched controller is supposed associated with a transfer matrix $G(s)$ with s a complex number. The motivation behind this quite new formulation is well described in the articles by Kouhi and al. ([11] and [12]). The main contribution in this paper consists on the use of the stochastic machinery in order to obtain some quite new stochastic stability results under verifiable conditions on the parameters and dynamics of the considered Markovian switched controller systems, and to give the suitable form to make the system switch to a finite number of states in a stochastic approach rather than two states in [11] and [12]. The justification of existence of a Lyapunov function is based on the Kalman-Yakubovic-Popov lemma.

This paper is organized as follows: We start by defining the Markovian switching system. Then, we give the conditions under which the system is considered in companion form. Next, the two different stochastic stability are recalled. After that, we establish under some conditions on state space realization of the transfer matrix G that the random-switched system is asymptotically almost surely stable. Finally, the examples are displayed to demonstrate the applicability and effectiveness of the theoretical result.

2. PROBLEM STATEMENT

Consider the closed loop system with state realization (A, B, C, D) associated with the transfer matrix $G(s) = C(sI - A)^{-1}B + D$

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t), \end{cases} \quad (1)$$

with state feedback of the form $u_r(t) = \frac{1}{N}(1 - r(t))Kx(t)$, where $r(t) = \{1, 2, \dots, N\}$ and $K = D^{-1}C$. Then, System (1) becomes

$$\dot{x}(t) = Ax(t) + \frac{1}{N}(1 - r(t))BD^{-1}Cx(t). \quad (2)$$

We can rewrite (2) in the following form

$$\dot{x}(t) = A_{r(t)}x(t), \quad (3)$$

with $A_{r(t)} = A + \frac{1}{N}(1 - r(t))BD^{-1}C$ and random signal $r(t) \in \mathcal{M} = \{1, 2, \dots, N\}$.

This system is represented schematically as

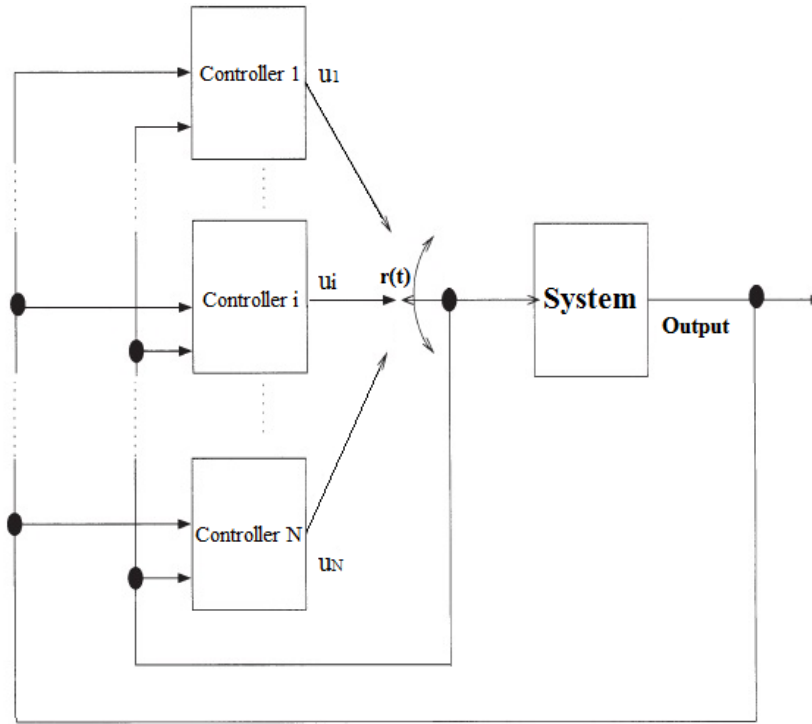


FIGURE 1. Dynamical system with Markovian switched controller.

We start a stochastic counterpart of the results obtained in [11] and [12] by considering a N states Markovian switched controller of the system in order to be able to perform some useful simulations, and prove stochastic stability under a Markovian transition rule rather than an arbitrary switching. Which means that we deal with the weaker notion of stability than the guaranteed one considered in [11].

3. FORMULATION

Throughout this paper, we denote n dimensional identity matrix by I_n or just I . \mathbb{R} and \mathbb{C} denote the field of real and complex numbers respectively. The space

of $n \times n$ matrices with real entries is denoted by $\mathbb{R}^{n \times n}$. We denote a state space representation of the $m \times m$ transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ by (A, B, C, D) , where we always assume that $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ has full column rank, $C \in \mathbb{R}^{m \times n}$ has full row rank, and $0 < D = D^T \in \mathbb{R}^{m \times m}$ for some integers $n \geq m$.

We also use $|y|$ to denote the Euclidean norm for a row or column vector y , v^T is the transpose of $v \in \mathbb{R}^{n_1 \times n_2}$ with $n_i \geq 1$. For $Z \in \mathbb{R}^{n \times n}$ being a square symmetric matrix, we use $\lambda_{max}(Z)$ and $\lambda_{min}(Z)$ to denote the maximum and minimum eigenvalue of Z respectively.

Let us consider a dynamical linear system with random switching in a probability space $(\Omega, \mathcal{F}, Pr)$, and assume that its state equation is described by the following differential equation

$$\dot{x}(t) = A_{r(t)}x(t), \quad (x(0), r(0)) = (x_0, r_0) \in \mathbb{R}^n \times \mathcal{M}, \quad (4)$$

where $x(t)$ is the continuous state, $A_{r(t)} = A + \frac{1}{N}(1-r(t))BD^{-1}C$, and $\{r(t), t \geq 0\}$ is a Markovian jump process with values in a finite state space $\mathcal{M} = \{1, 2, \dots, N\}$. $r(t)$ describes the switching between the N modes. Its evolution is governed by the following probability transition

$$Pr(r(t + \Delta t) = j / r(t) = i) = \begin{cases} q_{ij}\Delta t + o(\Delta t) & \text{if } i \neq j, \\ 1 + q_{ii}\Delta t + o(\Delta t) & \text{otherwise,} \end{cases} \quad (5)$$

where q_{ij} is the transition rate from mode i to j such that $i, j \in \mathcal{M}$ with $q_{ij} \geq 0$ when $i \neq j$, and $q_{ii} = -\sum_{j=1, j \neq i}^2 q_{ij}$ and $o(\Delta t)$ is such that $\lim_{\Delta t \rightarrow 0} \frac{o(\Delta t)}{\Delta t} = 0$.

The jump process $r(t)$ has a constant generator $\mathcal{Q} = (q_{ij})$ and the process $(x(t), r(t))$ defined by (4) and (5), is associated with an infinitesimal operator \mathcal{L} defined as follows:

For each $i \in \mathcal{M}$ and any $g(\cdot, i) \in \mathcal{C}^1(\mathbb{R}^n)$

$$\mathcal{L}g(x, i) = \langle A_i x, \nabla g(x, i) \rangle + \mathcal{Q}g(x, \cdot)(i),$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in \mathbb{R}^n , $\nabla g(x, i)$ denote the gradient (with respect to the variable x) of $g(x, i)$ and $\mathcal{Q}g(x, \cdot)(i) = \sum_{j \in \mathcal{M}} q_{ij}g(x, j)$.

Assume that the Markov chain $r(t)$ is irreducible in the sense that the system of equations

$$\begin{cases} \pi \mathcal{Q} = 0 \\ \pi \mathbf{1} = 1 \end{cases} \quad (6)$$

has a unique positive solution, where $\mathbf{1}$ is a column vector with all component being 1. The positive solution is termed a stationary distribution.

It is known that if P is a transition matrix of a irreducible Markov chain, then,

$$\lim_{n \rightarrow \infty} P^n = \begin{pmatrix} \pi_1 & \pi_2 & \cdot & \cdot & \cdot & \pi_N \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \pi_1 & \pi_2 & \cdot & \cdot & \cdot & \pi_N \end{pmatrix},$$

where $\pi = (\pi_1, \pi_2, \dots, \pi_N)$ is a stationary distribution satisfying

$$\pi P = \pi.$$

In the sequel, we will be interested by determining a Lyapunov function for the class of random switching Systems (4), with $A_{r(t)}$ is a Hurwitz matrix valued function taking values $A_i, i \in \{1, 2, \dots, N\}$ which are related through a transfer function matrix.

We recall that it is known that the theoretical justification of stability for this kind of random switching systems can be assured only through the existence of a suitable Lyapunov function.

Definition 3.1. [12] *An $m \times m$ rational transfer function matrix $G(s)$ is said to be strictly positive real (SPR) if there exists a real scalar $\alpha > 0$ such that $G(s)$ is analytic for $\text{Re}(s) \geq -\alpha$, and*

$$G(j\omega - \alpha) + G^T(-j\omega - \alpha) \geq 0, \quad \forall \omega \in \mathbb{R}.$$

Now, we give a lemma which relates the property of $G(s)$ is SPR and the state space realization (A, B, C, D) when G is a symmetric transfer matrix.

Lemma 3.1. [12] *Given a Hurwitz matrix A , the symmetric transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ with $D = D^T > 0$ is SPR if and only if $A(A - BD^{-1}C)$ has no real negative eigenvalue.*

Furthermore, if (A, B, C, D) is a minimal realization of $G(s)$, then the sufficient condition is also necessary.

Lemma 3.2. [8] *The pair (A, B) is controllable if and only if*

$$\text{rank}([B, AB, \dots, A^{n-1}B]) = n.$$

The pair (A, C) is observable if and only if

$$\text{rank}([C^T, A^T C^T, \dots, (A^T)^{n-1} C^T]) = n.$$

Let us now recall respectively the definitions of stochastically (mean square) stability and (asymptotic) almost surely stability.

Definition 3.2. [22] *A jump linear System (4) is said to be*

- (1) *stochastically (mean square) stable if for any initial state x_0 and initial distribution ρ , we have*

$$\int_0^{+\infty} \mathbb{E}_\rho\{|x(t; x_0, r_0)|^2\} dt < +\infty.$$

- (2) *(asymptotically) almost surely stable if for any initial state x_0 and initial distribution ρ , we have*

$$Pr\{\lim_{t \rightarrow +\infty} |x(t; x_0, r_0)| = 0\} = 1.$$

Lemma 3.3 ([22] page 81). *Any mean square stable jump linear system is almost surely stable.*

Before displaying the main result, let us give a basic result in systems theory by Kalman-Yakubovic-Popov (KYP). It gives algebraic conditions for the existence of a certain type of Lyapunov function for first state.

Lemma 3.4 (KYP). [12] *Let A be Hurwitz, (A, B) be controllable, and (A, C) be observable. Then, $G(s) = C(sI - A)^{-1}B + D$ is SPR if and only if there exist matrices $P = P^T > 0$, L , W , and a number $\alpha > 0$ satisfying*

$$\begin{aligned} A^T P + P A + \alpha P &= -L^T L \\ B^T P + W^T L &= C \\ D + D^T &= W^T W. \end{aligned}$$

4. MAIN RESULT

In this section, we present our main result in this paper concerning stochastic stability (Asymptotic almost surely stability) for System (4) by using Lyapunov function. We give sufficient conditions which guarantee stochastic stability under some conditions on the transfer matrix $G(s)$.

Theorem 4.1. *Assume that the transfer function matrix $G(s)$ is SPR, with (A, B) controllable and (A, C) observable. Then the dynamical random switching System (4) is asymptotically almost surely stable.*

Proof. Let us consider the arbitrary square matrices of order n , P_i and P_j satisfying $P_i = P_j = P = P^T > 0$ for $i \neq j$ on \mathcal{M} . We define a Lyapunov function by the following expression

$$V(x(t), r(t)) = x^T(t) P_{r(t)} x(t).$$

At time t , let $x(t) = x$ and $r(t) = i \in \mathcal{M}$. The infinitesimal operator acting on $V(\cdot, \cdot)$ and emanating from the point (x, i) at time t is given by

$$\begin{aligned} \mathcal{L}V(x, i) &= \langle A_i x, \nabla V(x, i) \rangle + \mathcal{Q}V(x, \cdot)(i) \\ &= x^T A_i^T \frac{\partial}{\partial x} V(x, i) + \sum_{j \in \mathcal{M}} q_{ij} V(x, j) \\ &= x^T (A_i^T P_i + P_i A_i) x + x^T \sum_{j \in \mathcal{M}} q_{ij} P_j x \\ &= x^T \left[\left(A + \frac{1}{N} (1-i) B D^{-1} C \right)^T P_i + P_i \left(A + \frac{1}{N} (1-i) B D^{-1} C \right) \right] x, \end{aligned}$$

with $\sum_{j \in \mathcal{M}} q_{ij} P_j = 0$, because $P_i = P_j, \forall i, j \in \mathcal{M}$ and $\sum_{j \in \mathcal{M}} q_{ij} = 0$. Now, we show that for any $i \in \mathcal{M}$, $\mathcal{L}V(x, i) \leq 0$. The matrix $G(s)$ is SPR, (A, B) and (A, C) are controllable and observable respectively, then, by Lemma 3.4, there exist $P = P^T > 0$, L, W and $\alpha > 0$ such that

$$A^T P + P A = -\alpha P - L^T L \quad (7)$$

$$B^T P + W^T L = C \quad (8)$$

$$D + D^T = W^T W. \quad (9)$$

We first take $i = 1$. By (7), we have

$$\begin{aligned} \mathcal{L}V(x, 1) &= x^T \left(A + \frac{1}{N} (1-1) B D^{-1} C \right)^T P_1 + P_1 \left(A + \frac{1}{N} (1-1) B D^{-1} C \right) x \\ &= x^T (A^T P + P A) x \\ &= -x^T (\alpha P + L^T L) x \\ &< 0. \end{aligned}$$

For $i = 2$, we get

$$\begin{aligned} \mathcal{L}V(x, 2) &= x^T \left[\left(A + \frac{1}{N} (1-2) B D^{-1} C \right)^T P_2 + P_2 \left(A + \frac{1}{N} (1-2) B D^{-1} C \right) \right] x \\ &= x^T \left[\left(A^T - \frac{1}{N} (B D^{-1} C)^T \right) P + P \left(A - \frac{1}{N} (B D^{-1} C) \right) \right] x \\ &= x^T \left[A^T P - \frac{1}{N} C^T D^{-1} B^T P + P A - \frac{1}{N} P B D^{-1} C \right] x. \end{aligned}$$

By (7), (8) and (9), we infer

$$\begin{aligned} \mathcal{L}V(x, 2) &= x^T \left[-\alpha P - L^T L - \frac{1}{N} C^T D^{-1} (C - W^T L) - \frac{1}{N} (C - W^T L)^T D^{-1} C \right] x \\ &= x^T \left[-\alpha P - L^T L - \frac{1}{N} C^T D^{-1} C + \frac{1}{N} C^T D^{-1} W^T L \right. \\ &\quad \left. - \frac{1}{N} C^T D^{-1} C + \frac{1}{N} L^T W D^{-1} C \right] x \end{aligned}$$

$$\begin{aligned}
&\leq x^T \left[-\alpha P - \frac{1}{N} L^T L - \frac{1}{N} C^T D^{-1} C + \frac{1}{N} C^T D^{-1} W^T L \right. \\
&\quad \left. - \frac{1}{N} C^T D^{-1} C + \frac{1}{N} L^T W D^{-1} C \right] x \\
&\leq x^T \left[-\alpha P - \frac{1}{N} L^T (L - W D^{-1} C) + \frac{1}{N} C^T D^{-1} W^T L \right. \\
&\quad \left. - \frac{1}{N} C^T D^{-1} W^T W D^{-1} C \right] x \\
&\leq x^T \left[-\alpha P - \frac{1}{N} L^T (L - W D^{-1} C) + \frac{1}{N} C^T D^{-1} W^T (L - W D^{-1} C) \right] x \\
&\leq x^T \left[-\alpha P - \frac{1}{N} (L - W D^{-1} C)^T (L - W D^{-1} C) \right] x \\
&\leq -x^T \left[\alpha P + \frac{1}{N} (L - W D^{-1} C)^T (L - W D^{-1} C) \right] x \\
&\leq 0.
\end{aligned}$$

For $i = 3$, we have

$$\begin{aligned}
\mathcal{L}V(x, 3) &= x^T \left[(A + \frac{1}{N} (1-3) B D^{-1} C)^T P_3 + P_3 (A + \frac{1}{N} (1-3) B D^{-1} C) \right] x \\
&= x^T \left[(A^T - \frac{2}{N} B D^{-1} C)^T P + P (A - \frac{2}{N} B D^{-1} C) \right] x \\
&= x^T \left[A^T P - \frac{2}{N} C^T D^{-1} B^T P + P A - \frac{2}{N} P B D^{-1} C \right] x.
\end{aligned}$$

By (7), (8) and (9), we infer

$$\begin{aligned}
\mathcal{L}V(x, 3) &= x^T \left[-\alpha P - L^T L - \frac{2}{N} C^T D^{-1} (C - W^T L) - \frac{2}{N} (C - W^T L)^T D^{-1} C \right] x \\
&= x^T \left[-\alpha P - L^T L - \frac{2}{N} C^T D^{-1} C + \frac{2}{N} C^T D^{-1} W^T L \right. \\
&\quad \left. - \frac{2}{N} C^T D^{-1} C + \frac{2}{N} L^T W D^{-1} C \right] x \\
&\leq x^T \left[-\alpha P - \frac{2}{N} L^T L - \frac{2}{N} C^T D^{-1} C + \frac{2}{N} C^T D^{-1} W^T L \right. \\
&\quad \left. - \frac{2}{N} C^T D^{-1} C + \frac{2}{N} L^T W D^{-1} C \right] x \\
&\leq x^T \left[-\alpha P - \frac{2}{N} L^T (L - W D^{-1} C) + \frac{2}{N} C^T D^{-1} W^T L \right. \\
&\quad \left. - \frac{2}{N} C^T D^{-1} W^T W D^{-1} C \right] x
\end{aligned}$$

$$\begin{aligned}
&\leq x^T \left[-\alpha P - \frac{2}{N} L^T (L - WD^{-1}C) + \frac{2}{N} C^T D^{-1} W^T (L - WD^{-1}C) \right] x \\
&\leq x^T \left[-\alpha P - \frac{2}{N} (L - WD^{-1}C)^T (L - WD^{-1}C) \right] x \\
&\leq -x^T \left[\alpha P + \frac{2}{N} (L - WD^{-1}C)^T (L - WD^{-1}C) \right] x \\
&\leq 0.
\end{aligned}$$

By the same argument as steps above, we can show that for any $i \in \mathcal{M} = \{2, \dots, N\}$

$$\mathcal{L}V(x, i) \leq -x^T \left[\alpha P + \frac{i-1}{N} (L - WD^{-1}C)^T (L - WD^{-1}C) \right] x \leq 0.$$

Let

$$M_i = \begin{cases} \alpha P + L^T L & \text{for } i = 1 \\ \alpha P + \frac{i-1}{N} (L - WD^{-1}C)^T (L - WD^{-1}C) & \text{for } i \geq 2, \end{cases}$$

then, for any $i \in \mathcal{M} = \{1, 2, \dots, N\}$, we have

$$\mathcal{L}V(x, i) \leq -x^T M_i x,$$

where M_i is positive defined. That is

$$\mathcal{L}V(x, i) \leq -\min_{i \in \mathcal{M}} \lambda_{\min}(M_i) x^T x.$$

By Dynkin's formula, we get

$$\begin{aligned}
\mathbb{E}(V(x, i)) - V(x_0, r_0) &= \mathbb{E} \left[\int_0^t \mathcal{L}V(x(s), r(s)) ds \right] \\
&\leq -\min_{i \in \mathcal{M}} \lambda_{\min}(M_i) \mathbb{E} \left[\int_0^t x^T(s) x(s) ds / (x_0, r_0) \right],
\end{aligned}$$

then

$$\begin{aligned}
\min_{i \in \mathcal{M}} \lambda_{\min}(M_i) \mathbb{E} \left[\int_0^t x^T(s) x(s) ds / (x_0, r_0) \right] &\leq V(x_0, r_0) - \mathbb{E}(V(x(t), i)) \\
&\leq V(x_0, r_0).
\end{aligned}$$

This yields that

$$\mathbb{E} \left[\int_0^t x^T(s) x(s) ds / (x_0, r_0) \right] \leq \frac{V(x_0, r_0)}{\min_{i \in \mathcal{M}} \lambda_{\min}(M_i)}.$$

Letting $t \rightarrow \infty$, then

$$\mathbb{E} \left[\int_0^{+\infty} |x(s)|^2 ds / (x_0, r_0) \right] \leq C(x_0, r_0),$$

where $C(x_0, r_0) = \frac{V(x_0, r_0)}{\min_{i \in \mathcal{M}} \lambda_{\min}(M_i)}$. This means that the trivial solution of System

(4) is stochastically (mean square) stable. By Lemma 3.3, System (4) is (asymptotically) almost surely stable. The proof of the theorem is complete. \square

Remark 4.1. We remark that if System (4) is 1-dimensional, then, $G(s)$ is a transfer function value, and the pairs (A, B) and (A, C) are always controllable and observable respectively. So, we have the following corollary basing on Lemma 3.1

Corollary 4.2. Let the continuous state $x(t)$ of System (4) be 1-dimensional and A be a Hurwitz matrix, then, the equilibrium point $x = 0$ is asymptotically almost surely stable if

$$A(A - BD^{-1}C) \geq 0.$$

Now, we display two examples to demonstrate the applicability and effectiveness of our theoretical result.

5. EXAMPLES

In this section, we give two examples to illustrate our result.

Example 5.1. In this example, we consider $G(s)$ symmetric in order to verify easily that it is SPR.

Consider System (4) associated to the symmetric transfer matrix $G(s) = C(sI - A)^{-1}B + D$, with

$$A = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ -0.3 & -0.3 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

The Markov jump process $r(t)$ takes values in $\mathcal{M} = \{1, 2, \dots, 5\}$ with generator

$$Q = \begin{pmatrix} -4 & 0 & 1 & 1 & 2 \\ 1 & -2 & 0 & 1 & 0 \\ 6 & 1 & -8 & 0 & 1 \\ 0 & 1 & 1 & -3 & 1 \\ 2 & 1 & 1 & 1 & -5 \end{pmatrix}.$$

The stationary distribution of irreducible Markov process $r(t)$ is $\pi = (0.27, 0.24, 0.09, 0.23, 0.17)$, which is obtained by solving Equation (6). Note that the five Hurwitz matrices associated to System (4) are given by

$$A_i = A + \frac{(1-i)}{5}BD^{-1}C, \quad \text{for } i \in \{1, 2, \dots, 5\}.$$

Then, we have

$$A_1 = \begin{pmatrix} -2.0000 & -1.0000 \\ 1.0000 & 0.0000 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -2.2350 & -1.6310 \\ 0.8330 & -0.3650 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -2.4700 & -2.2620 \\ 0.6660 & -0.7300 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -2.7050 & -2.8930 \\ 0.4990 & -1.0950 \end{pmatrix}$$

and

$$A_5 = \begin{pmatrix} -2.9400 & -3.5240 \\ 0.3320 & -1.4600 \end{pmatrix}.$$

The hybrid System (4) becomes a switching system associated with five ordinary differential equations

$$\dot{x}(t) = \begin{cases} A_1x(t) \\ \cdot \\ \cdot \\ \cdot \\ A_5x(t), \end{cases} \quad (10)$$

switching back and forth from one to another according to the movement of the jump process $r(t)$. Note that (A, B) and (A, C) are controllable and observable respectively; ($\text{rank}([B, AB]) = 2$ and $\text{rank}([C^T, A^T C^T]) = 2$). The transfer function $G(s) = C(sI - A)^{-1}B + D$ is symmetric

$$G(s) = \frac{1}{(s^2 + 2.9s + 2.425)} \begin{pmatrix} s^2 + 6.9s + 9.535 & -s - 1 \\ -s - 1 & 2s^2 + 5.8s + 4.7 \end{pmatrix}$$

and $A(A - BD^{-1}C)$ has no real negative eigenvalue, that means that $G(s)$ is SPR. Then, by Theorem 4.1 the hybrid System (10) is asymptotically almost surely stable.

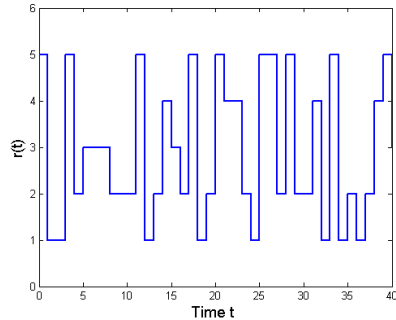


FIGURE 2. Markov jump $r(t)$ with initial condition $r(0) = 5$

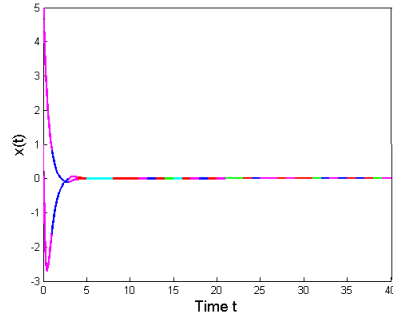


FIGURE 3. Trajectory solution of (10) with initial condition $x(0) = [1, 5]^T$

Example 5.2. Consider the Markovian jump $r(t)$ taking values in $\mathcal{M} = \{1, 2, 3\}$,

with infinitesimal operator $\mathcal{Q} = \begin{pmatrix} -3 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 3 & -3 \end{pmatrix}$.

Let the 1-dimensional System (4), with following specifications

$$A = (-1), B = (-3), C = (0.2), \text{ and } D = (0.6).$$

The expression of the transfer function G is $G(s) = \frac{0.6s}{s+1}$.

We note that

$$A_1 = A = (-1), A_2 = A - \frac{1}{3}BD^{-1}C = (-0.67) \text{ and } A_3 = A - \frac{2}{3}BD^{-1}C = (-0.33).$$

Then, we can rewrite System (4) as the following hybrid system consisting of three ordinary differential equations

$$\dot{x}(t) = \begin{cases} -x(t) & \text{if } r(t) = 1 \\ -0.67x(t) & \text{if } r(t) = 2 \\ -0.33x(t) & \text{if } r(t) = 3. \end{cases} \quad (11)$$

Note that $A(A - BD^{-1}C) = 0 \geq 0$. Then, by Corollary 4.2, System (11) is asymptotically almost surely stable.

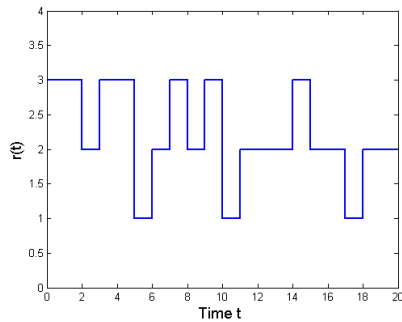


FIGURE 4. Markov jump $r(t)$ with initial condition $r(0) = 3$.

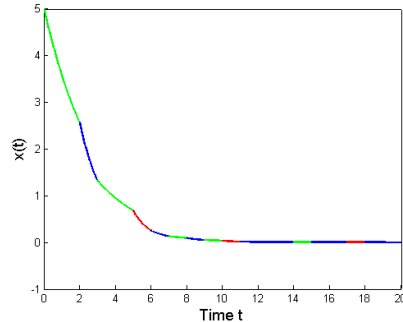


FIGURE 5. Trajectory solution of (11) with initial condition $x(0) = 5$

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