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ON NEW TYPES OF SETS VIA γ-OPEN SETS IN BITOPOLOGICAL SPACES

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Abstract. In this paper, the concept of (i,j)-γ-P-open sets in bitopological spaces are introduced and characterizations of their related notions are given.

1. Introduction

The notion of bitopological space (X, τ_1, τ_2) which is a nonempty set X endowed with two topologies τ_1 and τ_2 is introduced by Kelly [1]. Also in [1], some basic results of separation axioms in topological spaces are extended to bitopological spaces. In a bitopological space (X, τ_1, τ_2), a set $S \subseteq X$ is called (i,j)-preopen [2] if $S \subseteq \tau_i$-Int$(\tau_j$-Cl$(S))$, where $i, j = 1, 2$ and $i \neq j$. For a subset S of a bitopological space, τ_i-Int(S) and τ_i-Cl(S) stand for the interior and the closure of S with respect to the topology τ_i, respectively. A function $f : (X, \tau_1, \tau_2) \to (Y, \zeta_1, \zeta_2)$ is called (i,j)-precontinuous [2] if the inverse image of every ζ_i-open set in (Y, ζ_1, ζ_2) is (i,j)-preopen in (X, τ_1, τ_2).

Ogata [3] defined an operation γ on a topological space (X, τ) as a mapping from τ into the power set $\mathcal{P}(X)$ of X such that $U \subseteq \gamma(U)$ for each $U \in \tau$, where $\gamma(U)$ denotes the value of γ at U. By means of this operation, he defined the concept of operation-open sets called γ-open sets. A subset S of X is said to be γ-open if for each $x \in S$, there exists an open set U containing x such that $\gamma(U) \subseteq S$. It is clear that γ-open sets are open. A subset of X is called γ-closed if its complement is γ-open. τ_γ-interior of S [4] denoted by τ_γ-Int(S) is the union of all γ-open sets of X contained in S and τ_γ-closure of S [3] denoted by τ_γ-Cl(S) is the intersection of all γ-closed sets of X containing S.

In topological spaces, some generalizations of γ-open sets are introduced and some properties of these sets are investigated. Krishan and Balachandran [5] defined γ-preopen sets. A subset S of a topological space (X, τ) is said to be γ-preopen if $S \subseteq \tau_\gamma$-Int$(\tau_\gamma$-Cl$(S))$. Later, γ-P-open sets are defined and some weak separation

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axioms are introduced by Khalaf and Ibrahim [6]. A subset S of a topological space (X,τ) is said to be γ-P-open if $S\subseteq$Int(τ_γ-Cl(S)). By $\gamma PO(X,\tau)$, we denote the family of all γ-P-open sets in (X,τ).

Operations on bitopological spaces were studied in [7]. An operation γ on a bitopological space (X,τ_1,τ_2) is a mapping $\gamma:\tau_1\cup\tau_2\to\mathcal{P}(X)$ such that $U\subseteq\gamma(U)$ for each $U\in\tau_1\cup\tau_2$. In [8], the author studied some separation axioms in bitopological spaces by utilizing operations on such spaces.

By $\tau_{i\gamma}$-Int(S) and $\tau_{j\gamma}$-Cl(S), we denote the τ_γ-interior of S in (X,τ_i) and τ_γ-closure of S in (X,τ_j), respectively.

The main purpose of this paper is to extend the concepts of γ-preopen sets and γ-P-open sets which are weaker than γ-open sets to bitopological spaces. We give some properties related to these sets and also introduce some separation axioms in bitopological spaces. Further we define new types of functions in bitopological spaces, namely (i,j)-γ-pre-continuous and (i,j)-γ-P-continuous functions.

2. (i,j)-γ-P-Open Sets

Throughout the paper, we assume that $i,j=1,2$ and $i\neq j$.

Definition 1. *A subset S of a bitopological space (X,τ_1,τ_2) with an operation γ on $\tau_1\cup\tau_2$ is said to be:*
1. *(i,j)-γ-pre-open if $S\subseteq\tau_{i\gamma}$-Int($\tau_{j\gamma}$-Cl(S)).*
2. *(i,j)-γ-P-open if $S\subseteq\tau_i$-Int($\tau_{j\gamma}$-Cl(S)).*

By $\gamma PO(X,\tau_1,\tau_2)$, we denote the family of all (i,j)-γ-P-open sets in (X,τ_1,τ_2).

Theorem 1. *Every (i,j)-γ-pre-open set is (i,j)-γ-P-open.*

Proof. Let S be an (i,j)-γ-pre-open set. Then we have $S\subseteq\tau_{i\gamma}$-Int($\tau_{j\gamma}$-Cl(S)) $\subseteq\tau_i$-Int($\tau_{j\gamma}$-Cl(S)) and so S is (i,j)-γ-P-open. □

The following example shows that the converse of the above theorem is not true generally.

Example 1. *Consider $X=\{a,b,c,d\}$ with topologies $\tau_1=\{X,\emptyset,\{b\},\{d\},\{b,d\},\{a,b,c\}\}$, $\tau_2=\{X,\emptyset,\{a\},\{d\},\{a,b\},\{a,d\},\{a,b,d\}\}$ and a subset $\{a,b\}$ of X. Let γ be an operation on $\tau_1\cup\tau_2$ defined as follows:*

$$\gamma(U)=\begin{cases}U, & \text{if } U=\{d\}\\ X, & \text{if } U\neq\{d\}\end{cases}$$

Then $\{a,b\}$ is $(1,2)$-γ-P-open but it is not $(1,2)$-γ-pre-open.

Theorem 2. *Every (i,j)-pre-open set is (i,j)-γ-P-open.*

Proof. Let S be an (i,j)-pre-open set. Then we have $S\subseteq\tau_i$-Int(τ_j-Cl(S)) $\subseteq\tau_i$-Int($\tau_{j\gamma}$-Cl(S)) and so S is (i,j)-γ-P-open. □

The following example shows that the converse of the above theorem is not true generally.

Example 2. *Consider $X = \{a, b, c\}$ with topologies $\tau_1 = \{X, \emptyset, \{a\}\}$, $\tau_2 = \{X, \emptyset, \{b\}\}$ and a subset $\{c\}$ of X. Let γ be an operation on $\tau_1 \cup \tau_2$ defined by $\gamma(U) = X$. Then $\{c\}$ is $(1, 2)$-γ-P-open but it is not $(1, 2)$-pre-open.*

Conclusion 1. *If γ is identity operator on $\tau_1 \cup \tau_2$, then (i, j)-pre-open sets and (i, j)-γ-P-open sets coincide.*

Lemma 1. *There is no relation between (i, j)-pre-open sets and (i, j)-γ-pre-open sets.*

Example 3. *Let X, τ_1, τ_2 and γ be as in Example 1.*
Then $\{a, b\}$ is $(1, 2)$-pre-open which is not $(1, 2)$-γ-pre-open and $\{c, d\}$ is $(1, 2)$-γ-pre-open which is not $(1, 2)$-pre-open.

Theorem 3. *Every τ_i-open set is (i, j)-γ-P-open.*

It can be seen from the following example that (i, j)-γ-P-open set is not need to be τ_i-open.

Example 4. *Let X, τ_1, τ_2 and γ be as in Example 2.*
Then $\{b\}$ is $(1, 2)$-γ-P-open but $\{b\} \notin \tau_1$.

Corollary 1. *Every γ_i-open set is (i, j)-γ-P-open.*

Theorem 4. *Let $\{S_\alpha : \alpha \in \Lambda\}$ be a class of (i, j)-γ-P-open sets. Then $\cup_{\alpha\in\Lambda} S_\alpha$ is (i, j)-γ-P-open.*

Proof. Since S_α is an (i, j)-γ-P-open set, we have $S_\alpha \subseteq \tau_i\text{-Int}(\tau_{j\gamma}\text{-Cl}(S_\alpha))$ for all $\alpha \in \Lambda$. Hence it is obtained

$$\begin{aligned} \bigcup_{\alpha\in\Lambda} S_\alpha &\subseteq \bigcup_{\alpha\in\Lambda} \tau_i\text{-Int}(\tau_{j\gamma}\text{-Cl}(S_\alpha)) \\ &\subseteq \tau_i\text{-Int}\left(\bigcup_{\alpha\in\Lambda} \tau_{j\gamma}\text{-Cl}(S_\alpha)\right) \\ &\subseteq \tau_i\text{-Int}\left(\tau_{j\gamma}\text{-Cl}\left(\bigcup_{\alpha\in\Lambda} S_\alpha\right)\right). \end{aligned}$$

Therefore $\cup_{\alpha\in\Lambda} S_\alpha$ is also (i, j)-γ-P-open. □

The intersection of two (i, j)-γ-P-open sets is not need to be (i, j)-γ-P-open as shown in the following example.

Example 5. *Consider $X = \{a, b, c\}$ with topologies $\tau_1 = \{X, \emptyset, \{a\}, \{a, b\}\}$ and $\tau_2 = \{X, \emptyset, \{b\}, \{a, b\}\}$. Let γ be an operation on $\tau_1 \cup \tau_2$ defined as follows:*

$$\gamma(U) = \begin{cases} U, & \text{if } U = \{a, b\} \\ X, & \text{if } U \neq \{a, b\} \end{cases}$$

Then the sets $\{a, c\}$ and $\{b, c\}$ are $(1, 2)$-γ-P-open but their intersection $\{c\}$ is not $(1, 2)$-γ-P-open.

Lemma 2. *It is obtained that* $\gamma PO(X,\tau_1,\tau_2) \neq \gamma PO(X,\tau_1) \cup \gamma PO(X,\tau_2)$.

Example 6. *Let* $X = \{a,b,c\}$ *be given with topologies* $\tau_1 = \{X,\emptyset,\{a\},\{b,c\}\}$, $\tau_2 = \{X,\emptyset,\{b\}\}$ *and* γ *be defined as follows:*

$$\gamma(U) = \begin{cases} U, & \text{if } U = \{a\} \\ X, & \text{if } U \neq \{a\} \end{cases}$$

Then it is obtained

$$\gamma PO(X,\tau_1) = \{X,\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,c\}\},$$

$$\gamma PO(X,\tau_2) = \{X,\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,c\}\}$$

and

$$\gamma PO(X,\tau_1,\tau_2) = \{X,\emptyset,\{a\},\{a,b\},\{a,c\}\}.$$

Definition 2. *A subset* F *of a bitopological space* (X,τ_1,τ_2) *with an operation* γ *on* $\tau_1 \cup \tau_2$ *is said to be* (i,j)-γ-P*-closed ((*(i,j)-γ*-pre-closed) if* $X \backslash F$ *is* (i,j)-γ-P*-open ((*(i,j)-γ*-pre-open) in* X.

By $\gamma PC(X,\tau_1,\tau_2)$, we denote the family of all (i,j)-γ-P-closed sets in (X,τ_1,τ_2).

Theorem 5. *A subset* F *is* (i,j)-γ-P*-closed in a bitopological space* (X,τ_1,τ_2) *if and only if* $\tau_i\text{-}Cl(\tau_{j\gamma}\text{-}Int(F)) \subseteq F$.

Proof. Let F be an (i,j)-γ-P-closed set in X. Then $X\backslash F$ is (i,j)-γ-P-open that is $X\backslash F \subseteq \tau_i\text{-Int}(\tau_{j\gamma}\text{-Cl}(X\backslash F))$. It follows that

$$\begin{aligned} F &\supseteq X\backslash \tau_i\text{-Int}(\tau_{j\gamma}\text{-Cl}(X\backslash F)) \\ &= \tau_i\text{-Cl}(X\backslash \tau_{j\gamma}\text{-Cl}(X\backslash F)) \\ &= \tau_i\text{-Cl}(\tau_{j\gamma}\text{-Int}(F)). \end{aligned}$$

Conversely, we obtain

$$\begin{aligned} X\backslash F &\subseteq X\backslash \tau_i\text{-Cl}(\tau_{j\gamma}\text{-Int}(F)) \\ &= \tau_i\text{-Int}(X\backslash \tau_{j\gamma}\text{-Int}(F)) \\ &= \tau_i\text{-Int}(\tau_{j\gamma}\text{-Cl}(X\backslash F)) \end{aligned}$$

which completes the proof. □

Theorem 6. *Let* $\{F_\alpha : \alpha \in \Lambda\}$ *be a class of* (i,j)-γ-P*-closed sets. Then* $\cap_{\alpha\in\Lambda} F_\alpha$ *is* (i,j)-γ-P*-closed.*

Proof. The proof follows from Definition 2 and Theorem 4. □

The union of two (i,j)-γ-P-closed sets is not need to be (i,j)-γ-P-closed. By Example 5, the sets $\{b\}$ and $\{a\}$ are $(1,2)$-γ-P-closed but their union $\{a,b\}$ is not $(1,2)$-γ-P-closed.

Definition 3. *(i,j)-γ-P-interior ((i,j)-γ-pre-interior) point of a subset S of a bitopological space (X,τ_1,τ_2) is a point x in X satisfying the inclusion $V \subseteq S$ for an (i,j)-γ-P-open ((i,j)-γ-pre-open) set V containing x.*

By (i,j)-γ-P-Int(S) ((i,j)-γ-pre-Int(S)), we denote the (i,j)-γ-P-interior ((i,j)-γ-pre-interior) of S consisting of all (i,j)-γ-P-interior ((i,j)-γ-pre-interior) points of S.

Theorem 7. *The following properties hold for any subset S of a bitopological space (X,τ_1,τ_2):*

1. *(i,j)-γ-P-Int(S) is the union of all (i,j)-γ-P-open sets (the largest (i,j)-γ-P-open set) contained in S.*
2. *(i,j)-γ-P-Int(S) is an (i,j)-γ-P-open set.*
3. *S is (i,j)-γ-P-open if and only if $S = (i,j)$-γ-P-Int(S).*

Proof. The proof follows from definitions. □

Theorem 8. *The following properties hold for any subsets S_1, S_2 and any class of subsets $\{S_\alpha : \alpha \in \Lambda\}$ of a bitopological space (X,τ_1,τ_2):*

1. *If $S_1 \subseteq S_2$, then (i,j)-γ-P-Int$(S_1) \subseteq (i,j)$-γ-P-Int(S_2).*
2. *$\bigcup_{\alpha\in\Lambda} (i,j)$-$\gamma$-$P$-Int$(S_\alpha) \subseteq (i,j)$-$\gamma$-$P$-Int$(\bigcup_{\alpha\in\Lambda} S_\alpha)$.*
3. *(i,j)-γ-P-Int$(\bigcap_{\alpha\in\Lambda} S_\alpha) \subseteq \bigcap_{\alpha\in\Lambda} (i,j)$-$\gamma$-$P$-Int$(S_\alpha)$.*

Proof. The first statement is obvious. The second and the third ones follow from 1 which implies (i,j)-γ-P-Int$(S_\alpha) \subseteq (i,j)$-$\gamma$-$P$-Int$(\cup_{\alpha\in\Lambda} S_\alpha)$ and (i,j)-γ-P-Int$(\cap_{\alpha\in\Lambda} S_\alpha) \subseteq (i,j)$-$\gamma$-$P$-Int$(S_\alpha)$ for all $\alpha \in \Lambda$. Hence we have the results $\cup_{\alpha\in\Lambda}(i,j)$-$\gamma$-$P$-Int$(S_\alpha) \subseteq (i,j)$-$\gamma$-$P$-Int$(\cup_{\alpha\in\Lambda} S_\alpha)$ and (i,j)-γ-P-Int$(\cap_{\alpha\in\Lambda} S_\alpha) \subseteq \cap_{\alpha\in\Lambda}(i,j)$-$\gamma$-$P$-Int$(S_\alpha)$, respectively. □

The reverse inclusions in 2 and 3 of Theorem 8 may not be applicable as shown in the following examples.

Example 7. *Let X, τ_1, τ_2 and γ be as in Example 6. Then we have*

$$\{a,b\} = (i,j)\text{-}\gamma\text{-}P\text{-}Int\{a,b\} \nsubseteq (i,j)\text{-}\gamma\text{-}P\text{-}Int\{a\} \cup (i,j)\text{-}\gamma\text{-}P\text{-}Int\{b\} = \{a\}.$$

Example 8. *Let X, τ_1, τ_2 and γ be as in Example 5. Then we have*

$$\{c\} = (i,j)\text{-}\gamma\text{-}P\text{-}Int\{a,c\} \cap (i,j)\text{-}\gamma\text{-}P\text{-}Int\{b,c\} \nsubseteq (i,j)\text{-}\gamma\text{-}P\text{-}Int\{c\} = \emptyset.$$

Definition 4. *(i,j)-γ-P-cluster ((i,j)-γ-pre-cluster) point of a subset S of a bitopological space (X,τ_1,τ_2) is a point x in X satisfying $V \cap S \neq \emptyset$ for every (i,j)-γ-P-open ((i,j)-γ-pre-open) set V containing x.*

By (i,j)-γ-P-Cl(S) ((i,j)-γ-pre-Cl(S)), we denote the (i,j)-γ-P-closure ((i,j)-γ-pre-closure) of S consisting of all (i,j)-γ-P-cluster ((i,j)-γ-pre-cluster) points of S.

Theorem 9. *The following properties hold for any subset S of a bitopological space (X, τ_1, τ_2):*

1. (i,j)-γ-P-$Cl(S)$ is the intersection of all (i,j)-γ-P-closed sets (the smallest (i,j)-γ-P-closed set) containing S.

2. (i,j)-γ-P-$Cl(S)$ is an (i,j)-γ-P-closed set.

3. S is (i,j)-γ-P-closed if and only if $S = (i,j)$-γ-P-$Cl(S)$.

Proof. The proof follows from definitions. □

Theorem 10. *The following properties hold for any subsets S_1, S_2 and any class of subsets $\{S_\alpha : \alpha \in \Lambda\}$ of a bitopological space (X, τ_1, τ_2):*

1. If $S_1 \subseteq S_2$, then (i,j)-γ-P-$Cl(S_1) \subseteq (i,j)$-γ-P-$Cl(S_2)$.

2. $\bigcup\limits_{\alpha\in\Lambda} (i,j)$-$\gamma$-$P$-$Cl(S_\alpha) \subseteq (i,j)$-$\gamma$-$P$-$Cl(\bigcup\limits_{\alpha\in\Lambda} S_\alpha)$.

3. (i,j)-γ-P-$Cl(\bigcap\limits_{\alpha\in\Lambda} S_\alpha) \subseteq \bigcap\limits_{\alpha\in\Lambda} (i,j)$-$\gamma$-$P$-$Cl(S_\alpha)$.

Proof. It can be proved in a similar way with the proof of Theorem 8 □

The reverse inclusions in 2 and 3 of Theorem 10 may not be applicable as shown in the following examples.

Example 9. *Let X, τ_1, τ_2 and γ be as in Example 5. Then it is obtained $\gamma PC(X, \tau_1, \tau_2) = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{b,c\}, \{a,c\}\}$. Thus we have*

$$X = (i,j)\text{-}\gamma\text{-}P\text{-}Cl\{a,b\} \nsubseteq (i,j)\text{-}\gamma\text{-}P\text{-}Cl\{a\} \cup (i,j)\text{-}\gamma\text{-}P\text{-}Cl\{b\} = \{a,b\}.$$

Example 10. *Let X, τ_1, τ_2 and γ be as in Example 6. Then it is obtained $\gamma PC(X, \tau_1, \tau_2) = \{X, \emptyset, \{b\}, \{c\}, \{b,c\}\}$. Thus we have*

$$\{b\} = (i,j)\text{-}\gamma\text{-}P\text{-}Cl\{a\} \cap (i,j)\text{-}\gamma\text{-}P\text{-}Cl\{b\} \nsubseteq (i,j)\text{-}\gamma\text{-}P\text{-}Cl(\{a\} \cap \{b\}) = \emptyset.$$

Theorem 11. *The following properties hold for a subset S of a bitopological space (X, τ_1, τ_2):*

1. (i,j)-γ-P-$Int(X \backslash S) = X \backslash (i,j)$-$\gamma$-$P$-$Cl(S)$.

2. (i,j)-γ-P-$Cl(X \backslash S) = X \backslash (i,j)$-$\gamma$-$P$-$Int(S)$.

Proof. The proof follows from definitions. □

Definition 5. *(i,j)-γ-Pneighborhood of a point x in a bitopological space (X, τ_1, τ_2) is a subset N_x of X satisfying the inclusion $U \subseteq N_x$ for an (i,j)-γ-P-open set U containing x.*

Theorem 12. *A subset of a bitopological space (X, τ_1, τ_2) is (i,j)-γ-P-open if and only if one of the (i,j)-γ-Pneighborhoods of each points of the subset is itself.*

Proof. The proof follows from Definition 5. □

3. (i,j)-γ-P-T_k Spaces

Definition 6. *A bitopological space (X,τ_1,τ_2) is said to be (i,j)-γ-P-T_0 if for every distinct points x and y of X, there exists an (i,j)-γ-P-open set containing x but not y or vice versa.*

Theorem 13. *A bitopological space (X,τ_1,τ_2) is (i,j)-γ-P-T_0 if and only if for each distinct points x and y of X (i,j)-γ-P-$Cl\{x\} \neq (i,j)$-γ-P-$Cl\{y\}$.*

Proof. Let x and y be any two distinct points of X. Suppose that U contains x but not y for an (i,j)-γ-P-open set U of X. Then we have $\{y\} \cap U = \emptyset$ which implies $x \notin (i,j)$-γ-P-Cl$\{y\}$. Hence the result (i,j)-γ-P-Cl$\{x\} \neq (i,j)$-γ-P-Cl$\{y\}$ is clear.

Conversely, suppose that (i,j)-γ-P-Cl$\{x\} \neq (i,j)$-γ-P-Cl$\{y\}$ for every distinct points x and y of X. Let $z \in (i,j)$-γ-P-Cl$\{y\}$ but $z \notin (i,j)$-γ-P-Cl$\{x\}$. Then we have $\{y\} \cap U \neq \emptyset$ for every (i,j)-γ-P-open set U containing z and $\{x\} \cap U = \emptyset$ for at least one (i,j)-γ-P-open set U containing z. That is $y \in U$ and $x \notin U$ for an (i,j)-γ-P-open set. Thus the space X is (i,j)-γ-P-T_0. □

Definition 7. *A bitopological space (X,τ_1,τ_2) is said to be (i,j)-γ-P-T_1 if for every distinct points x and y of X, there exist two (i,j)-γ-P-open sets which one of them contains x but not y and the other one contains y but not x.*

Theorem 14. *A bitopological space (X,τ_1,τ_2) is (i,j)-γ-P-T_1 if and only if for each point x of X (i,j)-γ-P-$Cl\{x\} = \{x\}$.*

Proof. Let $y \notin \{x\}$ for x, y in X. Then by hypothesis, there exists an (i,j)-γ-P-open set U such that $y \in U$ but $x \notin U$. The result follows from $\{x\} \cap U = \emptyset$ which means $y \notin (i,j)$-γ-P-Cl$\{x\}$.

Conversely, let $x \neq y$ for x, y in X. Since $x \notin (i,j)$-γ-P-Cl$\{y\}$ and $y \notin (i,j)$-γ-P-Cl$\{x\}$, there exist (i,j)-γ-P-open sets U and V containing x and y, respectively such that $\{y\} \cap U = \emptyset$ and $\{x\} \cap V = \emptyset$. Hence we have $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Thus the proof is completed. □

Definition 8. *A bitopological space (X,τ_1,τ_2) is said to be (i,j)-γ-P-T_2 if for every distinct points x and y of X, there exist two disjoint (i,j)-γ-P-open sets which one of them contains x and the other one contains y.*

Theorem 15. *A bitopological space (X,τ_1,τ_2) is (i,j)-γ-P-T_2 if and only if for each distinct points x and y of X there exists an (i,j)-γ-P-open set U containing x such that $y \notin (i,j)$-γ-P-$Cl(U)$.*

Proof. The proof is obvious. □

Theorem 16. *A bitopological space (X,τ_1,τ_2) is (i,j)-γ-P-T_2 if and only the intersection of all (i,j)-γ-P-closed (i,j)-γ-Pneighborhood of each point of X consists of only that point.*

Proof. Let x be any point of X. Suppose that there exist (i,j)-γ-P-open sets U_y and V_y such that $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$ for each point y which is distinct from x. Hence $X \backslash V_y$ is an (i,j)-γ-P-closed (i,j)-γ-Pneighborhood of x which does not contain y since the inclusion $U_y \subseteq X \backslash V_y$ holds. This means that the equivalence $\cap\{X \backslash V_y : y \in X, y \neq x\} = \{x\}$ holds which completes the first part of the proof.

Let x and y be any two distinct points of X. By our hypothesis, there is an (i,j)-γ-P-closed (i,j)-γ-Pneighborhood of x in which y is not contained. Then we have $y \in X \backslash U$ and $X \backslash U$ is (i,j)-γ-P-open. Also, there exits an (i,j)-γ-P-open set V containing x such that $V \subseteq U$ since U is an (i,j)-γ-Pneighborhood of x. It is clear that the sets V and $X \backslash U$ are disjoint. This shows that X is an (i,j)-γ-P-T_2 space. □

4. (i,j)-γ-P-continuous functions

Definition 9. *Let $f : (X, \tau_1, \tau_2) \to (Y, \zeta_1, \zeta_2)$ be a function and x be any point of X. f is said to be (i,j)-γ-P-continuous (resp. (i,j)-γ-pre-continuous) at x if for every ζ_i open set H of Y containing $f(x)$ there exists an (i,j)-γ-P-open (resp. (i,j)-γ-pre-open) set G of X containing x such that $f(G) \subseteq H$.*

Theorem 17. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \zeta_1, \zeta_2)$, the following statements are equivalent:*

1. f is (i,j)-γ-P-continuous (resp. (i,j)-γ-pre-continuous).

2. For every ζ_i open subset O of Y, $f^{-1}(O)$ is an (i,j)-γ-P-open (resp. (i,j)-γ-pre-open) set in X.

3. For every ζ_i closed subset C of Y, $f^{-1}(C)$ is an (i,j)-γ-P-closed (resp. (i,j)-γ-pre-closed) set in X.

4. For every subset S of X, $f((i,j)\text{-}\gamma\text{-}P\text{-}Cl(S)) \subseteq \zeta_i\text{-}Cl(f(S))$ (resp. $f((i,j)\text{-}\gamma\text{-}pre\text{-}Cl(S)) \subseteq \zeta_i\text{-}Cl(f(S))$).

5. For every subset T of Y, $(i,j)\text{-}\gamma\text{-}P\text{-}Cl(f^{-1}(T)) \subseteq f^{-1}(\zeta_i\text{-}Cl(T))$ (resp. $(i,j)\text{-}\gamma\text{-}pre\text{-}Cl(f^{-1}(T)) \subseteq f^{-1}(\zeta_i\text{-}Cl(T))$).

Proof.

$1 \Rightarrow 2$ Let O be ζ_i open in Y and $x \in f^{-1}(O)$. Since f is (i,j)-γ-P-continuous on X, there exists an (i,j)-γ-P-open set G of X containing x such that $f(G) \subseteq O$. Hence we have $G \subseteq f^{-1}(O)$ which means x is an (i,j)-γ-P-interior point of $f^{-1}(O)$. Thus $f^{-1}(O)$ is an (i,j)-γ-P-open set in X.

$2 \Rightarrow 1$ Let x be any point of X and H be a ζ_i open set containing $f(x)$. By the assumption, $f^{-1}(H)$ is (i,j)-γ-P-open and $x \in f^{-1}(H)$. For $G = f^{-1}(H)$, we have $f(G) \subseteq H$ which concludes the proof.

$2 \Leftrightarrow 3$ It is obvious.

$1 \Rightarrow 4$ Let S be a subset of X and $f(x) \in f((i,j)\text{-}\gamma\text{-}P\text{-}\mathrm{Cl}(S))$, where $x \in (i,j)\text{-}\gamma\text{-}P\text{-}\mathrm{Cl}(S)$. Take any ζ_i open set H of Y containing $f(x)$. Since there exists an (i,j)-γ-P-open set G of X containing x such that $f(G) \subseteq H$ and also $G \cap S \neq \emptyset$, we have $H \cap f(S) \neq \emptyset$. This means $f(x) \in \zeta_i\text{-}\mathrm{Cl}(f(S))$. Thus the inclusion $f((i,j)\text{-}\gamma\text{-}P\text{-}\mathrm{Cl}(S)) \subseteq \zeta_i\text{-}\mathrm{Cl}(f(S))$ holds for any subset S of X.

$4 \Rightarrow 5$ Let T be a subset of Y. Then by the inclusion in 4, we have $f((i,j)\text{-}\gamma\text{-}P\text{-}\mathrm{Cl}(f^{-1}(T))) \subseteq \zeta_i\text{-}\mathrm{Cl}(T)$. Taking the pre-image of both sides, we obtain $(i,j)\text{-}\gamma\text{-}P\text{-}\mathrm{Cl}(f^{-1}(T)) \subseteq f^{-1}(\zeta_i\text{-}\mathrm{Cl}(T))$.

$5 \Rightarrow 3$ Let C be ζ_i-closed in Y. Then by the inclusion in 5, we have $(i,j)\text{-}\gamma\text{-}P\text{-}\mathrm{Cl}(f^{-1}(C)) \subseteq f^{-1}(C)$. Hence $f^{-1}(C)$ is (i,j)-γ-P-closed in X. □

Corollary 2.
1. *Every (i,j)-γ-pre-continuous function is (i,j)-γ-P-continuous.*
2. *Every (i,j)-pre-continuous function is (i,j)-γ-P-continuous.*

(i,j)-γ-P-continuous functions may not be (i,j)-γ-pre-continuous and (i,j)-pre-continuous as seen from the following examples.

Example 11. *Let X, τ_1, τ_2 and γ be as in Example 1. Define a function f on X as $f(a) = b$, $f(b) = b$, $f(c) = d$ and $f(d) = d$. Then for $\{a,b,c\} \in \tau_1$, the inverse image $f^{-1}(\{a,b,c\}) = \{a,b\}$ is not $(1,2)$-γ-pre-open and so f is not an $(1,2)$-γ-pre-continuous function. But f is $(1,2)$-γ-P-continuous.*

Example 12. *Let X, τ_1, τ_2 and γ be as in Example 2. Define a function f on X as $f(a) = c$, $f(b) = b$ and $f(c) = a$. Then for $\{a\} \in \tau_1$, the inverse image $f^{-1}(\{a\}) = \{c\}$ is not $(1,2)$-pre-open and so f is not an $(1,2)$-pre-continuous function. But f is $(1,2)$-γ-P-continuous.*

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