Available online: July 16, 2017

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 67, Number 1, Pages 235–241 (2018)
DOI: 10.1501/Commua1_0000000845
ISSN 1303–5991
http://communications.science.ankara.edu.tr/index.php?series=A1

COMMUNICATIONS
SERIES A1

SOME PROPERTIES OF SEQUENCE SPACE $\widehat{BV}_{\theta}(f,p,q,s)$

MAHMUT IŞIK

Abstract. In this paper, we define the sequence space $\widehat{BV}_{\theta}(f,p,q,s)$ on a seminormed complex linear space, by using a Modulus function. We give various properties and some inclusion relations on this space.

1. INTRODUCTION

Let ℓ_∞ and c denote the Banach spaces of real bounded and convergent sequences $x=(x_n)$ normed by $\|x\|=\sup\limits_n |x_n|$, respectively.

Let σ be a one to one mapping of the set of positive integers into itself such that $\sigma^k(n)=\sigma\left(\sigma^{k-1}(n)\right)$, $k=1,2,...$ .A continuous linear functional φ on ℓ_∞ is said to be an invariant mean or a $\sigma-$mean if and only if

(i) $\varphi(x)\geq 0$ when $x_n\geq 0$ for all n,
(ii) $\varphi(e)=1$, where $e=(1,1,1,...)$ and
(iii) $\varphi\left(\left\{x_{\sigma(n)}\right\}\right)=\varphi(\{x_n\})$ for all $x\in\ell_\infty$.

If σ is the translation mapping $n\to n+1$, a $\sigma-$mean is often called a Banach limit [3], and V_σ is the set of $\sigma-$convergent sequences, that is, the set of bounded sequences all of whose invariant means are equal, is the set $\hat{f}$ of almost convergent sequences [11].

It can be shown (see Schaefer [24]) that

$$V_\sigma=\left\{x=(x_n):\lim_r t_{rn}(x)=Le \text{ uniformly in } n,\ L=\sigma-\lim x\right\}, \tag{1.1}$$

where

$$t_{rn}(x)=\frac{1}{r+1}\sum_{j=0}^{r}T^j x_n.$$

The special case of (1.1), in which $\sigma(n)=n+1$ was given by Lorentz [11].

Received by the editors: June 08, 2016; Accepted: February 01, 2017.
2010 *Mathematics Subject Classification.* 40A05, 40C05, 40D05.
Key words and phrases. Modulus function, sequence spaces, seminorm.

Subsequently invariant means have been studied by Ahmad and Mursaleen[1], Mursaleen ([16],[17]), Raimi [20], Altinok et al. [2], Mohiuddine [13],[14], Mohiuddine and Mursaleen [15] many others.

We may remark here that the concept $\widehat{BV}$ of almost bounded variation have been introduced and investigated by Nanda and Nayak [19] as follows:

$$\widehat{BV} = \left\{ x : \sum_{r} |t_{rn}(x)| \text{ converges uniformly in } n \right\}$$

where

$$t_{rn}(x) = \frac{1}{r(r+1)} \sum_{v=1}^{r} v\left(x_{n+v} - x_{n+v-1}\right).$$

By a lacunary sequence $\theta = (k_r)_{r=0,1,2,\ldots}^{\infty}$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and we let $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will usually be denoted by q_r (see [7]).

Karakaya and Savaş [10] were defined sequence spaces $\widehat{BV}_\theta(p)$ and $\widehat{\widehat{BV}}_\theta(p)$ as follows:

$$\widehat{BV}_\theta(p) = \left\{ x : \sum_{r=1}^{\infty} |\varphi_{rn}(x)|^{p_r} \text{ converges uniformly in } n \right\},$$

$$\widehat{\widehat{BV}}_\theta(p) = \left\{ x : \sup_n \sum_{r=1}^{\infty} |\varphi_{rn}(x)|^{p_r} < \infty \right\},$$

where

$$\varphi_{r,n}(x) = \frac{1}{h_r+1} \sum_{j=k_{r-1}+1} x_{j+n} - \frac{1}{h_r} \sum_{j=k_{r-1}+1}^{k_r} x_{j+n}, \; r > 1.$$

Straightforward calculation shows that

$$\varphi_{r,n}(x) = \frac{1}{h_r(h_r+1)} \sum_{u=1}^{h_r} u\left(x_{k_{r-1}+u+1+n} - x_{k_{r-1}+u+n}\right),$$

and

$$\varphi_{r-1,n}(x) = \frac{1}{h_r(h_r-1)} \sum_{u=1}^{h_r-1} \left(x_{k_{r-1}+u+1+n} - x_{k_{r-1}+u+n}\right).$$

Note that for any sequences x, y and scalar λ, we have

$$\varphi_{r,n}(x+y) = \varphi_{r,n}(x) + \varphi_{r,n}(y) \text{ and } \varphi_{r,n}(\lambda x) = \lambda \varphi_{r,n}(x).$$

The notion of modulus function was introduced by Nakano [18] in 1953. We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x+y) \le f(x) + f(y)$, for all $x \ge 0, y \ge 0$,
(iii) f is increasing,
(iv) f is continuous from the right at 0.

A modulus may be bounded or unbounded. For example, $f(x) = x^p, (0 < p \le 1)$ is unbounded but $f(x) = \frac{x}{1+x}$ is bounded. Maddox [12] and Ruckle[21], Bhardwaj [4], Et ([5], [6]), Işık ([8], [9]), Savas ([22], [23]) used a modulus function to construct some sequence spaces.

A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ for all sequences (α_k) of scalars with $|\alpha_k| \le 1$.

It is well known that a sequence space E is normal implies that E is monotone .

Definition 1.1 Let q_1, q_2 be seminorms on a vector space X. Then q_1 is said to be stronger than q_2 if whenever (x_n) is a sequence such that $q_1(x_n) \to 0$, then also $q_2(x_n) \to 0$. If each is stronger than the others q_1 and q_2 are said to be equivalent (one may refer to Wilansky [25]).

Lemma 1.2 Let q_1 and q_2 be seminorms on a linear space X. Then q_1 is stronger than q_2 if and only if there exists a constant T such that $q_2(x) \le Tq_1(x)$ for all $x \in X$ (see for instance Wilansky [25]).

Let $p = (p_r)$ be a sequence of strictly positive real numbers, X be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q, f be a Modulus function and $s \ge 0$ be a fixed real number. Then we define the sequence space $\widehat{BV}_\theta(f,p,q,s)$ as follows:

$$\widehat{BV}_\theta(f,p,q,s) = \left\{ x = (x_k) \in X : \sum_{r=1}^{\infty} r^{-s} \left[f\left(q\left(\varphi_{rn}(x) \right) \right) \right]^{p_r} < \infty, \text{uniformly in } n, \right\}.$$

We get the following sequence spaces from $\widehat{BV}_\theta(f,p,q,s)$ by choosing some of the special p, f and s :
For $f(x) = x$, we get

$$\widehat{BV}_\theta(p,q,s) = \left\{ x = (x_k) \in X : \sum_{r=1}^{\infty} r^{-s} \left[\left(q\left(\varphi_{rn}(x) \right) \right) \right]^{p_r} < \infty, \text{ uniformly in } n \right\},$$

for $p_r = 1$ for all $r \in \mathbb{N}$, we get

$$\widehat{BV}_\theta(f,q,s) = \left\{ x = (x_k) \in X : \sum_{r=1}^{\infty} r^{-s} \left[f\left(q\left(\varphi_{rn}(x) \right) \right) \right] < \infty, \text{ uniformly in } n \right\},$$

for $s = 0$ we get

$$\widehat{BV}_\theta(f,p,q) = \left\{ x = (x_k) \in X : \sum_{r=1}^{\infty} \left[f\left(q\left(\varphi_{rn}(x) \right) \right) \right]^{p_r} < \infty, \text{ uniformly in } n \right\},$$

for $f(x)=x$ and $s=0$ we get

$$\widehat{BV_\theta}(p,q)=\left\{x=(x_k)\in X:\sum_{r=1}^{\infty}\left[\left(q\left(\varphi_{rn}(x)\right)\right)\right]^{p_r}<\infty,\ \text{uniformly in } n\right\},$$

for $p_r=1$ for all $r\in\mathbb{N}$, and $s=0$ we get

$$\widehat{BV_\theta}(f,q)=\left\{x=(x_k)\in X:\sum_{r=1}^{\infty}\left[f\left(q\left(\varphi_{rn}(x)\right)\right)\right]<\infty,\ \text{uniformly in } n\right\},$$

for $f(x)=x$, $p_r=1$ for all $r\in\mathbb{N}$, and $s=0$ we have

$$\widehat{BV_\theta}(q)=\left\{x=(x_k)\in X:\sum_{r=1}^{\infty}q\left(\varphi_{rn}(x)\right)<\infty,\ \text{uniformly in } n\right\}.$$

The following inequalities will be used throughout the paper. Let $p=(p_r)$ be a bounded sequence of strictly positive real numbers with $0<p_r\leq\sup p_r=H$, $D=\max\left(1,2^{H-1}\right)$, then

$$|a_r+b_r|^{p_r}\leq D\left\{|a_r|^{p_r}+|b_r|^{p_r}\right\}, \tag{1.2}$$

where a_r, $b_r\in\mathbb{C}$.

2. MAIN RESULTS

In this section we will prove the general results of this paper on the sequence space $\widehat{BV_\theta}(f,p,q,s)$, those characterize the structure of this space.

Theorem 2.1 The sequence space $\widehat{BV_\theta}(f,p,q,s)$ is a linear space over the field $\mathbb{C}$ of complex numbers.

Proof. Let $x,y\in\widehat{BV_\theta}(f,p,q,s)$. For $\lambda,\mu\in\mathbb{C}$ there exists M_λ and N_μ integers such that $|\lambda|\leq M_\lambda$ and $|\mu|\leq N_\mu$. Since f is subadditive, q is a seminorm

$$\begin{aligned}
&\sum_{r=1}^{\infty}r^{-s}\left[f\left(q\left(\lambda\varphi_{rn}(x)+\mu\varphi_{rn}(y)\right)\right)\right]^{p_r}\\
&\leq\sum_{r=1}^{\infty}r^{-s}\left[f\left(|\lambda|\,q\left(\varphi_{rn}(x)\right)\right)+f\left(q\left(|\mu|\,\varphi_{rn}(y)\right)\right)\right]^{p_r}\\
&\leq D\left(M_\lambda\right)^H\sum_{r=1}^{\infty}r^{-s}\left[f\left(q\left(\varphi_{rn}(x)\right)\right)\right]^{p_r}+D\left(N_\mu\right)^H\sum_{r=1}^{\infty}r^{-s}\left[f\left(q\left(\varphi_{rn}(y)\right)\right)\right]^{p_r}<\infty.
\end{aligned}$$

This proves that $\widehat{BV_\theta}(f,p,q,s)$ is a linear space.

Theorem 2.2 $\widehat{BV}_{\theta}(f,p,q,s)$ is a paranormed space (not necessarily totally paranormed), paranormed by

$$g(x)=\left(\sum_{r=1}^{\infty} r^{-s}\left[f\left(q\left(\varphi_{rn}(x)\right)\right)\right]^{p_r}\right)^{\frac{1}{M}},$$

where $M=\max(1,\sup p_r)$, $H=\sup_r p_r<\infty$.

Proof. It is clear that $g(\bar{\theta})=0$ and $g(x)=g(-x)$ for all $x\in\widehat{BV}_{\theta}(f,p,q,s)$, where $\bar{\theta}=(\theta,\theta,\theta,...)$. It also follows from (1.2), Minkowski's inequality and definition f that g is subadditive and

$$g(\lambda x)\leq K_{\lambda}^{H\backslash M}g(x),$$

where K_λ is an integer such that $|\lambda|<K_\lambda$. Therefore the function $(\lambda,x)\to\lambda x$ is continuous at $x=\bar{\theta}$ and that when λ is fixed, the function $x\to\lambda x$ is continuous at $x=\bar{\theta}$. If x is fixed and $\varepsilon>0$, we can choose r_0 such that

$$\sum_{r=r_0}^{\infty} r^{-s}\left[f\left(q\left(\varphi_{rn}(x)\right)\right)\right]^{p_r}<\frac{\varepsilon}{2}.$$

and $\delta>0$ so that $|\lambda|<\delta$ and definition of f gives

$$\sum_{r=1}^{r_0} r^{-s}\left[f\left(q\left(\lambda\varphi_{rn}(x)\right)\right)\right]^{p_r}=\sum_{r=1}^{r_0} r^{-s}\left[f\left(|\lambda|\,q\left(\varphi_{rn}(x)\right)\right)\right]^{p_r}<\frac{\varepsilon}{2}.$$

Therefore $|\lambda|<\min(1,\delta)$ implies that $g(\lambda x)<\varepsilon$. Thus the function $(\lambda,x)\to\lambda x$ is continuous at $\lambda=0$ and $\widehat{BV}_{\theta}(f,p,q,s)$ is paranormed space

Theorem 2.3 Let f,f_1,f_2 be modulus functions q,q_1,q_2 seminorms and $s,s_1,s_2\geq 0$. Then

(i) $\widehat{BV}_{\theta}(f_1,p,q,s)\cap\widehat{BV}_{\theta}(f_2,p,q,s)\subseteq\widehat{BV}_{\theta}(f_1+f_2,p,q,s)$,

(ii) If $s_1\leq s_2$ then $\widehat{BV}_{\theta}(f,p,q,s_1)\subseteq\widehat{BV}_{\theta}(f,p,q,s_2)$,

(iii) $\widehat{BV}_{\theta}(f,p,q_1,s)\cap\widehat{BV}_{\theta}(f,p,q_2,s)\subseteq\widehat{BV}_{\theta}(f,p,q_1+q_2,s)$,

(iv) If q_1 is stronger than q_2 then $\widehat{BV}_{\theta}(f,p,q_1,s)\subseteq\widehat{BV}_{\theta}(f,p,q_2,s)$.

Proof. i) The proof follows from the following inequality

$$r^{-s}\left[(f_1+f_2)\left(q\left(\varphi_{rn}(x)\right)\right)\right]^{p_r}\leq Dr^{-s}\left[f_1\left(q\left(\varphi_{rn}(x)\right)\right)\right]^{p_r}+Dr^{-s}\left[f_2\left(q\left(\varphi_{rn}(x)\right)\right)\right]^{p_r}.$$

ii), iii) and iv) follow easily.

Corollary 2.4 Let f be a modulus function, then we have

(i) If $q_1\cong$(equivalent to) q_2, then $\widehat{BV}_{\theta}(f,p,q_1,s)=\widehat{BV}_{\theta}(f,p,q_2,s)$,

(ii) $\widehat{BV}_{\theta}(f,p,q)\subseteq\widehat{BV}_{\theta}(f,p,q,s)$,

(iii) $\widehat{BV}_{\theta}(f,q)\subseteq\widehat{BV}_{\theta}(f,q,s)$.

Theorem 2.5. Suppose that $0 < m_r \leq t_r < \infty$ for each $r \in \mathbb{N}$. Then $\widehat{BV_\theta}(f,m,q) \subseteq \widehat{BV_\theta}(f,t,q)$.

Proof. Let $x \in \widehat{BV_\theta}(f,m,q)$. This implies that

$$[f(q(\varphi_{rn}(x)))]^{m_r} \leq 1$$

for sufficiently large values of k, say $k \geq k_0$ for some fixed $k_0 \in \mathbb{N}$. Since f is non decreasing, we have

$$\sum_{r=k_0}^{\infty} r^{-s}[f(q(\varphi_{rn}(x)))]^{t_r} \leq \sum_{r=k_0}^{\infty} r^{-s}[f(q(\varphi_{rn}(x)))]^{m_r}.$$

It gives $x \in \widehat{BV_\theta}(f,t,q)$.

The following result is a consequence of the above result.

Corollary 2.6

(i) If $0 < p_r \leq 1$ for each r, then $\widehat{BV_\theta}(f,p,q) \subseteq \widehat{BV_\theta}(f,q)$,

(ii) If $p_r \geq 1$ for all r, then $\widehat{BV_\theta}(f,q) \subseteq \widehat{BV_\theta}(f,p,q)$.

Theorem 2.7 The sequence space $\widehat{BV_\theta}(f,p,q,s)$ is solid.

Proof. Let $x \in \widehat{BV_\theta}(f,p,q,s)$, i.e.

$$\sum_{r=1}^{\infty} r^{-s}[f(q(\varphi_{rn}(x)))]^{p_r} < \infty.$$

Let (α_r) be sequence of scalars such that $|\alpha_r| \leq 1$ for all $r \in \mathbb{N}$. Then the result follows from the following inequality.

$$\sum_{r=1}^{\infty} r^{-s}[f(q(\alpha_r\varphi_{rn}(x)))]^{p_r} \leq \sum_{r=1}^{\infty} r^{-s}[f(q(\varphi_{rn}(x)))]^{p_r}.$$

Corollary 2.8 The sequence space $\widehat{BV_\theta}(f,p,q,s)$ is monotone.

References

[1] Ahmad, Z.U. and Mursaleen, M. An application of Banach limits, Proc. Amer. Math. Soc. 103, (1988), 244-246.

[2] Altinok, H. Altin, Y. Işik, M. The sequence space $BV_\sigma(M,p,q,s)$ on seminormed spaces. Indian J. Pure Appl. Math. 39(1) (2008), 49–58

[3] Banach, S. Theorie des Operations Linearies, Warszawa, 1932.

[4] Bhardwaj, V.K. A generalization of a sequence space of Ruckle, Bull. Calcutta Math. Soc. 95(5) (2003), 411-420.

[5] Et, M. Spaces of Cesàro difference sequences of order r defined by a modulus function in a locally convex space. Taiwanese J. Math. 10(4) (2006), 865–879.

[6] Et, M. : Strongly almost summable difference sequences of order m defined by a modulus. Studia Sci. Math. Hungar. 40(4) (2003), 463–476.

[7] Freedman, A.R. Sember, J. J. Raphael, M. Some Cesàro-type summability spaces. Proc. London Math. Soc. 3(3) 37 (1978), 508–520.
[8] Işik, M. Generalized vector-valued sequence spaces defined by modulus functions. J. Inequal. Appl. 2010, Art. ID 457892, 7 pp.
[9] Işik, M. Strongly almost $(w, \lambda, q)-$summable sequences. Math. Slovaca. 61(5) (2011), 779–788.
[10] Karakaya, V. and Savaş, E. On almost $p-$bounded variation of lacunary sequences. Comput. Math. Appl. 61(6) (2011), 1502–1506.
[11] Lorentz, G. G. A contribution the theory of divergent series, Acta Math. 80 (1948), 167-190.
[12] Maddox.I. J. Sequence spaces defined by a modulus, Math. Proc. Camb. Phil. Soc. 100 (1986), 161-166.
[13] Mohiuddine, S. A. An application of almost convergence in approximation theorems. Appl. Math. Lett. 24 (2011), no. 11, 1856–1860
[14] Mohiuddine, S. A. Matrix transformations of paranormed sequence spaces through de la Vallee-Pousion mean, Acta Scientiarum,Technology,37(1) (2015),71-75.
[15] Mursaleen, M. Mohiuddine, S. A. Some matrix transformations of convex and paranormed sequence spaces into the spaces of invariant means. J. Funct. Spaces Appl. 2012, Art. ID 612671, 10 pp
[16] Mursaleen, M. On some new invariant matrix methods of summability, Quart. J. Math. Oxford 34(2), (1983), 77-86.
[17] Mursaleen, M. Matrix transformations between some new sequence spaces, Houston J. Math. 9 , (1983), 505-509.
[18] Nakano,H. Concave modulars, J. Math. Soc. Japan. 5 (1953), 29-49.
[19] Nanda, S. and Nayak, K. C. Some new sequence spaces, Indian J.Pure Appl.Math. 9(8) (1978) 836-846.
[20] Raimi, R. A. Invariant means and invariant matrix method of summability, Duke Math. J. 30, (1963), 81-94.
[21] Ruckle,W. H. FK spaces in which the sequence of coordinate vectors is bounded, Canad. J. Math. 25 (1973), 973-978.
[22] Savaş, E. and Patterson, R. F. Double sequence spaces defined by a modulus. Math. Slovaca 61(2) (2011), 245–256.
[23] Savaş, E. On some new double sequence spaces defined by a modulus. Appl. Math. Comput. 187(1) (2007), 417–424.
[24] Schaefer, P. Infinite matrices and invariant means, Proc. Amer. Math. Soc. 36 (1972), 104-110.
[25] Wilansky, A. Functional Analysis, Blaisdell Publishing Company, New York, 1964.

Current address: Harran University, Faculty of Education, Sanliurfa-TURKEY
E-mail address: misik63@yahoo.com