Available online: July 17, 2017

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 67, Number 1, Pages 242–247 (2018)
DOI: 10.1501/Commua1_0000000846
ISSN 1303–5991

http://communications.science.ankara.edu.tr/index.php?series=A1

COMMUNICATIONS
SERIES A1

STABILITY AND LOWER-BOUND FUNCTIONS OF C_0-MARKOV SEMIGROUPS ON KB-SPACES

NAZIFE ERKURŞUN ÖZCAN

Abstract. In this paper, we investigate the relation between stability and lower-bound functions of Markov C_0-semigroups on KB-spaces.

1. Introduction

In a different fields, the investigation of Markov operators in L_p-spaces has significance. Particularly in ergodic theory, Markov operators play an important and incredulous role and it is a suitable generalization of a conditional expectation. On L_∞ and L_1-spaces, Markov operators are defined in a different ways. When we consider ergodic Markov operators, we also see a different kind of ergodic operator definitions. Actually, all of these definitions of Markov operators describe a subclass of the positive contraction operator class. In this paper, the main object is also a Markov operator which is defined on a Banach lattice and the definition is given by [5] and then the definition of Markov operator net is given in [3].

Moreover, the lower bound technique is also a useful tool in the ergodic theory of Markov processes. The technique was originated by Markov, 1906-1908 but it has been firstly used by Doeblin, see [2] to show mixing of a Markov chain whose transition probabilities possess a uniform lower bound. It was the main tool in proving the convergence of the iterates of some quadratic matrices and applied in the theory of Markov chains.

2. Preliminaries

Let E be a Banach lattice. Then $E_+ := \{x \in E : x \geq 0\}$ denotes the positive cone of E. On $\mathcal{L}(E)$, there is a canonical order given by $S \leq T$ if $Sx \leq Tx$ for all $x \in E_+$. If $0 \leq T$, then T is called positive. A Banach lattice E is called a KB-space whenever every increasing norm bounded sequence of E_+ is norm convergent. In

Received by the editors: August 05, 2016; Accepted: January 07, 2017.
2010 *Mathematics Subject Classification.* 37A30, 47A35.
Key words and phrases. Markov operator, C_0-semigroup, KB-space, asymptotic stability, lower-bound function, mean lower-bound function.

particular, it follows that every KB-space has order continuous norm. All reflexive Banach lattice and AL-space are examples of KB-spaces.

Before proving the following results, we need to define a Markov operator semigroup on a Banach lattice E.

Definition 2.1. Let E be a Banach lattice. A positive, linear, uniform bounded one-parameter semigroup $\Theta = (T_t)_{t\geq 0}$ is called an one-parameter Markov semigroup if there exists a strictly positive element $0 < e' \in E'_+$ such that $T'_t e' = e'$ for each t.

If we consider in the sequence case $\Theta = (T_n)_{n\in\mathbb{N}}$, each element of Θ is Markov operator, still we need a common fixed point e'. For instance even on L_1-space, the element T_m of Θ might not be a Markov operator on the new norm space (L_1, e'_n) for each $n \neq m \in \mathbb{N}$, [7].

As a remark, if we consider the net as $\Theta = (T^n)_{n\in\mathbb{N}}$ as the iteration of a single operator T, Markov operator sequence Θ means T is power-bounded. It is the general version of the definition in [5] where T is contraction. It is well known that if T is a positive linear operator defined on a Banach lattice E, then T is continuous. It is also well known that if the Banach lattice E has order continuous norm, then the positive operator T is also order continuous. We note that the Markov operators, according to this definition, are again contained in the class of all positive power-bounded and that the adjoint T' is also a positive and power-bounded. For more details, we refer to [5].

In the following theorem, we will establish the asymptotic properties of C_0 Markov semigroups on KB-spaces. For the proof, we refer to [1] and [3]

Theorem 2.2. *Let E be a KB-space with a quasi-interior point e, $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ be a C_0 Markov semigroup on E and $A_t^{\mathcal{T}}$ be the Cesaro averages of $\mathcal{T}$. Then the following assertions are equivalent:*

- i *There exists a function $g \in E_+$ and $\eta \in \mathbb{R}$, $0 \leq \eta < 1$ such that*

  $$\lim_{t\to\infty} \operatorname{dist}(A_t^{\mathcal{T}} x, [-g, g] + \eta B_E) = 0, \ \ \forall x \in B_E := \{x \in E : \|x\| \leq 1\}$$
- ii *The net $\mathcal{T}$ is strongly convergent and* $\dim\operatorname{Fix}(\mathcal{T}) < \infty$.

3. Mean Lower Bound Function

In this section, we give results about asymptotic stability in terms of lower-bounds on KB-spaceS. The results play an important role in the investigation of asymptotic behavior of many classes of Markov operators.

At first, we give the following two definitions used in [6].

Definition 3.1. Let $\mathcal{T}$ be an one-parameter Markov semigroup on KB-spaces. $\mathcal{T}$ is called asymptotically stable whenever there exists an element $u \in E_+ \cap U_E$ where $U_E := \{x \in E : \|x\| = 1\}$ such that

$$\lim_{t\to\infty} \|T_t x - u\| = 0$$

for every $x \in E_+ \cap U_E$.

Definition 3.2. An element $h \in E_+$ is called a lower-bound function for $\mathcal{T}$ if

$$\lim_{t\to\infty} \|(h - T_t x)_+\| = 0$$

for every element $x \in E_+ \cap U_E$. We say that a lower-bound function h is nontrivial if $h \neq 0$.

Before proving equivalence of asymptotic stability and existence of lower-bound function, we investigate A mean lower-bound function.

We call an $h \in E_+$ is called a mean lower-bound element for $\mathcal{T}$ if

$$\lim_{t\to\infty} \left\|(h - A_t^T x)_+\right\| = 0$$

for every element $x \in E_+ \cap U_E$.

Any lower-bound function is mean lower-bound.

Before main results we need a technical tool for proofs. The technical lemma connects norm convergence of order bounded sequences in KB-spaces with convergence in (E, e') for suitable linear forms $e' \in E'$. Recall that $e' \in E'$ is strictly positive if $\langle x, e'\rangle > 0$ for all $x \in E_+ \setminus \{0\}$. We refer to [7] for the proof of the lemma.

Lemma 3.3. *Let $(x_n)_{n\in\mathbb{N}}$ be an order bounded sequence in a KB-space and let $x' \in E'$ be strictly positive. Then $\lim_{n\to\infty} \|x_n\| = 0$ if and only if $\lim_{n\to\infty} \langle |x_n|, x'\rangle = 0$.*

Theorem 3.4. *Let T be a Markov operator on a KB-space with a quasi-interior point e. Then the following assertions are equivalent:*

i *There exists an element $g \in E_+ \cap U_E$ such that*

$$\lim_{n\to\infty} \left\|A_n^T x - g\right\| = 0$$

for every $x \in E_+ \cap U_E$.

ii *There exists a nontrivial mean lower-bound function h for T in the space (E, e').*

Proof: $(i) \Rightarrow (ii)$: Let $g \in E_+ \cap U_E$ satisfy

$$\lim_{t\to\infty} \left\|A_n^T x - g\right\| = 0$$

for every $x \in E_+ \cap U_E$, then g is automatically a nontrivial mean lower-bound function for T.

$(ii) \Rightarrow (i)$: Let h be a mean lower-bound function of T in (E, e'), namely

$$\lim_{n\to\infty} \left\langle (h - A_n^T x)_+, e'\right\rangle = 0.$$

Since the norm on (E, e') is an L_1-norm, then we can consider

$$\limsup_{n\to\infty} \left\langle (A_n^T x - h)_+, e'\right\rangle \leq \eta$$

where $\eta := 1 - \|h\|_{(E,e')} = 1 - \langle h, e'\rangle$ for each $x \in E_+ \cap U_E$.

By Theorem 2.2, $A_n^{\tilde{T}}$ where $\tilde{T}j_{e'}x = j_{e'}Tx$ for lattice homomorphism $j_{e'} : E \to (E, e')$. converges strongly to the finite dimensional fixed space of $\tilde{T}$. Therefore by Eberlein's Theorem

$$E_{e'} = \mathrm{Fix}(\tilde{T}) \oplus \mathrm{Ker}(\tilde{T}).$$

In addition by Theorem 4.1. in [8], $\mathrm{Fix}(\tilde{T})$ is a sublattice of (E, e') and by Judin's Theorem, it possesses a linear basis $(\tilde{u_i})_{i=1}^n$ where $n = \mathrm{dimFix}(\tilde{T})$ which consists of pairwise disjoint element with $\|\tilde{u_i}\|_{(E,e')} = 1$, $i = 1, \cdots, n$. Since $Tu_i = u_i$ for each $i = 1, \cdots, n$,

$$\langle (h - u_i)_+, e' \rangle = \langle (h - Tu_i)_+, e' \rangle = \lim_{n\to\infty} \left\langle (h - A_n^T u_i)_+, e' \right\rangle = 0$$

implies

$$u_i \geq h \geq 0 \quad i = 1, \cdots, n. \tag{3.1}$$

Since $(\tilde{u_i})_{i=1}^n$ is pairwise disjoint with $\|\tilde{u_i}\|_{(E,e')} = 1$ the condition 3.1 ensure that $\mathrm{dimFix}(\tilde{T}) = 1$. Therefore $\mathrm{Fix}(\tilde{T}) = \mathbb{R}\tilde{u_1}$ and for every element $x \in E_+ \cap B_E$, $\lim_{n\to\infty} A_n^T x = u_1$.

□

4. Lower Bound Function

The following theorem is the other main result of a one-parameter Markov semigroup.

Theorem 4.1. *Let $\mathcal{T}$ be a C_0 Markov semigroup on a KB-space with a quasi-interior point e. Then the following assertions are equivalent:*

i *$\mathcal{T}$ is asymptotically stable.*
ii *There exists a nontrivial lower-bound function h for T in the space (E, e').*

Proof: $(i) \Rightarrow (ii)$: Let $\mathcal{T}$ be asymptotically stable then u is automatically a nontrivial lower-bound function for $\mathcal{T}$.

$(ii) \Rightarrow (i)$: Let h be a nontrivial lower-bound function of $\mathcal{T}$ in the space (E, e')..

Case I. Assume T to be a Markov operator and $\mathcal{T} = (T^n)_{n=1}^\infty$ to be a discrete. Since any lower-bound function is a mean lower-bound function, by Theorem 3.4, T is mean ergodic and $E = \mathbb{R}u \oplus \overline{(I - T)E}$. It suffices to show that

$$\lim_{n\to\infty} \|T^n f\| = 0, \quad \forall f \in \overline{(I - T)E}).$$

If $f \in \overline{(I - T)E}$, then without loss of generality there exists $x \in E$ such that $f = (I - T)x$. Since T is Markov, then there exists $e' > 0$ such that $T'e' = e'$. By Lemma 3.3, we know that $\lim_{n\to\infty} \|T^n f\| = 0$ if and only if $\lim_{n\to\infty} \langle |T^n f|, e' \rangle = 0$ for strictly positive $e' \in E'$.

Since $\langle f, e' \rangle = \langle (I - T)x, e' \rangle = \langle x, (I - T')e' \rangle = 0$ and $\langle f_+ - f_-, e' \rangle = \langle ((I - T)x)_+, e' \rangle - \langle ((I - T)x)_-, e' \rangle$, then

$$\langle ((I - T)x)_+, e' \rangle = \langle ((I - T)x)_-, e' \rangle .$$

Notice that $(\langle |T^n f|, e'\rangle)_n$ is a decreasing sequence, since T is a contraction. Therefore for every $f \in \overline{(I-T)E}$, we obtain

$$\langle |f|, e'\rangle = \lim_{n\to\infty} \langle |T^n f|, e'\rangle = \inf_n \langle |T^n f|, e'\rangle$$

Suppose there exists an element f from $\overline{(I-T)E}$ with

$$L := \lim_{n\to\infty} \langle |T^n f|, e'\rangle > 0.$$

Then

$$\begin{aligned} L &= \lim_{n\to\infty} \langle |T^n x|, e'\rangle > 0 = L := \lim_{n\to\infty} \langle |T^n(x_+ - x_-)|, e'\rangle \\ &= L := \lim_{n\to\infty} \left\langle |(T^n x_+ - \frac{L}{2}h) - (T^n x_- - \frac{L}{2}h)|, e' \right\rangle \\ &\leq \lim_{n\to\infty} \left\langle |(T^n x_+ - \frac{L}{2}h)_+, e' \right\rangle + \lim_{n\to\infty} \left\langle |(T^n x_- - \frac{L}{2}h)_+, e' \right\rangle \\ &= L(1 - \|h\|) \end{aligned}$$

which is impossible. Therefore $\lim_{n\to\infty} \langle |T^n x|, e'\rangle = 0$, so $\lim_{n\to\infty} \|T^n x\| = 0$.

Case II. In this case, we reproduce the argument from [6]. Take any $t_0 > 0$ and consider the operator T_{t_0}. If we take the discrete semigroup $(T_{t_0}^n)_n$, h is a nontrivial lower-bound function for it. From the first case, there exists a unique T_{t_0}-invariant density u such that

$$\lim_{n\to\infty} \left\|T_{t_0}^n x - u\right\| = 0, \quad (\forall x \in E_+ \cap U_E).$$

Firstly, we show that $T_t u = u$, for every $t > 0$. Assume there exists $t' > 0$ with $x' = T_{t'} u$. Hence

$$\begin{aligned} \|T_{t'}u - u\| &= \lim_{n\to\infty} \|T_{t'}u - u\| \\ &= \lim_{n\to\infty} \|T_{t'}(T_{nt_0}u) - u\| \\ &= \lim_{n\to\infty} \|T_{nt_0}(T_{t'}u) - u\| \\ &= \lim_{n\to\infty} \left\|T_{t_0}^n x' - u\right\| \\ &= 0, \end{aligned}$$

because $x' \in E_+ \cap U_E$. Since t' is arbitrary, u is $\mathcal{T}$-invariant.

Consider any element $x \in E_+ \cap U_E$, and $\langle |T_t x - u|, e'\rangle$ is decreasing. Since for the subsequence $(nt_0)_n$, $\lim_{n\to\infty} \langle |T_{nt_0} x - u|, e'\rangle = 0$, then we obtain $\lim_{t\to\infty} \langle |T_t x - u|, e'\rangle = 0$. By Lemma 3.3, $\lim_{t\to\infty} \|T_t x - u\| = 0$.

□

In general, the lower bound technique helps to achieve ergodic properties of the Markov process from the fact that there exists a small set in the state space. The time averages of the mass of the process are concentrated on that set for all sufficiently large times. If this set is compact, the existence of an invariant probability measure can be obtained easily.

Acknowledgments The author would like to thank the referee for helpful comments.

References

[1] Alpay, S., A. Binhadjah, E. Yu Emelyanov, Z. Ercan, *Mean ergodicity of positive operators in KB-spaces*, J. Math. Anal. Appl. 323 (1) (2006) 371 – 378.

[2] Doeblin, W., *Elements dune theorie generale des chaines simples constantes de Markov*, Ann. Ecole Norm. 57 (1940) 61-111.

[3] Erkurşun Özcan, N., *Asymptotic behavior of operator sequences on KB-spaces*, arXiv:1605.03650

[4] Emel'yanov, E., M. P. H. Wolff, *Mean lower bounds for Markov operators.* Ann. Polon. Math. 83 (2004) 11–19.

[5] Hawke, P., Markov Operators on Banach Lattices, Thesis for the degree of Master of Science. School of Mathematics, University of the Witwatersrand, South Africa, 2006.

[6] Lasota, A., M. C. Mackey, *Chaos, Fractals and Noise. Stochastics Aspect of Dynamics, second edition*, Appl. Math. Sci. 97 Springer-Verlag, New York, xiv + 472 pp 1994.

[7] Räbiger, F., *Attractors and asymptotic periodicity of positive operators on Banach lattices*, Forum Math. 7 (6) (1995) 665–683.

[8] Schaefer, H. H., Banach Lattices and Positive Operators, Grundlehren Math. Wiss.; Band 215, Springer-Verlag, New York, 1974.

Current address: Department of Mathematics, Hacettepe University, Ankara, Turkey, 06800
E-mail address: erkursun.ozcan@hacettepe.edu.tr