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NONLINEAR $m-$SINGULAR INTEGRAL OPERATORS IN THE FRAMEWORK OF FATOU TYPE WEIGHTED CONVERGENCE

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Abstract. In the present paper, we prove some theorems concerning Fatou type weighted pointwise convergence of nonlinear $m-$singular integral operators of the form:

$$T_{\lambda}^{[m]}(f;x)=\int\limits_{\mathbb{R}} K_{\lambda}\left(t,\sum_{k=1}^{m}(-1)^{k-1}\left(\begin{array}{c} m \\ k \end{array}\right)f(x+kt)\right)dt,$$

where $x\in\mathbb{R}$, $m\geq 1$ is a finite natural number and $\lambda\in\Lambda$ which is a non-empty set of non-negative indices, at a common $m-p-\mu-$Lebesgue point of $f\in L_{p,\varphi}(\mathbb{R})$ $(1\leq p<\infty)$ and φ. Here, $\varphi:\mathbb{R}\rightarrow\mathbb{R}^{+}$ is a weight function endowed with some specific properties and $L_{p,\varphi}(\mathbb{R})$ is the space of all measurable functions for which $\left|\frac{f}{\varphi}\right|^{p}$ is integrable on $\mathbb{R}$.

1. Introduction

The approximation by singular integral operators is one of the highly studied and oldest topics of approximation theory. The researchers of this theory have investigated the limit behaviors of these type of operators while working on the problem of representing functions on some sets. The principal part of this survey belongs to approximation by linear singular integral operators on account of the comfort of the investigation. However, current problems of natural and applied sciences can not be interpreted by using only linear operator theory since nonlinear problems are included in the scope of those problems. Before giving brief information on nonlinear singular integrals, we first mention some of the studies and approximation technics used therein which have come to the fore in the literature of linear singular integral operators.

Fourier series of the functions play an important role in the representation of functions at their characteristic points, such as point of continuity, $d-$point and

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Lebesgue point. Especially, the connection between the Fourier series of the functions and the following integral operator

$$L_{\lambda}(f;x)=\int_{a}^{b}f(t)\,K_{\lambda}(t-x)\,dt,\quad x\in\langle a,b\rangle\,,\quad \lambda\in\Omega, \tag{1.1}$$

where Ω is a set of non-negative numbers with accumulation point λ_0, the symbol $\langle a,b\rangle$ stands for closed, semi-closed or open arbitrary interval and K_λ is the kernel function satisfying suitable conditions, is a well-known fact. Here, the kernel function satisfies the singularity assumption $\lim_{\lambda\to\lambda_0}K_\lambda(0)=\infty$. For some important studies concerning the pointwise convergence of the operators of type (1.1) in different settings, we refer the reader to [2, 6, 17, 22]. Also, we have to mention the studies [1, 17, 29] which contain many important results on weighted pointwise convergence of linear type of singular integral operators.

Now, we focus on Fatou type convergence, which is the heart of the matter, and give some related information. The great work by Fatou [9] gave an idea to the researchers such that the existence of one limit may indicate the existence of another one, and it was directly and indirectly used in the solutions of many approximation problems. The works directly based on this work with its theoretical background may be given as [8, 16].

Later on, Musielak [18] studied the nonlinear integral operators in the following setting:

$$T_w f(y)=\int_{G}K_w(x-y;f(x))dx,\quad y\in G,\quad w\in\Lambda, \tag{1.2}$$

where G is a locally compact Abelian group equipped with Haar measure and Λ is a non-empty index set with any topology. In this work, he used the Lipschitz condition for K_w with respect to second variable. Therefore, the solution technics developed for approximation problems for the linear case became applicable to nonlinear approximation problems. For some advanced studies about nonlinear integral operators in several settings, we refer the reader to [3, 4, 5, 11, 19, 20, 26].

Mamedov [17] obtained the following $m-$singular integral operators

$$L_{\lambda}^{[m]}(f;x)=(-1)^{m+1}\int_{\mathbb{R}}\left[\sum_{k=1}^{m}(-1)^{m-k}\begin{pmatrix} m\\ k\end{pmatrix}f(x+kt)\right]K_{\lambda}(t)dt, \tag{1.3}$$

where $x\in\mathbb{R}$, $m\geq 1$ is a finite natural number and $\lambda\in\Lambda$ which is a non-empty set of non-negative indices, by using $m-$th finite difference formulas. Here, the main concern is approximation of the $m-$th derivatives of the integral of the functions pointwise by using the corresponding $m-$singular integral operators. Later on, Karsli [15] studied the Fatou type convergence of nonlinear counterparts of the operators of type (1.3) in the following form:

$$T_{\lambda}^{[m]}(f;x)=\int\limits_{\mathbb{R}} K_{\lambda}\left(t,\sum_{k=1}^{m}(-1)^{k-1}\left(\begin{array}{c} m \\ k \end{array}\right)f(x+kt)\right)dt, \tag{1.4}$$

where $x\in\mathbb{R}$, $m\geq 1$ is a finite natural number and $\lambda\in\Lambda$ which is a non-empty set of non-negative indices, at $m-p-\mu-$Lebesgue point of the functions $f\in L_p(\mathbb{R})\ (1\leq p<\infty)$, where $L_p(\mathbb{R})$ is the space of all measurable functions for which $|f|^p$ is integrable on $\mathbb{R}$. For some advanced studies concerning approximation by $m-$singular integral operators in various settings, we refer the reader to [5, 13, 23].

As a continuation of the works [5, 15, 17], the current manuscript presents some results on the Fatou type weighted convergence of the operators of type (1.4). Our main concern is to prove that the family of the functions of type (1.4) converge to the functions $f\in L_{p,\varphi}(\mathbb{R})\ (1\leq p<\infty)$, where $\varphi:\mathbb{R}\to\mathbb{R}^+$ is a suitable weight function satisfying submultiplication property, i.e., $\varphi(t+x)\leq\varphi(t)\varphi(x)$, $\forall t,x\in\mathbb{R}$ (for some specific weight functions satisfying this algebraic property see, for example, [12, 17]), and $L_{p,\varphi}(\mathbb{R})$ is the space of all measurable functions for which $\left|\frac{f}{\varphi}\right|^p$ is integrable on $\mathbb{R}$ (see, e.g., [17]), at their $m-p-\mu-$Lebesgue points.

The paper is organized as follows: In Section 2, we introduce fundamental notions. In Section 3, we prove auxiliary results concerning existence and pointwise convergence of the operators of type (1.4). In Section 4, we present, as a main result, Fatou type convergence theorem for the indicated operators. In Section 5, we establish the rates of both pointwise and Fatou type convergences by using the results obtained in the previous two sections.

2. Preliminaries

Definition 1. *Let $1\leq p<\infty$ and $\delta_0\in\mathbb{R}^+$ be a fixed number. A point $x_0\in\mathbb{R}$ characterized with the following relation*

$$\lim_{h\to 0}\frac{1}{\mu(h)}\int\limits_{0}^{h}|\Delta_t^m g(x_0)|^p\,dt=0, \tag{2.1}$$

where

$$\Delta_t^m g(x_0)=\sum_{k=0}^{m}(-1)^{m-k}\left(\begin{array}{c} m \\ k \end{array}\right)g(x_0+kt),$$

is called $m-p-\mu-$Lebesgue point of locally $p-$integrable function (i.e., a function whose $p-th$ power is Lebesgue integrable on arbitrary bounded subsets of $\mathbb{R}$) $g:\mathbb{R}\to\mathbb{R}$. Here, $\mu:\mathbb{R}\to\mathbb{R}$ is an increasing and absolutely continuous function on $0<h\leq\delta_0$. Here, the relation(2.1) also holds when the integral is taken from $-h$ to 0.

Remark 1. *Definition 1 is similar to the characteristic point definition given and used in* [15]*, i.e., the difference is the domain of the integration. On the other hand,* *for some other $\mu-$generalized Lebesgue point characterizations, we refer the reader to* [10, 23] *and* [30].

Definition 2. (*Class A_{φ}*) *Let $1 \leq p < \infty$ and Λ be a non-empty set of non-negative λ indices with accumulation point λ_0. Let $\varphi : \mathbb{R} \to \mathbb{R}^{+}$ be a bounded weight function on arbitrary bounded subsets of $\mathbb{R}$ such that*

$$\varphi\left(t+x\right) \leq \varphi\left(t\right)\varphi\left(x\right) \tag{2.2}$$

holds for every $t, x \in \mathbb{R}$. We will say that the family of the functions $K_{\lambda} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, where $K_{\lambda}(\vartheta, u)$ is Lebesgue integrable on $\mathbb{R}$ for every $u \in \mathbb{R}$ and $\lambda \in \Lambda$, belongs to Class A_{φ} if the following conditions are satisfied:

a) $K_{\lambda}(\vartheta, 0) = 0$, for every $\vartheta \in \mathbb{R}$ and $\lambda \in \Lambda$.

b) There exists a function $L_{\lambda} : \mathbb{R} \to \mathbb{R}$ which is integrable on $\mathbb{R}$ for each $\lambda \in \Lambda$ such that the following inequality

$$\left|K_{\lambda}(t,u) - K_{\lambda}(t,v)\right| \leq L_{\lambda}(t)\left|u-v\right|$$

holds for every $t \in \mathbb{R}$, $u, v \in \mathbb{R}$ and $\lambda \in \Lambda$.

c) For every $u \in \mathbb{R}$, we have

$$\lim_{\lambda\to\lambda_0}\left|\int\limits_{\mathbb{R}} K_{\lambda}\left(t, \sum_{k=1}^{m}(-1)^{k-1}\left(\begin{array}{c} m \\ k \end{array}\right)\frac{u}{\varphi\left(x_0\right)}\varphi(x_0+kt)\right)dt - u\right| = 0,$$

provided $x_0 \in \mathbb{R}$ is a $m-p-\mu-$Lebesgue point of φ.

d) $\lim\limits_{\lambda\to\lambda_0}\left[\sup\limits_{|t|>\xi}\left[\varphi^{kp}(t)L_{\lambda}\left(t\right)\right]\right] = 0$, for every $\xi > 0$ and $k = 1, ..., m$.

e) $\lim\limits_{\lambda\to\lambda_0}\left[\int\limits_{|t|>\xi}\varphi^{kp}(t)L_{\lambda}\left(t\right)dt\right] = 0$, for every $\xi > 0$ and $k = 1, ..., m$.

f) For a given number $\delta_1 > 0$, the function $L_{\lambda}(t)$ is non-decreasing on $(-\delta_1, 0]$ and non-increasing on $[0, \delta_1)$ with respect to t, for any $\lambda \in \Lambda$.

g) $\left\|\varphi^{k}L_{\lambda}\right\|_{L_1(\mathbb{R})} \leq M_k < \infty$, for every $\lambda \in \Lambda$ and $k = 0, 1, ..., m$.

Throughout this article, we assume that K_{λ} belongs to *Class A_{φ}*.

Remark 2. *The studies* [3, 15, 17, 29]*, among others, are used as main reference works in the construction stage of Class A_{φ}. For the Lipschitz condition* (*b*)*, we refer the reader to* [3, 18, 19].

3. Auxiliary Results

Main results in this work are based on the following theorems. On the other hand, for the following type existence theorem, we refer the reader to [15].

Theorem 1. *If $f \in L_{p,\varphi}(\mathbb{R})$ $(1 \leq p < \infty)$, then the operator $T_{\lambda}^{[m]}(f) \in L_{p,\varphi}(\mathbb{R})$ and the inequality*

$$\left\| T_{\lambda}^{[m]}(f) \right\|_{L_{p,\varphi}(\mathbb{R})} \leq \sum_{k=1}^{m} \binom{m}{k} \left\| \varphi^{k} L_{\lambda} \right\|_{L_{1}(\mathbb{R})} \| f \|_{L_{p,\varphi}(\mathbb{R})}$$

holds for every $\lambda \in \Lambda$.

Proof. We prove the theorem for the case $1 < p < \infty$. The proof for the case $p = 1$ is similar.

The norm in the space $L_{p,\varphi}(\mathbb{R})$ (see, e.g., [17]) is given by the following equality:

$$\| f \|_{L_{p,\varphi}(\mathbb{R})} = \left(\int_{\mathbb{R}} \left| \frac{f(x)}{\varphi(x)} \right|^{p} dx \right)^{\frac{1}{p}}.$$

By condition (a), we may write

$$\begin{aligned} \left\| T_{\lambda}^{[m]}(f) \right\|_{L_{p,\varphi}(\mathbb{R})} &= \left(\int_{\mathbb{R}} \frac{1}{\varphi^{p}(x)} \left| \int_{\mathbb{R}} K_{\lambda}\left(t, \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f(x+kt)\right) dt \right|^{p} dx \right)^{\frac{1}{p}} \\ &\leq \left(\int_{\mathbb{R}} \frac{1}{\varphi^{p}(x)} \left| \int_{\mathbb{R}} L_{\lambda}(t) \left| \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f(x+kt) \right| dt \right|^{p} dx \right)^{\frac{1}{p}}. \end{aligned}$$

Now, applying generalized Minkowski inequality (see, e.g., [25]), we have

$$\begin{aligned} \left\| T_{\lambda}^{[m]}(f) \right\|_{L_{p,\varphi}(\mathbb{R})} &\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} L_{\lambda}^{p}(t) \left| \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} \left| \frac{f(x+kt)}{\varphi(x)} \right| \right|^{p} dx \right)^{\frac{1}{p}} dt \\ &= \int_{\mathbb{R}} L_{\lambda}(t) \left(\int_{\mathbb{R}} \left| \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} \left| \frac{f(x+kt)}{\varphi(x+kt)} \right| \frac{\varphi(x+kt)}{\varphi(x)} \right|^{p} dx \right)^{\frac{1}{p}} dt. \end{aligned}$$

Using inequality (2.2), we can write

$$\varphi(x+kt) \leq \varphi(x)\varphi(kt),$$

and since

$$\begin{aligned} \varphi(kt) &= \varphi((k-1+1)t) \\ &\leq \varphi((k-1)\,t)\varphi(t), \end{aligned}$$

inductively, we set

$$\varphi(kt) \leq \varphi^{k}(t).$$

It follows that

$$\begin{aligned}\left\|T_{\lambda}^{[m]}(f)\right\|_{L_{p,\varphi}(\mathbb{R})} &\leq \int_{\mathbb{R}} L_{\lambda}(t)\left(\int_{\mathbb{R}}\left|\sum_{k=1}^{m}\binom{m}{k}\left|\frac{f(x+kt)}{\varphi(x+kt)}\right|\varphi^{k}(t)\right|^{p}dx\right)^{\frac{1}{p}}dt \\ &\leq \sum_{k=1}^{m}\binom{m}{k}\int_{\mathbb{R}} L_{\lambda}(t)\varphi^{k}(t)dt\left(\int_{\mathbb{R}}\left|\frac{f(u)}{\varphi(u)}\right|^{p}du\right)^{\frac{1}{p}} \\ &= \sum_{k=1}^{m}\binom{m}{k}\left\|\varphi^{k}L_{\lambda}\right\|_{L_1(\mathbb{R})}\|f\|_{L_{p,\varphi}(\mathbb{R})}.\end{aligned}$$

The desired result follows from condition (g). Thus the proof is completed. □

Now, we give a theorem concerning weighted pointwise convergence of the operators of type (1.4).

Theorem 2. *If $x_0 \in \mathbb{R}$ is a common $m-p-\mu-$Lebesgue point of the functions $f \in L_{p,\varphi}(\mathbb{R})$ $(1 \leq p < \infty)$ and φ, then*

$$\lim_{\lambda\to\lambda_0}\left|T_{\lambda}^{[m]}(f;x_0)-f(x_0)\right|=0$$

provided that the function

$$\int_{-\delta}^{\delta} L_{\lambda}(t)\mu'(|t|)\,dt, \tag{3.1}$$

where $0 < \delta < \min\{\delta_0,\delta_1\}$ (for the definitions of the numbers δ_0 and δ_1, see Definition 1 and Definition 2, respectively), is bounded as λ tends to λ_0 on $\Lambda_1 \subseteq \Lambda$. Here, the set Λ_1, which is the subset of Λ (the definition of Λ is given in Definition 2), denotes the set of λ indices on which the function defined by (3.1) remains bounded for any fixed δ $(0 < \delta < \min\{\delta_0,\delta_1\})$ as λ tends to λ_0 on this set.

Proof. We prove the theorem for the case $1 < p < \infty$. The proof for the case $p = 1$ is similar. Let $|I_{\lambda}(x_0)| = \left|T_{\lambda}^{[m]}(f;x_0)-f(x_0)\right|$ and δ be a number such that $0 < \delta < \min\{\delta_0,\delta_1\}$.

In view of (c), we may write

$$|I_{\lambda}(x_0)| = \left|\int_{\mathbb{R}} K_{\lambda}\left(t,\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}f(x_0+kt)\right)dt\right.$$

$$+\int_{\mathbb{R}} K_{\lambda}\left(t,\sum_{k=1}^{m}(-1)^{k-1}\left(\begin{array}{c}m\\k\end{array}\right)\frac{f(x_0)}{\varphi(x_0)}\varphi(x_0+kt)\right)dt$$

$$\left.-\int_{\mathbb{R}} K_{\lambda}\left(t,\sum_{k=1}^{m}(-1)^{k-1}\left(\begin{array}{c}m\\k\end{array}\right)\frac{f(x_0)}{\varphi(x_0)}\varphi(x_0+kt)\right)dt-f(x_0)\right|.$$

From above equality, and using (b), we may easily get

$$\begin{aligned}|I_{\lambda}(x_0)| &\leq \int_{\mathbb{R}}\left|\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}\left(\frac{f(x_0+kt)}{\varphi(x_0+kt)}-\frac{f(x_0)}{\varphi(x_0)}\right)\varphi(x_0+kt)\right|L_{\lambda}(t)\,dt\\ &\quad+\left|\int_{\mathbb{R}} K_{\lambda}\left(t,\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}\frac{f(x_0)}{\varphi(x_0)}\varphi(x_0+kt)\right)dt-f(x_0)\right|.\end{aligned}$$

Since whenever y, z being positive numbers the inequality $(y+z)^p \leq 2^p(y^p+z^p)$ holds (see, e.g., [21]), we have

$$\begin{aligned}|I_{\lambda}(x_0)|^p &\leq 2^p\left(\int_{\mathbb{R}}\left|\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}\left(\frac{f(x_0+kt)}{\varphi(x_0+kt)}-\frac{f(x_0)}{\varphi(x_0)}\right)\varphi(x_0+kt)\right|L_{\lambda}(t)\,dt\right)^p\\ &\quad+2^p\left|\int_{\mathbb{R}} K_{\lambda}\left(t,\sum_{k=1}^{m}(-1)^{k-1}\left(\begin{array}{c}m\\k\end{array}\right)\frac{f(x_0)}{\varphi(x_0)}\varphi(x_0+kt)\right)dt-f(x_0)\right|^p\\ &= 2^pI_1+2^pI_2.\end{aligned}$$

We can write the integral I_1 as follows:

$$I_1=$$

$$\left(\left\{\int_{|t|>\delta}+\int_{-\delta}^{\delta}\right\}\left|\sum_{k=1}^{m}(-1)^{k-1}\left(\begin{array}{c}m\\k\end{array}\right)\left(\frac{f(x_0+kt)}{\varphi(x_0+kt)}-\frac{f(x_0)}{\varphi(x_0)}\right)\varphi(x_0+kt)\right|L_{\lambda}(t)\,dt\right)^p.$$

Let $\frac{1}{p}+\frac{1}{q}=1$. Now, we apply Hölder's inequality (see [21]) to the integral I_1 as follows:

$$I_1 \leq$$

$$M_0^{\frac{p}{q}}\left\{\int_{|t|>\delta}+\int_{-\delta}^{\delta}\right\}\left|\sum_{k=1}^{m}(-1)^{k-1}\left(\begin{array}{c}m\\k\end{array}\right)\left(\frac{f(x_0+kt)}{\varphi(x_0+kt)}-\frac{f(x_0)}{\varphi(x_0)}\right)\varphi(x_0+kt)\right|^p L_{\lambda}(t)\,dt$$

$$=M_0^{\frac{p}{q}}\left(I_{11}+I_{12}\right).$$

Let us show that $I_{11}\to 0$ as $\lambda\to\lambda_0$. The following inequality holds for I_{11} :

$$I_{11}=\int\limits_{|t|>\delta}\left|\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}\left(\frac{f(x_0+kt)}{\varphi(x_0+kt)}-\frac{f(x_0)}{\varphi(x_0)}\right)\varphi(x_0+kt)\right|^{p}L_\lambda(t)\,dt$$

$$\leq 2^p\left(\int\limits_{|t|>\delta}\sum_{k=1}^{m}\binom{m}{k}\left|\frac{f(x_0+kt)}{\varphi(x_0+kt)}\right|\varphi(x_0+kt)\right)^{p}L_\lambda(t)\,dt$$

$$+2^p\left|\frac{f(x_0)}{\varphi(x_0)}\right|^{p}\left(\int\limits_{|t|>\delta}\sum_{k=1}^{m}\binom{m}{k}\varphi(x_0+kt)\right)^{p}L_\lambda(t)\,dt.$$

Applying (2.2) to the above integral, we have

$$\begin{aligned}I_{11}&\leq 2^p\varphi^p(x_0)\int\limits_{|t|>\delta}\left(\sum_{k=1}^{m}\binom{m}{k}\left|\frac{f(x_0+kt)}{\varphi(x_0+kt)}\right|\varphi^k(t)\right)^{p}L_\lambda(t)\,dt\\&+2^p\left|\frac{f(x_0)}{\varphi(x_0)}\right|^{p}\varphi^p(x_0)\int\limits_{|t|>\delta}\left(\sum_{k=1}^{m}\binom{m}{k}\varphi^k(t)\right)^{p}L_\lambda(t)\,dt\\&=2^p\varphi^p(x_0)I_{111}+2^p\left|\frac{f(x_0)}{\varphi(x_0)}\right|^{p}\varphi^p(x_0)I_{112}.\end{aligned}$$

Expanding the summation inside the integral I_{111} and using the inequality $(y+z)^p\leq 2^p(y^p+z^p)$, we have

$$\begin{aligned}I_{111}&=\int\limits_{|t|>\delta}\left(\sum_{k=1}^{m}\binom{m}{k}\left|\frac{f(x_0+kt)}{\varphi(x_0+kt)}\right|\varphi^k(t)\right)^{p}L_\lambda(t)\,dt\\&\leq 2^{mp}\sum_{k=1}^{m}\binom{m}{k}^{p}\sup_{|t|>\delta}\left[\varphi^{kp}(t)L_\lambda(t)\right]\|f\|^p_{L_{p,\varphi}(\mathbb{R})}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}I_{112}&=\int\limits_{|t|>\delta}\left(\sum_{k=1}^{m}\binom{m}{k}\varphi^k(t)\right)^{p}L_\lambda(t)\,dt\\&\leq 2^{mp}\int\limits_{|t|>\delta}\left(\sum_{k=1}^{m}\binom{m}{k}^{p}\varphi^{kp}(t)\right)L_\lambda(t)\,dt.\end{aligned}$$

Since m is finite and using conditions (d) and (e), $I_{111}\to 0$ and $I_{112}\to 0$ as $\lambda\to\lambda_0$, respectively. Also, from (c), $I_2\to 0$ as $\lambda\to\lambda_0$.

Now, we focus on I_{12}. We have to show that $I_{12} \to 0$ as $\lambda \to \lambda_0$. Since φ is bounded, φ^k is bounded for each k. Set

$$D_\delta = \max_k \left\{ \sup_{-\delta \le t \le \delta} \left(\varphi^k(t)\right) \right\}_{k=1}^m , \quad 0 < \delta < \min\{\delta_0, \delta_1\}.$$

Therefore, the following inequality holds:

$$I_{12} \le \varphi^p(x_0) D_\delta^p \int_{-\delta}^{\delta} \left| \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \frac{f(x_0+kt)}{\varphi(x_0+kt)} \right|^p L_\lambda(t)\, dt$$

$$= \varphi^p(x_0) D_\delta^p \left\{ \int_{-\delta}^{0} + \int_{0}^{\delta} \right\} \left| \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \frac{f(x_0+kt)}{\varphi(x_0+kt)} \right|^p L_\lambda(t)\, dt$$

$$= \varphi^p(x_0) D_\delta^p \left\{ I_{121} + I_{122} \right\}.$$

For I_{121}, let us define

$$F(t) := \int_{t}^{0} \left| \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \frac{f(x_0+kv)}{\varphi(x_0+kv)} \right|^p dv.$$

According to Definition 1, for every $\varepsilon > 0$, there exists $\delta_2 > 0$ such that the inequality

$$F(t) \le \varepsilon \mu(-t) \tag{3.2}$$

holds for every δ_2 satisfying $0 < \delta_2 \le \delta < \min\{\delta_0, \delta_1\}$. Using integration by parts twice and (3.2), we have (for the similar situation, see [15, 17, 22])

$$|I_{121}| \le \varepsilon \int_{-\delta}^{0} \mu'(-t) L_\lambda(t)\, dt.$$

Similarly,

$$|I_{122}| \le \varepsilon \int_{0}^{\delta} \mu'(t) L_\lambda(t)\, dt.$$

Combining above results, we have

$$|I_{12}| \le \varepsilon D_\delta^p \varphi^p(x_0) \int_{-\delta}^{\delta} \mu'(|t|) L_\lambda(t)\, dt.$$

The remaining part follows from the arbitrariness of ε and boundedness of $\int_{-\delta}^{\delta} \mu'(|t|) L_\lambda(t)\, dt$ as λ tends to λ_0. This completes the proof. □

4. Fatou Type Convergence

Throughout years many approximation theory researchers including Siudut [24], Carlsson [7] and Karsli [15] investigated the pointwise convergence of the linear and nonlinear singular integral operators by using Fatou type convergence investigation method, i.e., restriction of the pointwise convergence to some subsets of the plane. Because the sensitive analysis is obtained via this method. For further reading concerning this method, we refer the reader to [10, 14, 22, 27, 28]. Although the expression Fatou type convergence is not directly mentioned in some works (e.g., [22]), they are evaluated in this concept by some contemporary researchers.

In this section we will prove the Fatou type weighted pointwise convergence of the operators of type (1.4), i.e., the convergence will be restricted to a bounded planar subsets of $\mathbb{R} \times \Lambda$. For this purpose, we suppose that for every $\varepsilon > 0$, there exists $\delta > 0$ such that the function Ω_δ given as

$$\Omega_\delta(x,\lambda) = \sum_{k=1}^{m} \binom{m}{k} \int_{-\delta}^{\delta} |f(x+kt) - f(x_0+kt)| \, L_\lambda(t) \, dt,$$

where $0 < \delta < \min\{\delta_0, \delta_1\}$, is bounded on the set defined as

$$Z_{C,\delta,m} = \{(x,\lambda) \in \mathbb{R} \times \Lambda_1 : \Omega_{\delta,m}(x,\lambda) < \varepsilon C\},$$

where C is positive constant, as (x,λ) tends to (x_0,λ_0).

Theorem 3. *Suppose that the hypotheses of Theorem 2 hold. If $x_0 \in \mathbb{R}$ is a common $m-p-\mu-$Lebesgue point of the functions $f \in L_{p,\varphi}(\mathbb{R})$ $(1 \le p < \infty)$ and φ, then*

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} \left| T_\lambda^{[m]}(f;x) - f(x_0) \right| = 0$$

provided that $(x,\lambda) \in Z_{C,\delta,m}$.

Proof. We prove the theorem for the case $1 < p < \infty$. The proof for the case $p = 1$ is similar. Let $0 < |x_0 - x| < \frac{\delta}{2}$ for a given $0 < \delta < \min\{\delta_0, \delta_1\}$.

Now, set $I_{\lambda}(x)=\left|T_{\lambda}^{[m]}(f;x)-f(x_0)\right|$. Let us write

$$\begin{aligned}
|I_{\lambda}(x)| &= \left|\int_{\mathbb{R}} K_{\lambda}\left(t,\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}f(x+kt)\right)dt-f(x_0)\right| \\
&= \left|\int_{\mathbb{R}} K_{\lambda}\left(t,\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}f(x+kt)\right)dt\right. \\
&\quad -\int_{\mathbb{R}} K_{\lambda}\left(t,\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}f(x_0+kt)\right)dt \\
&\quad \left.+\int_{\mathbb{R}} K_{\lambda}\left(t,\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}f(x_0+kt)\right)dt-f(x_0)\right|.
\end{aligned}$$

From above equality, we deduce that

$$\begin{aligned}
|I_{\lambda}(x)|^p &\leq 2^p\left(\int_{\mathbb{R}}\left|\sum_{k=1}^{m}\binom{m}{k}\left(f(x+kt)-f(x_0+kt)\right)\right|L_{\lambda}(t)\,dt\right)^p \\
&\quad +2^p\left|\int_{\mathbb{R}} K_{\lambda}\left(t,\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}f(x_0+kt)\right)dt-f(x_0)\right|^p \\
&= 2^p\left(I_1+I_2\right).
\end{aligned}$$

The following inequality holds for I_1 :

$$\begin{aligned}
I_1 &= \left(\left\{\int_{|t|>\delta}+\int_{-\delta}^{\delta}\right\}\sum_{k=1}^{m}\binom{m}{k}\left|f(x+kt)-f(x_0+kt)\right|L_{\lambda}(t)\,dt\right)^p \\
&\leq 2^p\left(\int_{|t|>\delta}\sum_{k=1}^{m}\binom{m}{k}\left|f(x+kt)-f(x_0+kt)\right|L_{\lambda}(t)\,dt\right)^p \\
&\quad +2^p\left(\int_{-\delta}^{\delta}\sum_{k=1}^{m}\binom{m}{k}\left|f(x+kt)-f(x_0+kt)\right|L_{\lambda}(t)\,dt\right)^p \\
&= 2^p\left(I_{11}+I_{12}\right).
\end{aligned}$$

It is easy to see that

$$\begin{aligned} I_{11} &\leq 2^p \left(\varphi(x) \int\limits_{|t|>\delta} \sum_{k=1}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) \left| \frac{f(x+kt)}{\varphi(x+kt)} \right| \varphi^k(t) L_\lambda(t)\, dt \right)^p \\ &\quad + 2^p \left(\varphi(x_0) \int\limits_{|t|>\delta} \sum_{k=1}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) \left| \frac{f(x_0+kt)}{\varphi(x_0+kt)} \right| \varphi^k(t) L_\lambda(t)\, dt \right)^p . \end{aligned}$$

Following same strategy as in Theorem 2, we have

$$\begin{aligned} I_{11} &\leq \varphi^p(x) 2^{(m+1)p} (M_0)^{\frac{p}{q}} \left(\sum_{k=1}^{m} \left(\begin{array}{c} m \\ k \end{array} \right)^p \sup_{|t|>\delta} \left[\varphi^{kp}(t) L_\lambda(t) \right] \right) \|f\|^p_{L_{p,\varphi}(\mathbb{R})} \\ &\quad + \varphi^p(x_0) 2^{(m+1)p} (M_0)^{\frac{p}{q}} \left(\sum_{k=1}^{m} \left(\begin{array}{c} m \\ k \end{array} \right)^p \sup_{|t|>\delta} \left[\varphi^{kp}(t) L_\lambda(t) \right] \right) \|f\|^p_{L_{p,\varphi}(\mathbb{R})} . \end{aligned}$$

Using the same method as in the proof of Theorem 2 and condition (d), we see that $I_{11} \to 0$ as $\lambda \to \lambda_0$. Clearly, by Theorem 2, $I_2 \to 0$ as λ tends to λ_0. The result follows from the hypothesis on the integral I_{12}.

Thus the proof is completed. □

5. Rate of Convergence

Theorem 4. *Suppose that the hypotheses of Theorem 2 are satisfied. Let*

$$\Delta(\lambda, \delta) = \int\limits_{-\delta}^{\delta} L_\lambda(t) \mu'(|t|)\, dt,$$

where $0 < \delta < \min\{\delta_0, \delta_1\}$, *and the following conditions are satisfied:*

(i) $\Delta(\lambda, \delta) \to 0$ as $\lambda \to \lambda_0$ for some $\delta > 0$.

(ii) For every $\delta > 0$ and $k = 1, ..., m,$ we have

$$\sup_{|t|>\delta} \varphi^{kp}(t) L_\lambda(t) = \mathbf{o}(\Delta(\lambda, \delta))$$

as $\lambda \to \lambda_0$.

(iii) For every $\delta > 0$ and $k = 1, ..., m,$ we have

$$\int\limits_{|t|>\delta} \varphi^{kp}(t) L_\lambda(t)\, dt = \mathbf{o}(\Delta(\lambda, \delta)) \text{ as } \lambda \to \lambda_0.$$

(*iv*) Letting $\lambda \to \lambda_0$, we have

$$\left| \int\limits_{\mathbb{R}} K_\lambda \left(t, \sum_{k=1}^{m} (-1)^{k-1} \begin{pmatrix} m \\ k \end{pmatrix} \frac{f(x_0)}{\varphi(x_0)} \varphi(x_0 + kt) \right) dt - f(x_0) \right|^p = \mathbf{o}(\Delta(\lambda, \delta)).$$

Then, at each common $m-p-\mu-$Lebesgue point of $f \in L_{p,\varphi}(\mathbb{R})$ $(1 \leq p < \infty)$ and φ, we have

$$\left| T_\lambda^{[m]}(f; x_0) - f(x_0) \right|^p = \mathbf{o}(\Delta(\lambda, \delta)), \quad \text{as } \lambda \to \lambda_0.$$

Proof. We prove the theorem for the case $1 < p < \infty$. The proof for the case $p = 1$ is similar. By the hypotheses of Theorem 2, we have

$$\left| T_\lambda^{[m]}(f; x_0) - f(x_0) \right|^p \leq \varepsilon 2^p \varphi^p(x_0) (M_0)^{\frac{p}{q}} D_\delta^p \int\limits_{-\delta}^{\delta} \mu'(|t|) L_\lambda(t) \, dt$$

$$+ 2^{(m+2)p} \varphi^p(x_0) (M_0)^{\frac{p}{q}} \sum_{k=1}^{m} \begin{pmatrix} m \\ k \end{pmatrix}^p \sup_{|t|>\delta} \left[\varphi^{kp}(t) L_\lambda(t) \right] \|f\|_{L_{p,\varphi}(\mathbb{R})}^p$$

$$+ 2^{(m+2)p} \varphi^p(x_0) (M_0)^{\frac{p}{q}} \left| \frac{f(x_0)}{\varphi(x_0)} \right|^p \int\limits_{|t|>\delta} \left(\sum_{k=1}^{m} \begin{pmatrix} m \\ k \end{pmatrix}^p \varphi^{kp}(t) \right) L_\lambda(t) \, dt$$

$$+ 2^p \left| \int\limits_{\mathbb{R}} K_\lambda \left(t, \sum_{k=1}^{m} (-1)^{k-1} \begin{pmatrix} m \\ k \end{pmatrix} \frac{f(x_0)}{\varphi(x_0)} \varphi(x_0 + kt) \right) dt - f(x_0) \right|^p.$$

The proof is completed by $(i) - (iv)$. □

Theorem 5. *Suppose that the hypotheses of Theorem 3 are satisfied. Let*

$$\Omega_\delta(x, \lambda) = \sum_{k=1}^{m} \begin{pmatrix} m \\ k \end{pmatrix} \int\limits_{-\delta}^{\delta} |f(x + kt) - f(x_0 + kt)| \, L_\lambda(t) \, dt,$$

where $0 < \delta < \min\{\delta_0, \delta_1\}$, *and the following conditions are satisfied:*

(*i*) $\Omega_\delta(x, \lambda) \to 0$ as $(x, \lambda) \to (x_0, \lambda_0)$ for some $\delta > 0$.

(*ii*) For every $\delta > 0$ and $k = 1, ..., m$, we have

$$\sup_{|t|>\delta} \varphi^{kp}(t) L_\lambda(t) = \mathbf{o}(\Omega_\delta(x, \lambda)), \text{ as } \lambda \to \lambda_0.$$

(*iii*) Letting $(x, \lambda) \to (x_0, \lambda_0)$,

$$\left| \int\limits_{\mathbb{R}} K_\lambda \left(t, \sum_{k=1}^{m} (-1)^{k-1} \begin{pmatrix} m \\ k \end{pmatrix} f(x_0 + t) \right) dt - f(x_0) \right|^p = \mathbf{o}(\Omega_\delta(x, \lambda)).$$

Then, at each common m–p–μ–Lebesgue point of $f \in L_{p,\varphi}(\mathbb{R})$ $(1 \leq p < \infty)$ and φ, we have

$$\left| T_{\lambda}^{[m]}(f;x) - f(x_0) \right|^p = \mathbf{o}\left(\Omega_{\delta}(x,\lambda)\right), \quad \text{as } (x,\lambda) \to (x_0,\lambda_0).$$

Proof. We prove the theorem for the case $1 < p < \infty$. The proof for the case $p = 1$ is similar. Under the hypotheses of Theorem 3, we may write

$$\begin{aligned}
&\left| T_{\lambda}^{[m]}(f;x) - f(x_0) \right|^p \leq \\
&\qquad \varphi^p(x) 2^{(m+3)p} (M_0)^{\frac{p}{q}} \left(\sum_{k=1}^{m} \binom{m}{k}^p \sup_{|t|>\delta} \left[\varphi^{kp}(t) L_{\lambda}(t) \right] \right) \|f\|_{L_{p,\varphi}(\mathbb{R})}^p \\
&\qquad + \varphi^p(x_0) 2^{(m+3)p} (M_0)^{\frac{p}{q}} \left(\sum_{k=1}^{m} \binom{m}{k}^p \sup_{|t|>\delta} \left[\varphi^{kp}(t) L_{\lambda}(t) \right] \right) \|f\|_{L_{p,\varphi}(\mathbb{R})}^p \\
&\qquad + 2^{2p} \left(\sum_{k=1}^{m} \binom{m}{k} \int_{-\delta}^{\delta} |f(x+kt) - f(x_0+kt)| \, L_{\lambda}(t)\, dt \right)^p \\
&\qquad + 2^p \left| \int_{\mathbb{R}} K_{\lambda}\left(t, \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f(x_0+kt) \right) dt - f(x_0) \right|^p .
\end{aligned}$$

From conditions $(i) - (iii)$, the proof is completed. □

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