



## ZERO-BASED INVARIANT SUBSPACES IN THE BERGMAN SPACE

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**ABSTRACT.** It is known that Beurling's theorem concerning invariant subspaces is not true in the Bergman space (in contrast to the Hardy space case). However, Aleman, Richter, and Sundberg proved that every cyclic invariant subspace in the Bergman space  $L_a^p(\mathbb{D})$ ,  $0 < p < +\infty$ , is generated by its extremal function. This implies, in particular, that for every zero-based invariant subspace in the Bergman space the Beurling's theorem stands true. Here, we calculate the reproducing kernel of the zero-based invariant subspace  $M_n$  in the Bergman space  $L_a^2(\mathbb{D})$  where the associated wandering subspace  $M_n \ominus zM_n$  is one-dimensional, and spanned by the unit vector  $G_n(z) = \sqrt{n+1}z^n$ .

### 1. INTRODUCTION

Let  $\mathbb{D}$  denote the open unit disk in the complex plane. The Bergman space  $L_a^p(\mathbb{D})$  is the space of all holomorphic functions  $f : \mathbb{D} \rightarrow \mathbb{C}$  such that

$$\|f\|_{L_a^p}^p = \int_{\mathbb{D}} |f(z)|^p dS(z) < +\infty, \quad (1.1)$$

where  $dS(z) = \pi^{-1}dxdy$  is the normalized area measure. It is well-known that for  $1 \leq p < +\infty$ , the Bergman space  $L_a^p(\mathbb{D})$  is a Banach space and for  $0 < p < 1$ , it is a complete metric space. For  $p = 2$ , the evaluation at  $z \in \mathbb{D}$  is a bounded linear functional on the Hilbert space  $L_a^2(\mathbb{D})$ . By the Riesz representation Theorem, there exists a unique function  $K_z$  in  $L_a^2(\mathbb{D})$  such that:

$$f(z) = \int_{\mathbb{D}} f(w) \overline{K_z(w)} dS(w) \quad (1.2)$$

for all  $f$  in  $L_a^2(\mathbb{D})$ . The function  $K(z, w)$  defined on  $\mathbb{D} \times \mathbb{D}$  by  $K(z, w) = \overline{K_z(w)}$  is called the Bergman kernel of  $\mathbb{D}$  (it's also called the reproducing kernel of  $L_a^2(\mathbb{D})$ ).

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Let  $e_n(z) = \sqrt{n+1}z^n$  for  $n \geq 0$ . Then,  $\{e_n\}$  forms an orthonormal basis for  $L_a^2(\mathbb{D})$ . Thus,

$$K(z, w) = \sum_{n=0}^{+\infty} (n+1)z^n \bar{w}^n = \frac{1}{(1-z\bar{w})^2}. \quad (1.3)$$

A closed subspace  $M \subset L_a^p(\mathbb{D})$  is said to be invariant if  $zM \subset M$ . A sequence  $\Lambda \subset \mathbb{D}$  is said to be a zero sequence if there exists a non-zero function  $f \in L_a^p(\mathbb{D})$  such that  $f$  vanishes precisely on  $\Lambda$ . An invariant subspace of the form

$$M = \{f \in L_a^p(\mathbb{D}) : f(z) = 0, z \in \Lambda\} \quad (1.4)$$

is called a zero-based invariant subspace. For a function  $f \in L_a^p(\mathbb{D})$ , the closure of all polynomial multiples of  $f$  in  $L_a^p(\mathbb{D})$  is an invariant subspace which is denoted by  $[f]$ ; this subspace is also known as the invariant subspace generated by  $f$ . An invariant subspace  $M$  is said to be cyclic if  $M = [f]$  for some  $f \in L_a^p(\mathbb{D})$ . It is known that every zero-based invariant subspace is cyclic. For an invariant subspace  $M$ , we consider the extremal problem

$$\sup \left\{ \operatorname{Re} G^{(j)}(0) : G \in M, \|G\|_{L_a^p} \leq 1 \right\}, \quad (1.5)$$

where  $j$  is the multiplicity of the common zero at the origin of all the functions in  $M$ . The solution to this problem is called the extremal function for  $M$ . This problem was first introduced by Hedenmalm [6] for the case  $p = 2$ , and subsequently by Duren et al. [4] for  $0 < p < +\infty$ . In the Hardy spaces, by Beurling's Theorem, every invariant subspace other than the trivial one  $\{0\}$  is generated by an inner function (which is an extremal function in that context). In other words, every invariant subspace of the Hardy space is cyclic. On the other hand, the invariant subspaces of the Bergman space  $L_a^2(\mathbb{D})$  need not be singly generated. Nevertheless, for the Bergman space  $L_a^2(\mathbb{D})$ , the Beurling-type Theorem holds true and every invariant subspace  $M$  is generated by  $M \ominus zM$ , that is,  $M = [M \ominus zM] = [M \cap (zM)^\perp]$ .

In [1], the author proved that every zero-based invariant subspace of  $L_a^p(\mathbb{D})$  is generated by its extremal function. The proof uses the density of the polynomials functions in some weighted Bergman spaces.

In this paper, we calculate the reproducing kernel of the wandering subspace  $M_n \ominus zM_n$  of the zero-based invariant subspace  $M_n$  in the Bergman space  $L_a^2(\mathbb{D})$ .

## 2. HARDY AND BERGMAN SPACES

The Hardy space  $H^2$  consists of all holomorphic functions defined on the open unit disk  $\mathbb{D}$  such that

$$\|f\|_{H^2} = \sup_{0 < r < 1} \left( \int_{\mathbb{T}} |f(rz)|^2 ds(z) \right)^{\frac{1}{2}} < +\infty, \quad (2.1)$$

where  $\mathbb{T}$  is the unit circle, and  $ds$  is the arc length measure, normalized so that the mass of  $\mathbb{T}$  equals 1. In terms of Taylor coefficients, the norm takes a more appealing

form. If  $f(z) = \sum_n a_n z^n$ , then

$$\|f\|_{H^2} = \left( \sum_n |a_n|^2 \right)^{\frac{1}{2}}. \tag{2.2}$$

On the other hand, the Bergman space  $L_a^2(\mathbb{D})$  consists of all holomorphic functions defined on  $\mathbb{D}$  such that

$$\|f\|_{L_a^2} = \left( \int_{\mathbb{D}} f(z)^2 dS(z) \right)^{\frac{1}{2}} < +\infty, \tag{2.3}$$

where  $dS$  is area measure normalized so that the mass of  $\mathbb{D}$  equals 1. Though the integral expression of the norm is more straightforward than that in the Hardy space, it is more complicated in terms of Taylor coefficients. If  $f(z) = \sum_n a_n z^n$ , then

$$\|f\|_{L_a^2} = \left( \sum_n \frac{|a_n|^2}{n+1} \right)^{\frac{1}{2}}. \tag{2.4}$$

The Bergman space  $L_a^2(\mathbb{D})$  contains  $H^2$  as a dense subspace. It is intuitively clear from the definition of the norm of  $H^2$  that functions have well-defined boundary values in  $L^2(\mathbb{T})$ . However, this is not the case for  $L_a^2(\mathbb{D})$ . In fact, there is a function in which it fails to have radial limits at every point of  $\mathbb{T}$ . This is a consequence of a more general statement due to MacLane [9]. Apparently, the spaces  $H^2$  and  $L_a^2(\mathbb{D})$  are very different from a function-theoretical perspective.

**2.1. Hardy space theory.** The classical factorization theory for the Hardy spaces (i.e., the spaces  $H^p$  with  $0 < p \leq +\infty$ ), which relies on work due to Blaschke, Riesz, and Szegö, requires some familiarity with the concepts of Blaschke product: singular inner function, inner function and outer function. Let  $H^\infty$  stands for the space of bounded analytic functions in  $\mathbb{D}$  supplied with the supremum norm. Given a sequence  $A = \{a_j\}_j$  of points in  $\mathbb{D}$  and consider the product

$$B_A(z) = \prod_j \frac{\bar{a}_j}{|a_j|} \frac{a_j - z}{1 - \bar{a}_j z} \quad \text{for } z \in \mathbb{D} \tag{2.5}$$

which converges to a function in  $H^\infty$  with norm 1 if and only if the Blaschke condition  $\sum_j 1 - |a_j| < +\infty$  is fulfilled. In this case,  $A$  is said the Blaschke sequence and  $B_A$  the Blaschke product. Note that, for Blaschke sequence  $A$ ,  $B_A$  vanishes precisely on  $A$  in  $\mathbb{D}$  with appropriate multiplicities depending on how many times a point is repeated in the sequence. Moreover, the function  $B_A$  has boundary values of modulus 1 almost everywhere, provided that the limits are taken in nontangential approach regions. We note also that if the sequence  $A$  fails to be Blaschke, the product  $B_A$  collapses to 0. Define the singular inner function in  $H^\infty$  as follows:

$$S_\mu(z) = \exp \left( - \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right) \quad \text{for } z \in \mathbb{D}, \tag{2.6}$$

where  $\mu$  is a finite positive Borel measure on the unit circle  $\mathbb{T}$ . This is the general criterion for a function in  $H^\infty$  to be inner; to have boundary values of modulus 1 almost everywhere. A product of an unimodular constant, a Blaschke product, and a singular inner function, is still inner, and all inner functions are obtained this way.

If  $h$  is a real-valued  $L^1$  function on  $\mathbb{T}$ , the associated outer function is

$$O_h(z) = \exp \left( \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} h(\zeta) ds(\zeta) \right) \quad \text{for } z \in \mathbb{D}, \quad (2.7)$$

which is an analytic function in  $\mathbb{D}$  with  $|O_h(z)| = \exp(h(z))$  almost everywhere on the unit circle. The boundary values of  $O_h$  being thought of in the non-tangential sense. The function  $O_h$  is in  $H^2$  if and only if  $\exp(h)$  is in  $L^2(\mathbb{T})$ . The factorization Theorem in  $H^2$  states that every nonidentically vanishing  $f$  in  $H^2$  has the form

$$f(z) = \gamma B_A(z) S_\mu(z) O_h(z) \quad \text{for } z \in \mathbb{D}, \quad (2.8)$$

where  $\gamma$  is an unimodular constant and  $\exp(h) \in L^2(\mathbb{T})$ .

The natural setting for the factorization theory is a larger class of functions known as the Nevanlinna class. It consists of all functions of the above type, where no additional requirement is made on  $h$ , and where the singular measure  $\mu$  is allowed to take negative values as well. It is well-known that  $f \in N$  if and only if the function  $f$  is holomorphic in  $\mathbb{D}$ , and

$$\sup_{0 < r < 1} \int_{\mathbb{T}} \log^+ |f(rz)| ds(z) < +\infty, \quad (2.9)$$

where  $N$  is the Nevanlinna class.

**2.2. Inner functions in Bergman space.** The Bergman space  $L_a^2(\mathbb{D})$  contains  $H^2$ . How then does it relate to  $N$ ? It turns out that there are functions in  $N$  that are not in  $L_a^2(\mathbb{D})$ , and there are functions in  $L_a^2(\mathbb{D})$  which are not in  $N$ . The latter statement follows from the fact alluded to above that there is a function in  $L_a^2(\mathbb{D})$  lacking nontangential boundary values altogether. On the other hand, all the functions in  $N$  have finite nontangential boundary values almost everywhere. The former statement follows from a much simpler example: Take  $\mu$  equal to a point mass at say 1, and consider the function  $1/S_\mu$ . It is in the Nevanlinna class  $N$ , but it is much bigger near 1 to be in  $L_a^2(\mathbb{D})$ .

The classical Nevanlinna factorization theory is ill-suited for the Bergman space. This is particularly apparent from the fact that there are zero sequences for  $L_a^2(\mathbb{D})$  that are not Blaschke. The question is which functions can replace the Blaschke products or more general inner functions in the Bergman space setting. There may be several ways to do this, but only one is canonical from the point of view of operator theory.

A subspace  $M$  of  $H^2$  is invariant if it is closed and  $zM \subset M$ , and the inner functions in  $H^2$  are characterized as elements of unit norm in some  $M \ominus zM$ , where  $M$  is a nonzero invariant subspace. We call  $M \ominus zM$  the wandering subspace for

$M$ . For a collection  $L$  of functions in  $H^2$ , we let  $[L]$  stands for the smallest invariant subspace containing  $L$ . We note that  $u \in H^2$  is an inner function if and only if

$$h(0) = \int_{\mathbb{T}} h(z)|u(z)|^2 ds(z) \quad \text{for } h \in L_h^\infty(\mathbb{D}). \tag{2.10}$$

Here,  $L_h^\infty(\mathbb{D})$  is the Banach space of bounded harmonic functions on  $\mathbb{D}$ . We say a function  $G \in L_a^2(\mathbb{D})$  is  $L_a^2(\mathbb{D})$ -inner provided that

$$h(0) = \int_{\mathbb{D}} h(z)|G(z)|^2 dS(z) \quad \text{for } h \in L_h^\infty(\mathbb{D}). \tag{2.11}$$

A function  $G$  of unit norm in  $L_a^2(\mathbb{D})$  is  $L_a^2(\mathbb{D})$ -inner if and only if it is in a wandering subspace  $M \ominus zM$  for some nonzero invariant subspace  $M$  of  $L_a^2(\mathbb{D})$ . In contrast, with the  $H^2$  case, where  $M \ominus zM$  always has dimension 1 (unless  $M$  is the zero subspace), this time the dimension may take any value in the range  $1, 2, 3, \dots, +\infty$ . This follows from the dilation theory developed by Apostol, Bercovici, Foias, and Pearcy [2]. The dimension of  $M \ominus zM$  will be referred to the index of the invariant subspace  $M$ .

For the space  $H^2$ , Beurling’s invariant subspace Theorem yields to a complete description:

**Theorem 1** (Beurling 1949). *Let  $M$  be an invariant subspace of  $H^2$ , and  $M \ominus zM$  be called the associated wandering subspace. Then  $M = [M \ominus zM]$ .*

*If  $M$  is not the zero subspace, then  $M \ominus zM$  is one-dimensional and spanned by an inner function  $\phi$  and  $M = [\phi] = \phi H^2$ .*

A natural question is whether the analogous statement  $M = [M \ominus zM]$  (with the brackets referring to the invariant subspace lattice of  $L_a^2(\mathbb{D})$ ) holds for general invariant subspaces  $M$  of  $L_a^2(\mathbb{D})$ .

### 3. BEURLING’S THEOREM

Let  $\Delta = \frac{1}{4}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$  stand for the Laplace operator in the complex plane. Then, we have

$$\Delta|f|^2 = |f'|^2 \tag{3.1}$$

and

$$\Delta|f|^p = \frac{p^2}{4}|f|^{p-2}|f'|^2. \tag{3.2}$$

Let  $M$  be a zero-based invariant subspace in  $L_a^p(\mathbb{D})$  and let  $G$  be its extremal function. It was shown by Hedenmalm [5] for  $p = 2$  and by other authors for arbitrary values of  $0 < p < +\infty$  that  $G$  satisfies the equation

$$\Delta\phi(z) = |G(z)|^p - 1, \quad z \in \mathbb{D}, \tag{3.3}$$

where  $\phi$  is a  $C^\infty$  function in  $\mathbb{D}$ , it vanishes on the boundary of the unit disk. Moreover,  $\phi$  satisfies the inequalities  $0 \leq \phi(z) \leq 1 - |z|^2$ . To study the invariant

subspaces of the Bergman spaces, Hedenmalm introduced the space

$$\mathcal{A}^p = \left\{ f \in L_a^p(\mathbb{D}) : \int_{\mathbb{D}} \phi(z) \Delta |f(z)|^p dS(z) < +\infty \right\} \tag{3.4}$$

for  $0 < p < \infty$ .

For  $f \in \mathcal{A}^p$ , he defined the following norm:

$$\|f\|_{\mathcal{A}^p}^p = \|f\|_{\mathbb{D}^p}^p + \int_{\mathbb{D}} \phi(z) \Delta |f(z)|^p dS(z). \tag{3.5}$$

It can be proved that for  $1 \leq p < +\infty$ , the set  $\mathcal{A}^p$  is a normed vector space. Moreover, for  $0 < p < 1$ , it enjoys the induced metric

$$d(f, g) = \|f - g\|_{\mathbb{D}^p}^p + \int_{\mathbb{D}} \phi(z) \Delta |(f - g)(z)|^p dS(z). \tag{3.6}$$

Let  $\mathcal{A}_0^p$  denote the closure of the polynomials in  $\mathcal{A}^p$  (with respect to the norm or metric defined above). It was shown by Hedenmalm [5] for  $p = 2$  and by Khavinson and Shapiro [8] for  $p \neq 2$  that  $[G] = G \cdot \mathcal{A}_0^p$  and

$$\|Gf\|_{L_a^p}^p = \|f\|_{L_a^p}^p + \int_{\mathbb{D}} \phi(z) \Delta |f(z)|^p dS(z), \quad f \in \mathcal{A}_0^p. \tag{3.7}$$

Moreover, Khavinson and Shapiro [8] left the following open question : Is  $\mathcal{A}^p = \mathcal{A}_0^p$  ? It is clear that  $[G] \subset M$ , and it was already observed that  $M \subset G \cdot \mathcal{A}^p$ . Therefore, if  $\mathcal{A}^p = \mathcal{A}_0^p$ , then the Beurlings Theorem is true for  $M$ , because

$$M \subset G \cdot \mathcal{A}^p = G \cdot \mathcal{A}_0^p = [G]. \tag{3.8}$$

**Theorem 2.** *Let  $M$  be a zero-based invariant subspace of  $L_a^p(\mathbb{D})$ ,  $0 < p < +\infty$ . Then  $M$  is generated by its extremal function  $G$ , that is,  $M = [G]$ .*

*Proof.* We have already mentioned that it suffices to show  $\mathcal{A}^p = \mathcal{A}_0^p$ . Let  $f \in \mathcal{A}^p$ ,  $0 < r < 1$ , and consider the dilated functions  $f_r(z) = f(rz)$ . Since every  $f_r$  can be approximated uniformly by the polynomials, it is enough to show that  $\|f_r f\|_{\mathcal{A}^p} \rightarrow 0$  as  $r \rightarrow 1^-$ . To do this, let us take

$$\|f_r\|_{\mathcal{A}^p}^p = \|f_r\|_{L_a^p}^p + \int_{\mathbb{D}} \phi(z) \Delta |f_r(z)|^p dS(z). \tag{3.9}$$

However,

$$\|f_r\|_{L_a^p}^p = \int_{\mathbb{D}} |f_r(z)|^p dS(z) \tag{3.10}$$

$$= \int_{r\mathbb{D}} |f(z)|^p \frac{dS(z)}{r^2} \tag{3.11}$$

$$= \frac{1}{r^2} \int_{r\mathbb{D}} |f(z)|^p dS(z). \tag{3.12}$$

Therefore,

$$\lim_{r \rightarrow 1^-} \|f_r\|_{L_a^p}^p = \int_{\mathbb{D}} |f(z)|^p dS(z) = \|f\|_{L_a^p}^p. \tag{3.13}$$

We now manage to show that

$$\lim_{r \rightarrow 1^-} \int_{\mathbb{D}} \phi(z) |f_r(z)|^p dS(z) = \int_{\mathbb{D}} \phi(z) |f(z)|^p dS(z). \tag{3.14}$$

From (3.2), we have

$$\Delta^2 \phi(z) = \Delta(|G(z)|^p - 1) \tag{3.15}$$

$$= \frac{p^2}{4} |G(z)|^{p-2} |G'(z)|^2 \tag{3.16}$$

$$\geq 0, \tag{3.17}$$

then  $\phi$  is a superbiharmonic function in the unit disk. Moreover,

$$0 \leq \phi(z) \leq 1 - |z|^2 \leq 2(1 - |z|), \quad z \in \mathbb{D}. \tag{3.18}$$

□

The main result of this paper is given by the following Theorem.

**Theorem 3.** *Let  $M_n$  be a zero-based invariant subspace of  $L_a^2(\mathbb{D})$ , where the associated wandering subspace  $M_n \ominus zM_n$  is one-dimensional and spanned by the unit vector  $G_n(z) = \sqrt{n+1}z^n$ . The reproducing kernel of  $M_n \ominus zM_n$  is given by the formula:*

$$K_w^{G_n}(z) = \frac{1 - (1-n)(\bar{w}z)^n + n(\bar{w}z)^{n-1}}{(1 - \bar{w}z)^2}. \tag{3.19}$$

*Proof.* We prove that

- (1)  $K_w^{G_n} \in M_n \ominus zM_n$ ,
- (2)  $\langle f, K_w^{G_n} \rangle_{L_a^2} = f(z)$  for all  $f$  in  $M_n \ominus zM_n$ , where  $\langle \cdot, \cdot \rangle_{L_a^2}$  denotes the inner product in the Bergman space, i.e.,

$$\langle f, g \rangle_{L_a^2} = \frac{1}{\pi} \int_{\mathbb{D}} f(z) \bar{g}(z) dS(z), \quad f, g \in L_a^2(\mathbb{D}). \tag{3.20}$$

For the proof of (1), note that for fixed  $w \in \mathbb{D}$ , the function  $K_w^{G_n} \in L_a^2(z)$ . Moreover,  $z \mapsto K_w^{G_n}(z)$  is a bounded analytic function. To show that  $K_w^{G_n} \in L_a^2(z)$ , we need to verify that

$$\langle G_n g, K_w^{G_n} \rangle_{L_a^2} = 0, \quad g \in L_a^2(\mathbb{D}). \tag{3.21}$$

The kernel function of  $L_a^2(\mathbb{D})$  is

$$K_w(z) = \frac{1}{(1 - \bar{w}z)^2}, \tag{3.22}$$

and its reproducing property is

$$\langle f, K_w \rangle = f(w), \quad \text{for } f \in L_a^2(\mathbb{D}). \tag{3.23}$$

Then,

$$\langle G_n g, K_w^{G_n} \rangle = \langle G_n g, K_w \rangle - G_n(w) \langle G_n g, G_n K_w \rangle \quad (3.24)$$

$$= G_n(w)g(w) - G_n(w) \langle g, K_w \rangle \quad (3.25)$$

$$= G_n(w)g(w) - G_n(w)g(w) \quad (3.26)$$

$$= 0 \quad (3.27)$$

which proves (1).

The proof of (2) follows from

$$\langle f, K_w^{G_n} \rangle = \langle f, K_w \rangle - G_n(w) \langle f, G_n K_w \rangle \quad (3.28)$$

$$= f(w) + 0 \quad (3.29)$$

$$= f(w). \quad (3.30)$$

□

#### 4. CONCLUSION

The kernel functions play an essential role in the theory of Bergman spaces. In this paper, we calculated the reproducing kernel of the wandering subspace  $M_n \ominus zM_n$  of the zero-based invariant subspace  $M_n$  in the Bergman space  $L_a^2(\mathbb{D})$ . In the other cases, the problem remains unsolved.

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