



ENERGY DECAY RATE OF THE SOLUTIONS OF A MARINE RISER EQUATION WITH A VARIABLE COEFFICIENT

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ABSTRACT. In this work the initial boundary value problem for a fourth order non linear equation which describes the marine riser is studied :

$$u_{tt} + ku_{xxxx} - [a(x)u_x]_x + \gamma u_{tx} + b(t)u_t|u_t|^p = 0, \quad x \in [0, l], \quad t > 0,$$

Under appropriate conditions on $a(x)$ and $b(t)$, we prove that the energy of the problem tends to zero as $t \rightarrow \infty$.

1. INTRODUCTION

We work on the decay properties of solutions to the initial boundary value problem of the marine riser equation:

$$u_{tt} + ku_{xxxx} - [a(x)u_x]_x + \gamma u_{tx} + b(t)u_t|u_t|^p = 0, \quad x \in [0, l], \quad t > 0, \quad (1.1)$$

$$u(0, t) = u_{xx}(0, t) = u(l, t) = u_{xx}(l, t) = 0, \quad t > 0, \quad (1.2)$$

where k, p, γ are given positive numbers, a, b are given functions. This equation without the variable damping coefficient is studied in [1] and [2]. This problem about the offshore drilling operations which done by a long slender vertical pipe that is including a drilling string and drilling mud, which is so called Marine riser.

The problem of riser stability, that is the stability of pipes conveying fluid has caught the attention of many authors (see e.g. [1]-[11]).

Since our equation includes a variable coefficient $b(t)$ the techniques used in above articles is not applicable to our problem. Therefore we adapt the study of Martinez [8], in this article a new weighted integral inequality method was used to estimate the decay rate of solutions of the wave equation. This method is originated a result of Haraux [3].

In [9], the following simplest equation that can be used in modeling of marine riser:

$$u_{tt} + u_{xxxx} - Nu_{xx} = 0 \quad x \in (0, 1), \quad t > 0,$$

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under the homogeneous boundary conditions (1.2) is considered. Where N is a positive number. Lyapunov's direct method is used in detail.

In [7], the following nonlinear marine riser equation:

$$mu_{tt} + EIu_{xxxx} - (Nu_x)_x + au_{xt} + bu_t |u_t| = 0 \quad x \in (0, l), \quad t > 0,$$

under the boundary conditions (1.2) is studied and the stability of zero solution of this problem is established.

In [11], the initial boundary value problem for the fourth order equation

$$mu_{tt} + (EIu_{xx})_{xx} + P(t)u_{xx} = 0, \quad x \in (0, l), \quad t > 0,$$

under the boundary conditions (1.2) is considered. The necessary conditions on $P(t)$ for the stability of solutions are obtained. In [6], the initial boundary value problem for the marine riser equation:

$$mu_{tt} + ku_{xxxx} - (a(x)u_x)_x + \gamma u_{tx} + bu_t |u_t|^p = 0 \quad x \in (0, l), t > 0,$$

under the boundary conditions (1.2) is considered. The global asymptotic stability of solutions and the estimates for the rate of decay of the solutions were obtained.

In [1], the globally asymptotically stability of the zero solution to the problem for multidimensional marine riser equation:

$$u_{tt} + k\Delta^2 u + a\Delta u + \vec{g} \cdot \nabla u_t + bu_t |u_t|^p = 0, \quad x \in \Omega, t > 0,$$

under the initial boundary conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,$$

$$u(x, t) = \frac{\partial u(x, t)}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0,$$

where $\Omega \subset R^N$, $N \leq 3$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, ν is the unit outward normal vector to the boundary, $k > 0$, $p \geq 1$, $b > 0$ and $a \in R$ are given numbers and $\vec{g} = (g_1, g_2, \dots, g_N) \in R^N$, is studied. Furthermore, continuous dependence of the weak and the strong solutions of the problem on the coefficients a , b and g were proved.

There are many articles devoted to the study of boundary control of initial boundary value problems for marine riser type equations (see, e.g. [4], [5], [10]). In what follows, we will use the following notations:

$$\|u(t)\| := \left(\int_0^l u^2(x, t) dx \right)^{\frac{1}{2}}, \quad \|u(t)\|_q := \left(\int_0^l u^q(x, t) dx \right)^{\frac{1}{q}}.$$

The proof of our main result will be based on the following pre mentioned Lemma.

Lemma 1.1. (Martinez, [8]) *Let $E : R^+ \rightarrow R^+$ be a non increasing function and $\phi : R^+ \rightarrow R^+$ a strictly increasing function of class C^1 such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty. \quad (1.3)$$

Assume that there exist $\sigma \geq 0$ and $\omega > 0$ such that

$$\int_S^{+\infty} E(t)^{1+\sigma} \phi'(t) dt \leq \frac{1}{\omega} E(0)^\sigma E(S). \quad (1.4)$$

Then $E(t)$ has the following decay property:

$$\text{if } \sigma = 0, \quad \text{then } E(t) \leq E(0)e^{1-\omega\phi(t)}, \forall t \geq 0, \quad (1.5)$$

$$\text{if } \sigma > 0, \quad \text{then } E(t) \leq E(0) \left(\frac{1+\sigma}{1+\omega\sigma\phi(t)} \right)^{\frac{1}{\sigma}}, \forall t \geq 0. \quad (1.6)$$

2. ASYMPTOTIC BEHAVIOR

Theorem 2.1. Suppose that $b(t)$ is a nonincreasing function of class C^1 on R^+ satisfying $\int_0^t b(s) ds \rightarrow \infty$ as $t \rightarrow \infty$ and there exists a positive number a_0 such that

$$a(x) \leq a_0.$$

Then each solution of the problem (1.1)-(1.2) satisfies the following energy decay property:

$$E(t) \leq E(0) \left(\frac{p+2}{2+\omega p \int_0^t b(s) ds} \right)^{\frac{2}{p}}, \quad \forall t > 0,$$

where

$$\omega^{-1} = \frac{2}{\theta} \max \left\{ c\mu, \frac{(2^{\frac{3p+2}{p}} pl)^{\frac{p}{2}}}{(p+2)(\theta(p+2))^{\frac{p}{2}} E^q(0)}, \frac{(p+1)(4(d_2\mu)^{p+2})^{\frac{1}{p+1}}}{(p+2)(\theta(p+2))^{\frac{1}{p+1}} E^{\frac{q\beta-p-1}{\beta}}(0)} \right\},$$

and

$$\theta = \frac{1}{2} - \frac{c\gamma^2}{k^{\frac{1}{2}}} > 0.$$

Proof. Suppose that u is a solution to the problem (1.1)-(1.2). Multiplying equation (1.1) by u_t and integrating over $(0, l)$ we get

$$\frac{d}{dt} E(t) = -2 \int_0^l b(t) |u_t(x, t)|^{p+2} dx, \quad (2.1)$$

where

$$E(t) := \|u_t(t)\|^2 + k \|u_{xx}(t)\|^2 + \int_0^l a(x) u_x^2(x, t) dx. \quad (2.2)$$

Now, multiplying equation (1.1) by $\phi'^q u$ and integrating over $(0, l) \times (S, T)$ and using boundary conditions we get

$$\begin{aligned} & \phi'(t)E^q \int_0^l (u(x, t)u_t(x, t)) \Big|_S^T dx - \int_S^T \phi'(t)E^q(t) \|u_t(t)\|^2 dt \\ & \quad - \int_S^T \int_0^l [\phi''(t)E^q(t) + q\phi'(t)E^{q-1}(t)E'(t)]u(x, t)u_t(x, t)dxdt \\ & + \int_S^T \phi'(t)E^q(t)[E(t) - \|u_t(t)\|^2]dt + \int_S^T \phi'(t)E^q(t) \int_0^l \gamma u(x, t)u_{tx}(x, t)dxdt \\ & \quad + \int_S^T \phi'(t)E^q(t) \int_0^l b(t)u(x, t)|u_t(x, t)|^{p+1}dxdt = 0. \end{aligned}$$

So we have

$$\begin{aligned} & \int_S^T \phi'(t)E^{q+1}(t)dt = -\phi'(t)E^q(t) \int_0^l (u(x, t)u_t(x, t)) \Big|_S^T dx \\ & \quad + 2 \int_S^T \phi'(t)E^q(t) \|u_t(t)\|^2 dt - \int_S^T \phi'(t)E^q(t) \int_0^l \gamma u(x, t)u_{tx}(x, t)dxdt \\ & \quad + \int_S^T \int_0^l [\phi''(t)E^q(t) + q\phi'(t)E^q(t)E'(t)]u(x, t)u_t(x, t)dxdt \\ & \quad - \int_S^T \phi'(t)E^q(t) \int_0^l b(t)u(x, t)|u_t(x, t)|^{p+1}dxdt. \end{aligned} \tag{2.3}$$

Using Cauchy inequality, Hölder's inequality, definition of $E(t)$ and $\phi'(t)\mu$ we get

$$\left| \int_0^l u(x, t)u_t(x, t)dx \right| \leq \left(\int_0^l u^2(x, t)dx \right) \left(\int_0^l u_t^2(x, t)dx \right) \leq cE(t), \tag{2.4}$$

$$\left| \phi'(t)E^q \int_0^l u(x, t)u_t(x, t)dx \Big|_S^T \right| \leq c\mu E^{q+1}(S), \tag{2.5}$$

$$\begin{aligned} & 2 \int_S^T \phi'(t)E^q(t) \|u_t(t)\|^2 dt \leq 2 \int_S^T \phi'(t)E^q(t) l^{\frac{p}{p+2}} \left(\int_0^l |u_t(x, t)|^{p+2} dx \right)^{\frac{2}{p+2}} dt \\ & \leq \frac{p(2\epsilon_1)^{\frac{p+2}{p}}}{p+2} \int_S^T (\phi'(t)l) E^{\frac{q(p+2)}{p}} dt + \frac{2}{(p+2)\epsilon_1^{\frac{p+2}{2}}} \int_S^T \left(\frac{-E'(t)}{2} \right) dt, \end{aligned} \tag{2.6}$$

$$\begin{aligned} & \left| \int_S \int_0^l [\phi''(t)E^q(t) + q\phi'(t)E^{q-1}(t)E'(t)]u(x,t)u_t(x,t)dxdt \right| \\ & \leq \int_S^T |\phi''(t)E^q(t) + q\phi'(t)E^{q-1}(t)E'(t)| cE(t)dt \\ & \leq \frac{qc\mu}{q+1} E^{q+1}(S), \end{aligned} \quad (2.7)$$

$$\left| \int_S^T \phi'(t)E^q \int_0^l \gamma u(x,t)u_{tx}(x,t)dxdt \right| \leq \frac{d_1}{2} \int_S^T \phi'^{q+1}(t)dt, \quad (2.8)$$

$$\begin{aligned} & \left| \int_S^T \phi'(t)E^q(t) \int_0^l b(t)u(x,t)|u_t(x,t)|^{p+1}dxdt \right| \\ & \leq \frac{(\mu d_2 \epsilon_2)^{p+2}}{p+2} \int_S^T \phi'(t)E^{q(q+\frac{1}{2})(p+2)}(t)dt + \frac{p+1}{(p+2)\epsilon_2^{\frac{p+2}{p+1}}} E(S), \end{aligned} \quad (2.9)$$

where

$$c = \frac{l^2}{\pi^2 k^{\frac{1}{2}}}, \quad d_1 = 1 + \frac{\gamma^2 l^2}{\pi^2 k}, \quad d_2 = \frac{l^{\frac{3}{2} + \frac{1}{p+2}}}{\pi k^{\frac{1}{2}} 2^{\frac{p+1}{p+2}}}.$$

Thanks to Sobolev inequality $\|u(t)\|_{p+2} \leq l^{\frac{1}{2} + \frac{1}{p+2}} \|u_x(t)\|$ (ref. [6]) and the definition of $E(t)$ we have

$$\|u(t)\|_{p+2} \leq \frac{l^{\frac{1}{2} + \frac{1}{p+2}}}{\pi} E(t)^{\frac{1}{2}}.$$

Employing the inequalities 2.5–2.9 and 2.3 we get

$$\begin{aligned} \int_S^T \phi'(t)E^{q+1}(t)dt & \leq c\mu E(0)^q E(S) + \frac{pl(2\epsilon_1)^{\frac{p+2}{p}}}{p+2} \int_S^T \phi'(t)E^{\frac{q(p+2)}{2}} dt \\ & + \frac{1}{(p+2)\epsilon_1^{\frac{p+2}{2}}} \int_S^T (-E'(t))dt + \frac{qc\mu}{q+1} E^{q+1}(S) + \frac{d_1}{2} \int_S^T \phi'(t)E^{q+1}(t)dt \\ & + \frac{(\mu d_2 \epsilon_2)^{p+2}}{p+2} \int_S^T \phi'(t)E^{(q+\frac{1}{2})(p+2)}(t)dt + \frac{p+1}{(p+2)\epsilon_2^{\frac{p+2}{p+1}}} E(S), \end{aligned}$$

If we choose $q = \frac{p}{2}$ we get

$$\begin{aligned} \int_S^T \phi'(t)E^{q+1}(t)dt & \leq c\mu E(0)^q E(S) + \frac{pl(2\epsilon_1)^{\frac{p+2}{p}}}{p+2} \int_S^T \phi'(t)E^{\frac{q(p+2)}{2}}(t)dt \\ & + \frac{1}{(p+2)\epsilon_1^{\frac{p+2}{2}}} \int_S^T (-E'(t))dt + \frac{qc\mu}{q+1} E^{q+1}(S) + \frac{d_1}{2} \int_S^T \phi'(t)E^{q+1}(t)dt \\ & + \frac{(\mu d_2 \epsilon_2)^{p+2}}{p+2} \int_S^T \phi'(t)E^{q+1+\beta}(t)dt + \frac{p+1}{(p+2)\epsilon_2^{\frac{p+2}{p+1}}} E(S), \end{aligned}$$

Here $\beta = \frac{p(p+2)}{2}$. Choosing $k > \frac{2l^2\gamma^2}{\pi^2}$ we get $\theta = \frac{1}{2} - \frac{c\gamma^2}{k^{\frac{1}{2}}} > 0$,

$$\begin{aligned} \theta \int_S^T \phi'(t)E^{q+1}(t)dt &\leq c\mu E(0)^q E(S) + \frac{pl(2\epsilon_1)^{\frac{p+2}{p}}}{p+2} \int_S^T \phi'(t)E^{\frac{q(p+2)}{p}}(t)dt \\ &+ \frac{1}{(p+2)\epsilon_1^{\frac{p+2}{2}}} \int_S^T (-E'(t))dt + \frac{qc\mu}{q+1} E^q(0)E(S) \\ &+ \frac{(\mu d_2 \epsilon_2)^{p+2}}{p+2} \int_S^T \phi'(t)E^{q+1+\beta}(t)dt + \frac{p+1}{(p+2)\epsilon^{\frac{p+2}{p+1}}} E(S), \end{aligned}$$

If we choose $\epsilon_2^{p+2} = \frac{\theta(p+2)}{4(d_2\mu)^{p+2}E^\beta(0)}$ and $\epsilon_1^{\frac{p+2}{p}} = \frac{\theta(p+2)}{2^{\frac{3p+2}{p}}pl}$ we get

$$\begin{aligned} \frac{\theta}{2} \int_S^T \phi'(t)E^{q+1}(t)dt &\leq c\mu E^q(0)E(S) + \frac{qc\mu}{q+1} E^q(0)E(S) \\ &+ \frac{(2^{\frac{3p+2}{p}}pl)^{\frac{p}{2}}}{(p+2)(\theta(p+2))^{\frac{p}{2}}} E(S) + \frac{(p+1)(4(d_2\mu)^{p+2})^{\frac{1}{p+1}} E^{\frac{\beta}{p+1}}(0)}{(p+2)(\theta(p+2))^{\frac{1}{p+1}}} E(S), \end{aligned}$$

Thus we obtain

$$\int_S^T \phi'(t)E^{q+1}(t)dt \leq \frac{1}{\omega} E^q(0)E(S).$$

Now, using Lemma 1.1 we get

$$E(t) \leq E(0) \left(\frac{p+2}{2 + \omega p \int_0^t b(s)ds} \right)^{\frac{2}{p}}, \quad \forall t > 0,$$

Here

$$\omega^{-1} = \frac{2}{\theta} \max \left\{ c\mu, \frac{(2^{\frac{3p+2}{p}}pl)^{\frac{p}{2}}}{(p+2)(\theta(p+2))^{\frac{p}{2}}} E^q(0), \frac{(p+1)(4(d_2\mu)^{p+2})^{\frac{1}{p+1}}}{(p+2)(\theta(p+2))^{\frac{1}{p+1}}} E^{\frac{q\beta-p-1}{\beta}}(0) \right\}.$$

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