



## ANALYSIS OF TWO DIMENSIONAL PARABOLIC EQUATION WITH PERIODIC BOUNDARY CONDITIONS

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**ABSTRACT.** In this paper two dimensional parabolic equation with Dirichlet type boundary condition is considered. The existence and uniqueness of solution are shown. Also we construct an iteration algorithm for the numerical solution of this problem.

### 1. INTRODUCTION

Consider the following mixed problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y, t), \quad (1)$$

$$(x, y, t) \in \Omega \quad : \quad = \{0 < x < \pi, 0 < y < \pi, 0 < t < T\}$$

$$u(0, y, t) = u(\pi, y, t) = 0, \quad t \in [0, T] \quad (2)$$

$$u(x, 0, t) = u(x, \pi, t) = 0, \quad t \in [0, T] \quad (3)$$

$$u(x, y, 0) = \varphi(x, y), \quad x \in [0, \pi] \quad (4)$$

for a two dimensional parabolic equation with the Dirichlet type boundary condition. The function  $\varphi(x, y)$  and  $f(x, y, t)$  are given functions on  $[0, \pi]$  and  $\bar{\Omega}$  respectively. Denote the solution of problem (1)-(4) by  $u(x, y, t)$ .

Two dimensional parabolic equation arise in many areas of science and engineering and wide scope and applications in heat conduction [5, 6, 7]. Srivastava et al [1] discuss analytical solutions of two-dimensional rectangular heat equation. The description of various numerical and other methods with useful bibliography may be found in the surveys of [8, 2, 3]. Compact difference scheme for solving wave equations in two-space dimensions is discussed in [4].

In this study we prove the existence, uniqueness of the solution and we construct an iteration algorithm for the numerical solution. We will use Fourier method for the considered problem (1)-(4).

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The paper is organized as follows. In Section 2, the existence and the uniqueness of the solution of the problem are proved by using the Fourier method and iteration method. In Section 3, stability of method for the solution is shown. In Section 4, stability of method for the solution is given. In Section 5, the numerical procedure for the solution of the problem is given.

## 2. EXISTENCE AND UNIQUENESS OF THE SOLUTION

The main result on the existence and uniqueness of the solution of problems(1)-(4) is presented as follows.

We have the following assumptions on the data of problems (1)-(4).

(F1) Let the function  $f(x, y, t)$  be continuous with respect to all arguments in  $\bar{\Omega}$  and  $f(x, y, t) \in L_2([0, \pi] \times [0, \pi])$ .

(F2)  $\varphi(x, y) \in C([0, \pi] \times [0, \pi])$ .

By applying the standard procedure of the Fourier method, we obtain the following representation for the solution of (1)-(3).

$$u(x, y, t) = \sum_{m,n=1}^{\infty} C_{mn} \sin mx \sin ny$$

$$C_{mn}(t) = \varphi_{mn} e^{-(m^2+n^2)t} + \int_0^t \int_0^{\pi} \int_0^{\pi} e^{-(m^2+n^2)(t-\tau)} f(\xi, \eta, \tau) \sin m\xi \sin n\eta d\xi d\eta d\tau$$

$$u(x, y, t) = \sum_{m,n=1}^{\infty} \left( \varphi_{mn} e^{-(m^2+n^2)t} \right) \sin mx \sin ny$$

$$+ \sum_{m,n=1}^{\infty} \left( \int_0^t f_{mn}(\tau) d\tau \right) \sin mx \sin ny \quad (5)$$

where

$$\varphi_{mn} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} \varphi(x, y) \sin mx \sin ny dx dy,$$

$$f_{mn} = \int_0^{\pi} \int_0^{\pi} e^{-(m^2+n^2)(t-\tau)} f(\xi, \eta, \tau) \sin m\xi \sin n\eta d\xi d\eta.$$

Under the assumptions (F1) and (F2), the solution  $u(x, y, t)$  of the problems (1)-(4) is a unique solution.

## 3. CONTINUOUS DEPENDENCE UPON THE DATA

The following result on continuous dependence on the data of the solution of (1)–(4) holds.

**Theorem 1.**  $\Phi = \{\varphi, f\}$  satisfy the assumptions (F1)–(F2) of theorem 1 then the solution of the problem (1)–(4) depends continuously upon the data  $f, \varphi$ .

*Proof.* Let  $\Phi = \{\varphi, f\}$  and  $\bar{\Phi} = \{\bar{\varphi}, \bar{f}\}$  be two sets of the data, which satisfy the assumptions (F1)–(F2).

Let us denote  $\|\Phi\| = (\|\varphi\|_{C^{2,2}([0,\pi] \times [0,\pi])} + \|f\|_{C^{2,2,0}(\bar{\Omega})})$ .

$u - \bar{u} =$

$$\begin{aligned} & \sum_{m,n=1}^{\infty} \left( (\varphi_{mn} - \bar{\varphi}_{mn}) e^{-(m^2+n^2)t} \right) \sin mx \sin ny \\ & + \sum_{m,n=1}^{\infty} \left( \int_0^t \int_0^\pi \int_0^\pi e^{-(m^2+n^2)(t-\tau)} \left( f(\xi, \eta, \tau) - \bar{f}(\xi, \eta, \tau) \right) \sin m\xi \sin n\eta \, d\xi d\eta d\tau \right) \\ & \quad \times \sin mx \sin ny \end{aligned}$$

$$\|u - \bar{u}\| \leq \frac{1}{\pi\sqrt{6}} (\|\varphi - \bar{\varphi}\| + T \|f - \bar{f}\|)$$

where  $T > 0$ .

$$\|u - \bar{u}\| \leq \frac{1}{\pi\sqrt{6}} \|\Phi - \bar{\Phi}\|$$

For  $\Phi \rightarrow \bar{\Phi}$  then  $u \rightarrow \bar{u}$ . □

## 4. FULLY IMPLICIT BACKWARD-DIFFERENCE SCHEME

Consider the following advection-dispersion equation with forcing function  $f(x, y, t)$ .

Using five point difference scheme and fully implicit backward-difference equation, we obtain the following discrete form for (1)–(4).

$$u_{i,j}^{n+1} = \frac{\Delta t}{h^2} \left\{ u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n+1)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n+1)} \right\} + \left( \frac{h^2}{\Delta t} - 4 \right) u_{i,j}^{(n)} + h^2 f_{i,j}^{(n)}$$

and than this equation can be write

$$\begin{aligned} & -ru_{i-1,j}^{(n+1)} + \left( \frac{1}{2} + 2r \right) u_{i,j}^{(n+1)} - ru_{i+1,j}^{(n+1)} - ru_{i,j-1}^{(n+1)} + \left( \frac{1}{2} + 2r \right) u_{i,j}^{(n+1)} - ru_{i,j+1}^{(n+1)} \\ & = u_{i,j}^{(n)} + h^2 f_{i,j}^{(n)} \end{aligned} \tag{6}$$

and equation (6) can be written matrix form as, let  $\Psi(i) = \Psi(j) = \Psi$

$$\begin{pmatrix} -r\Psi & \frac{1}{2} + 2r & -r & 0 & \dots & u_{0,j}^{(n+1)} \\ 0 & -r & \frac{1}{2} + 2r & -r & 0 & u_{1,j}^{(n+1)} \\ \vdots & & & & & \vdots \\ & & & \ddots & & \\ 0 & \dots & -r & \frac{1}{2} + 2r & -r\Psi & u_{N,j}^{(n+1)} \end{pmatrix} + \begin{pmatrix} -r\Psi & \frac{1}{2} + 2r & -r & 0 & \dots & u_{0,j}^{(n+1)} \\ 0 & -r & \frac{1}{2} + 2r & -r & 0 & u_{1,j}^{(n+1)} \\ \vdots & & & & & \vdots \\ & & & \ddots & & \\ 0 & \dots & -r & \frac{1}{2} + 2r & -r\Psi & u_{N,j}^{(n+1)} \end{pmatrix} = u_{i,j}^{(n)} + h^2 f_{i,j}^{(n)}$$

$$Au_i^{n+1} + Au_j^{n+1} = u_{i,j}^{(n)} + h^2 f_{i,j}^{(n)}$$

$$A[u_i^{n+1} + u_j^{n+1}] = u_{i,j}^{(n)} + h^2 f_{i,j}^{(n)} \tag{7}$$

and with boundary conditions

$$u(0, y, t) = u(\pi, y, t) = \Psi$$

$$u(x, 0, t) = u(x, \pi, t) = \Psi$$

equation (7) can be written as

$$[u_i^{n+1} + u_j^{n+1}] = A^{-1}u_{i,j}^{(n)} + A^{-1}h^2 f_{i,j}^{(n)}$$

from superposition principle equation (7) is equivalent to the (8) equation

$$u_i^{n+1} + u_j^{n+1} = u_{i,j}^{(n+1)} \tag{8}$$

Computationally, the implicit method defined by (8) can now solved by the following iterative scheme. At time  $t = t_{n+1}$  :

Step 1: Solve the problem in the  $x$ -direction for each fixed  $y_j$  to obtain an intermediate solution  $u_{i,j}^{n+\frac{1}{2}}$ .

Step 2: Then solve it in the  $y$ -direction for each fixed  $x_i$ .

The initial and boundary conditions for numerical solution  $u_{i,j}^{n+1}$  and  $u_{i,j}^n$  are defined from the given initial and boundary conditions.

5. STABILITY OF METHOD

**Theorem 2.** (Gerschgorin's Theorem) *The largest eigenvalues of the square matrix A module does not exceed the total of any row or column modules for any of the terms.*

*Proof.* Let  $\lambda_i$  be an eigenvalue of the  $N \times N$  matrix  $A$ , and  $x_i$  the corresponding eigenvector with components  $\nu_1, \nu_2, \dots, \nu_s$ . Then the equation

$$Ax_i = \lambda x_i$$

in detail, is

$$\begin{aligned} a_{1,1}v_1 + a_{1,2}v_2 + \cdots + a_{1,n}v_n &= \lambda_i v_1 \\ a_{2,1}v_1 + a_{2,2}v_2 + \cdots + a_{2,n}v_n &= \lambda_i v_2 \\ &\vdots \\ a_{s,1}v_1 + a_{s,2}v_2 + \cdots + a_{s,n}v_n &= \lambda_i v_s \\ &\vdots \\ a_{n,1}v_1 + a_{n,2}v_2 + \cdots + a_{n,n}v_n &= \lambda_i v_n \end{aligned}$$

Let  $v_s$  be largest in modulus of  $v_1, v_2, \dots, v_n$ . Select the  $s$ th equation and divide by  $v_s$ , giving

$$\lambda_i = a_{s,1}\left(\frac{v_1}{v_s}\right) + a_{s,2}\left(\frac{v_2}{v_s}\right) + \cdots + a_{s,n}\left(\frac{v_n}{v_s}\right)$$

therefore  $\left|\frac{v_i}{v_s}\right| \leq 1$   $i = 1, 2, \dots, n$ .  $\square$

**Theorem 3.** (*Brauer's Theorem*) Let  $P_s$  be sum of the moduli of the terms along the  $s$ th row excluding the diagonal element  $a_{s,s}$ . Then every eigenvalue of  $A$  lies inside or on the boundary of at least one of the circles.

$$|\lambda - a_{s,s}| = P_s.$$

*Proof.* The proof of the Gerschgorin's theorem

$$\lambda_i = a_{s,1}\left(\frac{v_1}{v_s}\right) + a_{s,2}\left(\frac{v_2}{v_s}\right) + \cdots + a_{s,s} + a_{s,n}\left(\frac{v_n}{v_s}\right)$$

hence

$$\begin{aligned} |\lambda_i - a_{s,s}| &= \left| a_{s,1}\left(\frac{v_1}{v_s}\right) + a_{s,2}\left(\frac{v_2}{v_s}\right) + \cdots + 0 + a_{s,n}\left(\frac{v_n}{v_s}\right) \right| \\ |\lambda_i - a_{s,s}| &\leq |a_{s,1}| + |a_{s,2}| + \dots + 0 + \dots + |a_{s,n}| \\ |\lambda_i - a_{s,s}| &\leq P_s \end{aligned}$$

this completes the proof.  $\square$

Application of Brauer's theorem to this  $A$  matrix with  $a_{s,s} = -r\Psi$  and  $P_s = 2r$  shows that its eigenvalues  $\lambda$  lie on or within the circle  $|\lambda - \Psi| \leq P_s$

using Fig.1.  $\lambda_1 = -r(\Psi + 2)$  and  $\lambda_2 = r(\Psi + 2)$  and for stability  $|\lambda_1| \leq 1$  and  $|\lambda_2| \leq 1$ .

For  $|\lambda_1| \leq 1$

$$-1 \leq r(\Psi + 2) \leq 1 \Rightarrow r \leq \frac{1}{(\Psi + 2)}$$

For  $|\lambda_2| \leq 1$

$$r \leq \frac{1}{(\Psi + 2)}$$

For overall stability  $r \leq \frac{1}{(\Psi + 2)}$ .

The finite difference equations will be stable when the modulus of every eigenvalue of  $A^{-1}$  does not exceed one, that is when

$$\left| \frac{1}{\lambda} \right| \leq 1 \Rightarrow -\lambda \leq 1 \leq \lambda$$

$$\lambda \geq 1$$

proving that the equations are unconditionally stable as  $\lambda \geq 1$  for all values of  $r$ .

## 6. NUMERICAL EXAMPLES

If we consider the advection-dispersion equation (1)-(4) with initial conditions

$$u(x, y, 0) = \text{Si } nx \cdot \text{Si } ny \cdot (1 - x)^2 \cdot (1 - y)^2, x, y \in [0, \pi] \times [0, \pi]$$

and forcing function

$$f(x, y, t) = -(1 + 2xy) e^{-t} x^3 y^{3.6}$$

and Dirichlet boundary conditions on  $[0, \pi] \times [0, \pi]$  in the form

$$u(0, y, t) = u(\pi, y, t) = 0$$

$$u(x, 0, t) = u(x, \pi, t) = 0$$

for all  $t \geq 0$ .

The exact solution to this two-dimensional advection-dispersion equation is where

$$u(x, y, t) = e^t \text{Si } nx \cdot \text{Si } ny \cdot (1 - x)^2 \cdot (1 - y)^2$$

Figure 1 shows exact solutions and numerical solutions with  $\Delta t = \frac{1}{1000}$ ,  $h = 50$  for  $t \in [0, 1]$ .

From Figure 1, it can be seen that the numerical results are in good agreement with theoretical results. Specifically, for homogeneous Dirichlet boundary conditions, we have

$$\begin{aligned} u_{0,j}^{n+1} &= u(0, y_j, t_{n+1}) = 0, u_{m,j}^{n+1} = u(\pi, y_j, t_{n+1}) = 0 \\ u_{i,0}^{n+1} &= u(x_i, 0, t_{n+1}) = 0, u_{i,m}^{n+1} = u(x_i, \pi, t_{n+1}) = 0. \end{aligned}$$

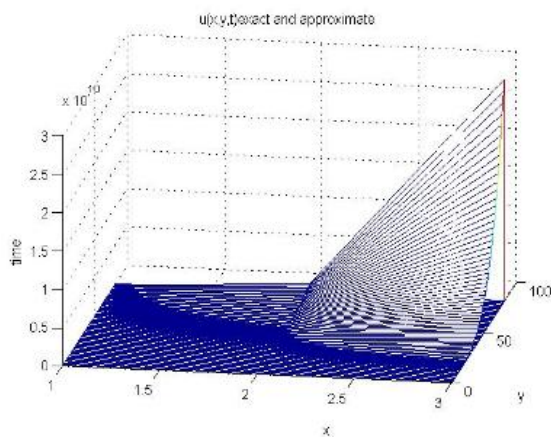


FIGURE 1. Exact solutions and numerical solutions with  $\Delta t = \frac{1}{1000}$ ,  $h = 50$  for  $t \in [0, 1]$ .

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