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GENERALIZED FRACTIONAL HERMITE-HADAMARD TYPE INEQUALITIES FOR $m-$CONVEX AND $(\alpha, m)-$CONVEX FUNCTIONS

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Abstract. In the present article, we derive some new inequalities of Hermite-Hadamard type involving left-sided and right-sided generalized fractional integral operators for products of two m-convex and (α, m)- convex functions, respectively. It is worth mentioning that the presented results have close connection with those in [6]. These new results generalize the existing Hermite-Hadamard type inequalities for products of two functions. Therefore the ideas of this article may stimulate further research in this field.

1. Introduction and Preliminaries

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g.,[11, p.137],[7]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The inequality (1.1) has evoked the interest of many mathematicians. Especially in the last three decades numerous generalizations, variants and extensions of this inequality have been obtained, to mention a few, see ([2, 3, 4, 5, 7, 8, 10, 11, 13, 14]) and the references cited therein.

$m-$ convexity was defined by Toader as follows:

Definition 1. (see [17]) *The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be $m-$convex, where $m \in [0, 1]$, if we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

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for all $x, y \in [0, b]$ *and* $t \in [0, 1]$.

One says that f is $m-$ concave if $(-f)$ is $m-$ convex. Denote by $K_m(b)$ the class of all $m-$ convex functions on $[0, b]$ for which $f(0) \leq 0$.

Obviously, for $m = 1$, Definition 1 recaptures concept of standard convex functions on $[0, b]$ and for $m = 0$ the concept of starshaped functions. The notion of $m-$ convexity has been further generalized in [9] as it is stated in the following definition.

Definition 2. (see [9]) *The function* $f : [0, b] \to \mathbb{R}$*,* $b > 0$*, is said to be* $(\alpha, m)-$ *convex, where* $(\alpha, m) \in [0, 1]^2$*, if one has*

$$f(tx + m(1 - t)y) \leq t^{\alpha} f(x) + m(1 - t^{\alpha})f(y)$$

for all $x, y \in [0, b]$ *and* $t \in [0, 1]$.

Denote by $K_m^{\alpha}(b)$ the class of $(\alpha, m)-$ convex functions on $[0, b]$ for which $f(0) \leq 0$.

It can be easily seen that when $(\alpha, m) \in \{(1, 1), (1, m)\}$ one obtains the following classes of functions: convex and $m-$ convex, respectively. Note that $K_1^1(b)$ is proper subclass of $m-$ convex and $(\alpha, m)-$ functions on $[0, b]$. The interested reader can find more about partial ordering of convexity in [11].

We recall some necessary definitions and preliminary results which are used and referred to throughout this paper as follows:

Definition 3. *Let* $f \in L_1[a, b]$*. The Riemann-Liouville integrals* $J_{a+}^{\alpha} f$ *and* $J_{b-}^{\alpha} f$ *of order* $\alpha > 0$ *with* $a \geq 0$ *are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha - 1} f(t) dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha - 1} f(t) dt, \quad x < b$$

respectively where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} u^{\alpha - 1} du$*. Here is* $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

Some Hermite-Hadamard type inequalities for products of two functions are proposed by Chen in [6] as follows:

Theorem 1. *Let* $f, g : [0, \infty) \to [0, \infty)$*,* $0 \leq a < b$*, be functions such that* $fg \in L_1[a, b]$*. If* f *is* m_1- *convex and* g *is* m_2- *convex on* $[a, b]$ *with* $m_1, m_2 \in (0, 1]$*, then one has*

$$\begin{aligned} & \frac{\Gamma(\alpha)}{(b - a)^{\alpha}} J_{a+}^{\alpha} f(b) g(b) \\ \leq \; & \frac{f(a)g(a)}{\alpha + 2} + \frac{m_2}{(\alpha + 1)(\alpha + 2)} f(a) g\left(\frac{b}{m_2}\right) \\ & + \frac{m_1}{(\alpha + 1)(\alpha + 2)} g(a) f\left(\frac{b}{m_1}\right) + \frac{2 m_1 m_2}{\alpha(\alpha + 1)(\alpha + 2)} f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned}&\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}J^{\alpha}_{b^-}f(a)g(a)\\ \leq\ &\frac{f(b)g(b)}{\alpha+2}+\frac{m_2}{(\alpha+1)(\alpha+2)}f(b)g\left(\frac{a}{m_2}\right)\\ &+\frac{m_1}{(\alpha+1)(\alpha+2)}g(b)f\left(\frac{b}{m_1}\right)+\frac{2m_1m_2}{\alpha(\alpha+1)(\alpha+2)}f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right).\end{aligned} \tag{1.3}$$

Theorem 2. *Let $f,g:[0,\infty)\to[0,\infty)$, $0\leq a<b$, be functions such that $fg\in L_1[a,b]$. If f is $(\alpha_1,m_1)-$ convex and g is $(\alpha_2,m_2)-$ convex on $[a,b]$ with $(\alpha_1,m_1),(\alpha_2,m_2)\in(0,1]^2$, respectively, then one has*

$$\begin{aligned}&\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}J^{\alpha}_{a^+}f(b)g(b)\\ \leq\ &\frac{1}{\alpha_1+\alpha_2+\alpha}f(a)g(a)+\frac{\alpha_2}{(\alpha+\alpha_1)(\alpha+\alpha_1+\alpha_2)}m_2f(a)g\left(\frac{b}{m_2}\right)\\ &+\frac{\alpha_1}{(\alpha+\alpha_2)(\alpha+\alpha_1+\alpha_2)}m_1g(a)f\left(\frac{b}{m_1}\right)\\ &+\left(\frac{1}{\alpha}-\frac{1}{\alpha+\alpha_1}-\frac{1}{\alpha+\alpha_2}+\frac{1}{\alpha+\alpha_1+\alpha_2}\right)m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right),\end{aligned} \tag{1.4}$$

and

$$\begin{aligned}&\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}J^{\alpha}_{b^-}f(a)g(a)\\ \leq\ &\frac{1}{\alpha_1+\alpha_2+\alpha}f(b)g(b)+\frac{\alpha_2}{(\alpha+\alpha_1)(\alpha+\alpha_1+\alpha_2)}m_2f(b)g\left(\frac{a}{m_2}\right)\\ &+\frac{\alpha_1}{(\alpha+\alpha_2)(\alpha+\alpha_1+\alpha_2)}m_1g(b)f\left(\frac{a}{m_1}\right)\\ &+\left(\frac{1}{\alpha}-\frac{1}{\alpha+\alpha_1}-\frac{1}{\alpha+\alpha_2}+\frac{1}{\alpha+\alpha_1+\alpha_2}\right)m_1m_2f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right).\end{aligned} \tag{1.5}$$

In [12], Raina introduced a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x)=\mathcal{F}^{\sigma(0),\sigma(1),\dots}_{\rho,\lambda}(x)=\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k \quad (\rho,\lambda>0;|x|<\mathbf{R}), \tag{1.6}$$

where the coefficients $\sigma(k)\,(k\in\mathbb{N}=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers and $\mathbf{R}$ is the set of real numbers. With the help of (1.6), Raina [12] and Agarwal *et al.* [1] defined the following left-sided and right-sided fractional integral operators respectively, as follows:

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right)(x)=\int_a^x(x-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(x-t)^{\rho}]\varphi(t)dt \qquad (x>a>0), \tag{1.7}$$

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right)(x)=\int_x^b (t-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\varphi(t)dt \qquad (0<x<b), \tag{1.8}$$

where $\lambda,\rho>0$, $w\in\mathbb{R}$ and $\varphi(t)$ is such that the integral on the right side exits.

It is easy to verify that $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi(x)$ and $\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi(x)$ are bounded integral operators on $L(a,b)$, if

$$\mathfrak{M}:=\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]<\infty. \tag{1.9}$$

In fact, for $\varphi\in L(a,b)$, we have

$$||\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi(x)||_1\le \mathfrak{M}(b-a)^{\lambda}||\varphi||_1 \tag{1.10}$$

and

$$||\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi(x)||_1\le \mathfrak{M}(b-a)^{\lambda}||\varphi||_1 \tag{1.11}$$

where

$$||\varphi||_p:=\left(\int_a^b |\varphi(t)|^p dt\right)^{\frac{1}{p}}.$$

Here, many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. For instance the classical Riemann-Liouville fractional integrals J^{α}_{a+} and J^{α}_{b-} of order α follow easily by setting $\lambda=\alpha$, $\sigma(0)=1$ and $w=0$ in (1.7) and (1.8). Some recent results and properties concerning the fractional integral operators can be found [15, 16, 18, 19].

In this paper, some new Hermite-Hadamard type inequalities for products of two different convex functions via generalized fractional integral operator are obtained.

2. Inequalities for product of m-convex and (α,m)-convex functions

Theorem 3. *Let $f,g:[0,\infty)\to[0,\infty)$, $0\le a<b$, be functions such that $fg\in L_1[a,b]$. If f is m_1- convex and g is m_2- convex on $[a,b]$ with $m_1,m_2\in(0,1]$, then one has*

$$\begin{aligned}
&\frac{1}{(b-a)^{\alpha}}(\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w})(fg(b)) \qquad (2.1)\\
\le\ & f(a)g(a)\mathcal{F}^{\sigma_1}_{\rho,\alpha}[w(b-a)^{\rho}]+f(a)g\left(\frac{b}{m_2}\right)\mathcal{F}^{\sigma_2}_{\rho,\alpha}[w(b-a)^{\rho}]\\
&+g(a)f\left(\frac{b}{m_1}\right)\mathcal{F}^{\sigma_3}_{\rho,\alpha}[w(b-a)^{\rho}]+f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\mathcal{F}^{\sigma_4}_{\rho,\alpha+1}[w(b-a)^{\rho}]
\end{aligned}$$

and

$$\begin{aligned} & \frac{1}{(b-a)^{\alpha}}(\mathcal{J}^{\sigma}_{\rho,\alpha,b-;w})(fg(a)) \qquad (2.2) \\ \leq\ & f(b)g(b)\mathcal{F}^{\sigma_1}_{\rho,\alpha}\left[w(b-a)^{\rho}\right] + f(b)g\left(\frac{a}{m_2}\right)\mathcal{F}^{\sigma_2}_{\rho,\alpha}\left[w(b-a)^{\rho}\right] \\ & + g(b)f\left(\frac{a}{m_1}\right)\mathcal{F}^{\sigma_3}_{\rho,\alpha}\left[w(b-a)^{\rho}\right] + f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\mathcal{F}^{\sigma_4}_{\rho,\alpha+1}\left[w(b-a)^{\rho}\right], \end{aligned}$$

where $\alpha > 0$ *and*

$$\sigma_1(k) := \sigma(k)\frac{1}{\alpha+\rho k+2}, \qquad \sigma_2(k) := \sigma(k)\frac{m_2}{(\alpha+\rho k+1)(\alpha+\rho k+2)},$$
$$\sigma_3(k) := \sigma(k)\frac{m_1}{(\alpha+\rho k+1)(\alpha+\rho k+2)}, \qquad \sigma_4(k) := \sigma(k)\frac{2m_1m_2}{(\alpha+\rho k+1)(\alpha+\rho k+2)}.$$

Proof. By using the definitions of f and g, we can write

$$f(ta+(1-t)b) \leq tf(a) + m_1(1-t)f\left(\frac{b}{m_1}\right) \qquad (2.3)$$

and

$$g(ta+(1-t)b) \leq tg(a) + m_2(1-t)g\left(\frac{b}{m_2}\right). \qquad (2.4)$$

By multiplying (2.3) and (2.4), we get

$$\begin{aligned} & f(ta+(1-t)b)g(ta+(1-t)b) \qquad (2.5) \\ \leq\ & t^2f(a)g(a) + m_2f(a)g\left(\frac{b}{m_2}\right)t(1-t) \\ & + m_1g(a)f\left(\frac{b}{m_1}\right)t(1-t) + m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)(1-t)^2. \end{aligned}$$

If we multiply both sides of (2.5) by $t^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]$, then integrating with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]f(ta+(1-t)b)g(ta+(1-t)b)dt\\
=\ &\int_b^a \left(\frac{b-u}{b-a}\right)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}\left[w(b-u)^{\rho}\right]f(u)g(u)\frac{du}{a-b}\\
=\ &\frac{1}{(b-a)^{\alpha}}(\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w})(fg(b))\\
\leq\ &f(a)g(a)\int_0^1 t^{\alpha+1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]dt\\
&+m_2f(a)g\left(\frac{b}{m_2}\right)\int_0^1 t^{\alpha}(1-t)\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]dt\\
&+m_1g(a)f\left(\frac{b}{m_1}\right)\int_0^1 t^{\alpha}(1-t)\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]dt\\
&+m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\int_0^1 t^{\alpha-1}(1-t)^2\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]dt\\
=\ &f(a)g(a)\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\alpha+\rho k)}\int_0^1 t^{\alpha+\rho k+1}dt\\
&+m_2f(a)g\left(\frac{b}{m_2}\right)\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\alpha+\rho k)}\int_0^1 t^{\alpha+\rho k}(1-t)dt\\
&+m_1g(a)f\left(\frac{b}{m_1}\right)\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\alpha+\rho k)}\int_0^1 t^{\alpha+\rho k}(1-t)dt\\
&+m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\alpha+\rho k)}\int_0^1 t^{\alpha+\rho k-1}(1-t)^2dt\\
=\ &f(a)g(a)\mathcal{F}^{\sigma_1}_{\rho,\alpha}\left[w(b-a)^{\rho}\right]+f(a)g\left(\frac{b}{m_2}\right)\mathcal{F}^{\sigma_2}_{\rho,\alpha}\left[w(b-a)^{\rho}\right]\\
&+g(a)f\left(\frac{b}{m_1}\right)\mathcal{F}^{\sigma_3}_{\rho,\alpha}\left[w(b-a)^{\rho}\right]+f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\mathcal{F}^{\sigma_4}_{\rho,\alpha+1}\left[w(b-a)^{\rho}\right].
\end{aligned}$$

Analogously, we obtain

$$\begin{aligned}
&f((1-t)a+tb)g((1-t)a+tb)\\
\leq\ &t^2f(b)g(b)+m_2f(b)g\left(\frac{a}{m_2}\right)t(1-t)\\
&+m_1g(b)f\left(\frac{a}{m_1}\right)t(1-t)+m_1m_2f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)(1-t)^2.
\end{aligned}\tag{2.6}$$

If we multiply both sides of (2.6) by $t^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]$, then integrating with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
& \int_0^1 t^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]f((1-t)a+tb)g((1-t)a+tb)dt \\
= & \int_a^b \left(\frac{v-a}{b-a}\right)^{\alpha-1} \mathcal{F}^{\sigma}_{\rho,\alpha}\left[w(v-a)^{\rho}\right] f(v)g(v)\frac{dv}{b-a} \\
= & \frac{1}{(b-a)^{\alpha}}(\mathcal{J}^{\sigma}_{\rho,\alpha,b-;w})(fg(a)) \\
\leq & f(b)g(b)\int_0^1 t^{\alpha+1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]dt \\
& +m_2 f(b)g\left(\frac{a}{m_2}\right)\int_0^1 t^{\alpha}(1-t)\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]dt \\
& +m_1 g(b)f\left(\frac{a}{m_1}\right)\int_0^1 t^{\alpha}(1-t)\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]dt \\
& +m_1 m_2 f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\int_0^1 t^{\alpha-1}(1-t)^2\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]dt \\
= & f(b)g(b)\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\alpha+\rho k)}\int_0^1 t^{\alpha+\rho k+1}dt \\
& +m_2 f(b)g\left(\frac{a}{m_2}\right)\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\alpha+\rho k)}\int_0^1 t^{\alpha+\rho k}(1-t)dt \\
& +m_1 g(b)f\left(\frac{a}{m_1}\right)\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\alpha+\rho k)}\int_0^1 t^{\alpha+\rho k}(1-t)dt \\
& +m_1 m_2 f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\alpha+\rho k)}\int_0^1 t^{\alpha+\rho k-1}(1-t)^2dt \\
= & f(b)g(b)\mathcal{F}^{\sigma_1}_{\rho,\alpha}\left[w(b-a)^{\rho}\right]+f(b)g\left(\frac{a}{m_2}\right)\mathcal{F}^{\sigma_2}_{\rho,\alpha}\left[w(b-a)^{\rho}\right] \\
& +g(b)f\left(\frac{a}{m_1}\right)\mathcal{F}^{\sigma_3}_{\rho,\alpha}\left[w(b-a)^{\rho}\right]+f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\mathcal{F}^{\sigma_4}_{\rho,\alpha+1}\left[w(b-a)^{\rho}\right].
\end{aligned}$$

Here, we used the facts that

$$\int_0^1 t^{\alpha+\rho k+1}dt = \frac{1}{\alpha+\rho k+2},$$

$$\int_0^1 t^{\alpha+\rho k}(1-t)dt = \frac{1}{(\alpha+\rho k+1)(\alpha+\rho k+2)},$$

$$\int_0^1 t^{\alpha+\rho k-1}(1-t)^2 dt = \frac{2}{(\alpha+\rho k)(\alpha+\rho k+1)(\alpha+\rho k+2)}.$$

This completes the proof. □

Remark 1. *If we take $\sigma(0)=1$ and $w=0$ in the Theorem 3, then the inequalities* (2.1) *and* (2.2) *reduces to the inequalities* (1.2) *and* (1.3)*, respectively.*

Theorem 4. *Let $f,g:[0,\infty)\to[0,\infty)$, $0\le a<b$, be functions such that $fg\in L_1[a,b]$. If f is $(\alpha_1,m_1)-$ convex and g is $(\alpha_2,m_2)-$ convex on $[a,b]$ with $(\alpha_1,m_1),(\alpha_2,m_2)\in(0,1]^2$, respectively, then one has*

$$\begin{aligned}&\frac{1}{(b-a)^\alpha}(\mathcal{J}^\sigma_{\rho,\alpha,a+;w})(fg(b)) \qquad (2.7)\\ \le\;& f(a)g(a)\mathcal{F}^{\sigma_5}_{\rho,\alpha}\left[w(b-a)^\rho\right]+f(a)g\left(\frac{b}{m_2}\right)\mathcal{F}^{\sigma_6}_{\rho,\alpha}\left[w(b-a)^\rho\right]\\ &+g(a)f\left(\frac{b}{m_1}\right)\mathcal{F}^{\sigma_7}_{\rho,\alpha}\left[w(b-a)^\rho\right]+f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\mathcal{F}^{\sigma_8}_{\rho,\alpha}\left[w(b-a)^\rho\right]\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{(b-a)^\alpha}(\mathcal{J}^\sigma_{\rho,\alpha,b-;w})(fg(a)) \qquad (2.8)\\ \le\;& f(b)g(b)\mathcal{F}^{\sigma_5}_{\rho,\alpha}\left[w(b-a)^\rho\right]+f(b)g\left(\frac{a}{m_2}\right)\mathcal{F}^{\sigma_6}_{\rho,\alpha}\left[w(b-a)^\rho\right]\\ &+g(b)f\left(\frac{a}{m_1}\right)\mathcal{F}^{\sigma_7}_{\rho,\alpha}\left[w(b-a)^\rho\right]+f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\mathcal{F}^{\sigma_8}_{\rho,\alpha}\left[w(b-a)^\rho\right],\end{aligned}$$

where $\alpha>0$ and

$$\sigma_5(k):=\sigma(k)\frac{1}{\alpha_1+\alpha_2+\alpha+\rho k},$$

$$\sigma_6(k):=\sigma(k)\frac{\alpha_2 m_2}{(\alpha+\rho k+\alpha_1)(\alpha+\rho k+\alpha_1+\alpha_2)},$$

$$\sigma_7(k):=\sigma(k)\frac{\alpha_1 m_1}{(\alpha+\rho k+\alpha_2)(\alpha+\rho k+\alpha_1+\alpha_2)},$$

$$\sigma_8(k):=\sigma(k)\left(\frac{1}{\alpha+\rho k}-\frac{1}{\alpha+\rho k+\alpha_1}-\frac{1}{\alpha+\rho k+\alpha_2}+\frac{1}{\alpha+\rho k+\alpha_1+\alpha_2}\right)m_1m_2.$$

Proof. By using the definitions of f and g, we can write

$$f(ta+(1-t)b)\le t^{\alpha_1}f(a)+m_1(1-t^{\alpha_1})f\left(\frac{b}{m_1}\right) \qquad (2.9)$$

and

$$g(ta+(1-t)b)\le t^{\alpha_2}g(a)+m_2(1-t^{\alpha_2})g\left(\frac{b}{m_2}\right). \qquad (2.10)$$

By multiplying (2.9) and (2.10), we get

$$\begin{aligned} & f(ta+(1-t)b)g(ta+(1-t)b) \\ \leq\ & t^{\alpha_1+\alpha_2}f(a)g(a)+m_2f(a)g\left(\frac{b}{m_2}\right)t^{\alpha_1}(1-t^{\alpha_2}) \\ & +m_1g(a)f\left(\frac{b}{m_1}\right)t^{\alpha_2}(1-t^{\alpha_1}) \\ & +m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)(1-t^{\alpha_1})(1-t^{\alpha_2}). \end{aligned} \tag{2.11}$$

If we multiply both sides of (2.11) by $t^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]$, then integrating with respect to t over $[0,1]$, we obtain

$$\begin{aligned} & \int_0^1 t^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]f(ta+(1-t)b)g(ta+(1-t)b)dt \\ =\ & \int_b^a \left(\frac{b-u}{b-a}\right)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}\left[w(b-u)^{\rho}\right]f(u)g(u)\frac{du}{a-b} \\ =\ & \frac{1}{(b-a)^{\alpha}}(\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w})(fg(b)) \\ \leq\ & f(a)g(a)\int_0^1 t^{\alpha_1+\alpha_2+\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]dt \\ & +m_2f(a)g\left(\frac{b}{m_2}\right)\int_0^1 t^{\alpha-1}t^{\alpha_1}(1-t^{\alpha_2})\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]dt \\ & +m_1g(a)f\left(\frac{b}{m_1}\right)\int_0^1 t^{\alpha-1}t^{\alpha_2}(1-t^{\alpha_1})\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]dt \\ & +m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\int_0^1 t^{\alpha-1}(1-t^{\alpha_1})(1-t^{\alpha_2})\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]dt \\ =\ & f(a)g(a)\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\alpha+\rho k)}\int_0^1 t^{\alpha_1+\alpha_2+\alpha+\rho k-1}dt \\ & +m_2f(a)g\left(\frac{b}{m_2}\right)\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\alpha+\rho k)}\int_0^1 t^{\alpha+\rho k-1}t^{\alpha_1}(1-t^{\alpha_2})dt \\ & +m_1g(a)f\left(\frac{b}{m_1}\right)\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\alpha+\rho k)}\int_0^1 t^{\alpha+\rho k-1}t^{\alpha_2}(1-t^{\alpha_1})dt \\ & +m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\alpha+\rho k)}\int_0^1 t^{\alpha+\rho k-1}(1-t^{\alpha_1})(1-t^{\alpha_2})dt \end{aligned}$$

$$
\begin{aligned}
= \quad & f(a)g(a)\mathcal{F}^{\sigma_5}_{\rho,\alpha}\left[w(b-a)^{\rho}\right] + f(a)g\left(\frac{b}{m_2}\right)\mathcal{F}^{\sigma_6}_{\rho,\alpha}\left[w(b-a)^{\rho}\right] \\
& + g(a)f\left(\frac{b}{m_1}\right)\mathcal{F}^{\sigma_7}_{\rho,\alpha}\left[w(b-a)^{\rho}\right] + f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\mathcal{F}^{\sigma_8}_{\rho,\alpha}\left[w(b-a)^{\rho}\right].
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& f((1-t)a+tb)g((1-t)a+tb) \\
\leq \quad & t^{\alpha_1+\alpha_2}f(b)g(b) + m_2 f(b)g\left(\frac{a}{m_2}\right)t^{\alpha_1}(1-t^{\alpha_2}) \\
& + m_1 g(b)f\left(\frac{a}{m_1}\right)t^{\alpha_2}(1-t^{\alpha_1}) \\
& + m_1 m_2 f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)(1-t^{\alpha_1})(1-t^{\alpha_2}). \qquad (2.12)
\end{aligned}
$$

If we multiply both sides of (2.12) by $t^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]$, then integrating with respect to t over $[0,1]$, we obtain

$$
\begin{aligned}
& \int_0^1 t^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]f((1-t)a+tb)g((1-t)a+tb)dt \\
& = \int_a^b \left(\frac{v-a}{b-a}\right)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}\left[w(v-a)^{\rho}\right]f(v)g(v)\frac{dv}{b-a} \\
& = \frac{1}{(b-a)^{\alpha}}(\mathcal{J}^{\sigma}_{\rho,\alpha,b-;w})(fg(a)) \\
& \leq f(b)g(b)\int_0^1 t^{\alpha_1+\alpha_2+\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]dt \\
& \quad + m_2 f(b)g\left(\frac{a}{m_2}\right)\int_0^1 t^{\alpha-1}t^{\alpha_1}(1-t^{\alpha_2})\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]dt \\
& \quad + m_1 g(b)f\left(\frac{a}{m_1}\right)\int_0^1 t^{\alpha-1}t^{\alpha_2}(1-t^{\alpha_1})\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]dt \\
& \quad + m_1 m_2 f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\int_0^1 t^{\alpha-1}(1-t^{\alpha_1})(1-t^{\alpha_2})\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}t^{\rho}]dt
\end{aligned}
$$

$$\begin{aligned}
&= f(b)g(b)\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\alpha+\rho k)}\int_0^1 t^{\alpha_1+\alpha_2+\alpha+\rho k-1}dt \\
&\quad + m_2 f(b)g\left(\frac{a}{m_2}\right)\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\alpha+\rho k)}\int_0^1 t^{\alpha+\rho k-1}t^{\alpha_1}(1-t^{\alpha_2})dt \\
&\quad + m_1 g(b)f\left(\frac{a}{m_1}\right)\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\alpha+\rho k)}\int_0^1 t^{\alpha+\rho k-1}t^{\alpha_2}(1-t^{\alpha_1})dt \\
&\quad + m_1 m_2 f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\alpha+\rho k)}\int_0^1 t^{\alpha+\rho k-1}(1-t^{\alpha_1})(1-t^{\alpha_2})dt \\
&= f(b)g(b)\mathcal{F}^{\sigma_5}_{\rho,\alpha}\left[w(b-a)^{\rho}\right] + f(b)g\left(\frac{a}{m_2}\right)\mathcal{F}^{\sigma_6}_{\rho,\alpha}\left[w(b-a)^{\rho}\right] \\
&\quad + g(b)f\left(\frac{a}{m_1}\right)\mathcal{F}^{\sigma_7}_{\rho,\alpha}\left[w(b-a)^{\rho}\right] + f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\mathcal{F}^{\sigma_8}_{\rho,\alpha}\left[w(b-a)^{\rho}\right].
\end{aligned}$$

Here, we used the facts that

$$\begin{aligned}
\int_0^1 t^{\alpha_1+\alpha_2+\alpha+\rho k-1}dt &= \frac{1}{\alpha_1+\alpha_2+\alpha+\rho k}, \\
\int_0^1 t^{\alpha+\rho k-1}t^{\alpha_1}(1-t^{\alpha_2})dt &= \frac{\alpha_2}{(\alpha+\rho k+\alpha_1)(\alpha+\rho k+\alpha_1+\alpha_2)}, \\
\int_0^1 t^{\alpha+\rho k-1}t^{\alpha_2}(1-t^{\alpha_1})dt &= \frac{\alpha_1}{(\alpha+\rho k+\alpha_2)(\alpha+\rho k+\alpha_1+\alpha_2)}, \\
\int_0^1 t^{\alpha+\rho k-1}(1-t^{\alpha_1})(1-t^{\alpha_2})dt &= \frac{1}{\alpha+\rho k} - \frac{1}{\alpha+\rho k+\alpha_1} \\
&\quad - \frac{1}{\alpha+\rho k+\alpha_2} + \frac{1}{\alpha+\rho k+\alpha_1+\alpha_2}.
\end{aligned}$$

This completes the proof. □

Remark 2. *If we take* $\sigma(0) = 1$ *and* $w = 0$ *in the Theorem 4, then the inequalities* (2.7) *and* (2.8) *reduces to the inequalities* (1.4) *and* (1.5)*, respectively.*

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