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CH-TRANSFORMATIONS ON THE TANGENT BUNDLE OF A RIEMANNIAN MANIFOLD

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Abstract. In this paper we study a conformal-holomorphic change of the complete lift of a Riemannian metric, and we prove a necessary and sufficient condition for the existence of such a metric. We show that the complete lift of the curvature tensor on the base in the tangent bundle is CH-invariant under a conformal-holomorphic change.

We suppose that M is a Riemannian (pseudo-Riemannian) manifold with a metric g, and that TM is its tangent bundle with a metric g^c, which is the complete lift of g. An integrable nilpotent tensor field f of type $(1,1)$ arises on TM. If $(x^1, x^2, \ldots, x^n)$ are local coordinates in $U \subset M$ and $(x^1, x^2, \ldots, x^n;\ y^1, y^2, \ldots, y^n)$ are the corresponding ones on TM, the matrix of f is

$$f : \left(\begin{array}{c|c} 0 & 0 \\ \hline E & 0 \end{array} \right),$$

where E is the unit matrix of order n.

The vertical lift g^v and g^c are connected by f in the following way

$$g^v(x,y) = g^c(fx,y) = g^c(x,fy)$$

(1)
$$g^v_{\alpha\beta} = g^c_{\sigma\beta} f^\sigma_\alpha = g^c_{\alpha\sigma} f^\sigma_\beta ,$$

where

$$(g^c)_{ik} = y^h \frac{\partial}{\partial x^h} g_{ik},\ (g^c)_{n+i,k} = g_{ik},\ (g^c)_{n+i,n+k} = 0;\ i,j = 1,2,\ldots,n\,.$$

Definition 1. The functions $u = u(x,y)$, $v = v(x,y)$ define an ordered holomorphic pair (u,v) if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,,\quad \frac{\partial u}{\partial y} = 0\,.$$

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By this definition the ordered pairs $(g^v_{\alpha\beta}, g^c_{\alpha\beta})$, $\alpha, \beta = 1, 2, \ldots, 2n$, are holomorphic with respect to the variables (x^i, y^i).

For any point p on TM, the pair of tangent vectors

$$(v, \tilde{v})_p := (v, fv)_p, \;\; \tilde{v} = fv$$

defines a holomorphic plane through the point p. The holomorphic plane is called complete if the rank of the matrix defined by the coordinates of v and $\tilde{v}$ is complete (equal to 2).

Two holomorphic planes define a pair of stationary angles between them. The squares of the cosines of these angles are the eigenvalues of the matrix

$$W = (A^T A)^{-1}.(A^T B).(B^T B)^{-1}.(B^T A) \; ,$$

where

$$A^T A = \begin{pmatrix} 1 & x\tilde{x} \\ x\tilde{x} & 0 \end{pmatrix} , \; A^T B = \begin{pmatrix} xy & x\tilde{y} \\ \tilde{x}y & 0 \end{pmatrix}, \quad xy := g^c(x, y)$$

(see [1]). By the notation introduced above,

$$W = \begin{pmatrix} W_1 & 0 \\ W_2 & W_1 \end{pmatrix} ,$$

where

$$W_1 = \frac{(x\tilde{y})^2}{(x\tilde{x})^2.(y\tilde{y})^2} ,$$

$$W_2 = -\frac{(x\tilde{y})^2(y\tilde{y} + x\tilde{x})}{(x\tilde{x})^2.(y\tilde{y})^2} .$$

The eigenvalues of W coincide and $\cos^2\theta = W_1$, where θ is the stationary angle between the holomorphic planes $(x, \tilde{x})$ and $(y, \tilde{y})$.

A two-dimensional plane from TM defines a bivector. If U and V are bivectors determined by the holomorphic planes $(u, \tilde{u})_p$ and $(v, \tilde{v})_p$, then according to the theory developed by Grassman (see [2, p. 396], for example), the angle between U and V is

$$\cos(U, V) := \pm \cos\theta_1 \cos\theta_2 \; , \tag{2}$$

where $\cos^2\theta_1$ and $\cos^2\theta_2$ are the eigenvalues of the matrix W.

Theorem 1. *Two holomorphic planes, determined by vectors that are lifts of vector fields from the base, are orthogonal if and only if these vector fields are orthogonal.*

Proof. The equations that determine the orthogonality of the two planes are

$$g^c(u^c, v^c) = 0 \, , \quad g^c(u^c, v^v) = 0.$$

Then

$$g^c(u^v, v^c) = 0 \, , \quad g^c(u^v, v^v) = 0$$

are implied by (1) and the relations

$$f\, u^c = u^v \, , \quad f\, u^v = f\, v^v = 0 \, .$$

Since

$$g^c(u^c, v^c)_p = [g(u, v)]^c_p \;\text{ and }\; g^c(u^c, v^v)_p = [g(u, v)]^v_p$$

(see [4]), the proof is complete. □

Corollary 1. *The orthogonal vector fields*

$$e_1, e_2, \ldots, e_n,$$

which determine a basis in each point on the base M, generate a system of mutually orthogonal pairs

$$(e_1^c, e_1^v), \ldots, (e_n^c, e_n^v),$$

with respect to which g^c has a canonical form

$$g^c : \left[\begin{array}{ccc|ccc} \partial g_{11} & & & g_{11} & & \\ & \ddots & & & \ddots & \\ & & \partial g_{nn} & & & g_{nn} \\ \hline g_{11} & & & & & \\ & \ddots & & & & \\ & & g_{nn} & & & \end{array}\right],$$

where $\partial g_{ik} = y^s \frac{\partial}{\partial x^s} g_{ik}$. The blank spaces denote zeros.

We suppose that σ is a function on the base M. The conformal change of the metric $g \to \sigma g$ leads to a change of the metric on $TM : g^c \to (\sigma g)^c$. As

$$(\sigma g)^c = \sigma^c g^v + \sigma^v g^c \ ,$$

the change of the metric on TM is called a conformal-holomorphic change (CH-change) [3]. It has not been investigated thoroughly so far and therefore it is the main object of this paper.

Theorem 2. *The conformal-holomorphic change of the metric g^c preserves the angle between two bivectors.*

Proof. We assume that U and V are bivectors, initiated by $(u, \tilde{u})$ and $(v, \tilde{v})$, respectively. If σ is a function we have

$$(\sigma g)^c(u, v) = 0 \ ,$$

$$(\sigma g)^c(u, \tilde{v}) = (\sigma^c g^v + \sigma^v g^c)(u, \tilde{v}) = \sigma^v g^v(u, v) \, .$$

Hence, from (2) we get

$$\overline{\cos(U, V)} = \frac{[\sigma^v g^v(u, v)]^2}{\sigma^v g^v(u, u) \sigma^v g^v(v, v)} = \cos(U, V) \ . \tag{3}$$

Here $\overline{\cos(U, V)}$ is calculated by means of $(\sigma g)^c$. □

We note that the last theorem remains valid for an arbitrary $\tau, \nu : g^c \to \tau g^c + \nu g^v$, since in such a change g^v is replaced by τg^v.

Theorem 3. *If two metrics g^c and $\overline{g}^c$ preserve the angles between the bivectors, then they are conformal-holomorphic.*

Proof. We suppose that u and v are two random vector fields from the base M, and

$$U := (u^c, \tilde{u})\,,\ V := (v^c, \tilde{v})$$

are the corresponding bivectors. These metrics preserve the angle between U and V. Then from (3),

$$\frac{\overline{g}(u,v)}{[\overline{g}(u,u)\overline{g}(v,v)]^{\frac{1}{2}}} = \frac{g(u,v)}{[g(u,u)g(v,v)]^{\frac{1}{2}}}\ ,$$

which shows conformal equivalence of g and $\overline{g} : \overline{g} = \sigma g$, and thus we have

$$\overline{g}^c = \sigma^c g^v + \sigma^v g^c\ .$$

□

The condition (1) for purity of the metric g^c with respect to f plays an important role for the differential-geometrical properties of TM. For example, the tensor field f should be constant with respect to the connection generated by g^c.

Theorem 4. *The nilpotent structure f is covariant constant with respect to the Riemann connection generated by g^c if and only if the partial derivatives*

$$\frac{\partial}{\partial z^\alpha} g_{\beta\sigma} (z^\alpha = x^1, x^2, \dots x^n;\ y^1, y^2, \dots, y^n)$$

are pure with respect to f.

Proof. The proof is standard and we omit it. It is sufficient to use $\frac{\partial}{\partial z^\sigma} f^\alpha_\beta = 0$. □

Therefore there exist many connections on TM and f is covariant constant with respect to them.

Theorem 5. *If σ is a differentiable function on the base, then the pair*

$$((\sigma g_{\alpha\beta})^v, (\sigma g_{\alpha\beta})^c)$$

is holomorphic.

Proof. It is sufficient to consider the special case $\alpha = k$, $\beta = j$. From

$$\sigma^c = y^i \frac{\partial \sigma}{\partial x^i} \quad \text{and} \quad (g^c)_{kj} = y^i \frac{\partial}{\partial x^i} g_{kj}$$

we find that

$$\frac{\partial \sigma^c}{\partial y^k} = \frac{\partial \sigma}{\partial x^k} \quad \text{and} \quad \frac{\partial}{\partial y^i}(g^c)_{kj} = \frac{\partial}{\partial x^i}(g^v)_{kj}\ ,$$

and therefore

$$\begin{aligned} \frac{\partial}{\partial y^i}\,(\sigma g)^c_{kj} &= \frac{\partial}{\partial y^i}(\sigma^c g^v_{kj} + \sigma^v g^c_{kj}) = \frac{\partial}{\partial x^i}\sigma g^v_{kj} + \sigma^v \frac{\partial}{\partial x^i} g_{kj} \\ &= \frac{\partial}{\partial x^i}(\sigma g)_{kj} = \frac{\partial}{\partial x^i}(\sigma g)^v_{kj} \end{aligned}$$

and

$$\frac{\partial}{\partial y^i}(\sigma g)^v_{kj} = 0\,.$$

□

Theorem 6. *The purity of the partial derivatives of $g^c_{\alpha\beta}$ is a sufficient condition for the holomorphicity of the pairs $(g^v_{\alpha\beta}, g^c_{\alpha\beta})$.*

Proof. Suppose that $G : (G_{\alpha\beta})$ is a pure metric on TM, i.e. $G_{\sigma\beta}f^{\sigma}_{\alpha} = G_{\alpha\sigma}f^{\sigma}_{\beta}$. Then

$$G_{n+i,k} = G_{i,n+k}\,, \quad G_{n+i,n+k} = 0\,.$$

For the partial derivatives $\partial_\sigma G_{\alpha\beta}$ we also assume purity with respect to f:

$$f^{\lambda}_{\sigma}\partial_\lambda G_{\alpha\beta} = f^{\nu}_{\alpha}\partial_\sigma G_{\nu\beta}\,, \quad \partial_\lambda G_{\alpha\beta} = \frac{\partial}{\partial z^\lambda}G_{\alpha\beta}\,,$$

$$\alpha, \beta, \dots \ = 1, 2, \dots, 2n\,,$$

where z^ν is one of the variables

$$x_1, x_2, \dots, x_n;\ y_1, y_2, \dots, y_n\,.$$

The above condition is equivalent to

$$f^{n+k}_{\sigma}\partial_{n+k}G_{\alpha\beta} = f^{n+p}_{\alpha}\partial_\sigma G_{n+p,\beta}\ .$$

Taking into consideration that

$$f^{n+l}_{k} = \delta^{l}_{k}$$

for

$$(\sigma = s,\ \alpha = i,\ \beta = k)\ \text{ and }\ (\sigma = n+s,\ \alpha = i,\ \beta = k)\,,$$

we have

$$\frac{\partial}{\partial y^\sigma}G_{ik} = \frac{\partial}{\partial x^\sigma}G_{n+i,k}\,, \quad 0 = \frac{\partial}{\partial y^s}G_{n+i,k}\ .$$

Solutions to this system are the components of the metrics g^c and g^v. Indeed,

$$\frac{\partial}{\partial y^s}g^{c}_{i,k} = \frac{\partial}{\partial x^s}g^{v}_{i,k}\,, \quad 0 = \frac{\partial}{\partial y^s}g^{c}_{n+i,k} = \frac{\partial}{\partial y^s}g_{ik}$$

and

$$\frac{\partial}{\partial y^s}g^{v}_{i\alpha} = \frac{\partial}{\partial x^s}0 = 0\,, \quad 0 = \frac{\partial}{\partial y^s}g^{v}_{n+i,k}\ .$$

□

Theorem 7. *A necessary and sufficient condition the metric*

$$G_{\alpha\beta} = \lambda g^{c}_{\alpha\beta} + \mu g^{v}_{\alpha\beta}$$

to have pure partial derivatives with respect to f is holomorphicity of the pair functions (λ, μ).

Proof. With respect to the local coordinates $z^i = x^i, z^{n+j} = y^j$, the condition for purity of the partial derivatives

$$\frac{\partial}{\partial z^\sigma}G_{\alpha\beta}f^{\sigma}_{\nu} = \frac{\partial}{\partial z^\nu}G_{\theta\beta}f^{\theta}_{\alpha}$$

is

$$\left(\frac{\partial\lambda}{\partial z^\sigma}g^{c}_{\alpha\beta} + \frac{\partial\mu}{\partial z^\sigma}g^{v}_{\alpha\beta}\right)f^{\sigma}_{\nu} \quad + \quad \left(\lambda\frac{\partial g^{c}_{\alpha\beta}}{\partial z^\sigma} + \mu\frac{\partial g^{v}_{\alpha\beta}}{\partial z^\sigma}\right)f^{\sigma}_{\nu}$$

$$= \quad \left(\frac{\partial\lambda}{\partial z^\nu}g^{c}_{\theta\beta} + \frac{\partial\mu}{\partial z^\nu}g^{v}_{\theta\beta}\right)f^{\theta}_{\alpha} + \left(\lambda\frac{\partial}{\partial z^\nu}g^{c}_{\theta\beta} + \mu\frac{\partial}{\partial z^\nu}g^{v}_{\theta\beta}\right)f^{\theta}_{\alpha}.$$

From (1) and the holomorphicity of $(g^{v}_{\alpha\beta}, g^{c}_{\alpha\beta})$, the above equation is equivalent to

$$\left(\frac{\partial\lambda}{\partial z^{n+s}}g^{c}_{\alpha\beta}+\frac{\partial\mu}{\partial z^{n+s}}g^{v}_{\alpha\beta}\right)f^{n+s}_{\nu}=\frac{\partial\lambda}{\partial z^{\nu}}g^{v}_{\alpha\beta}=0. \tag{4}$$

First, assume that $\frac{\partial}{\partial z^{\sigma}}G_{\alpha\beta}$ are pure. For $(\nu=i,\ \beta=n+k,\ \alpha=h)$ we have

$$\frac{\partial\lambda}{\partial z^{n+i}}g^{c}_{h,n+k}=0\ \Leftrightarrow\ \frac{\partial\lambda}{\partial y^{i}}g_{nk}=0\,,$$

from where we get

$$\frac{\partial\lambda}{\partial y^{i}}=0\ .$$

For $(\nu=i,\ \beta=k,\ \alpha=h)$ the last equation implies

$$\frac{\partial\lambda}{\partial z^{n+i}}y^{s}\frac{\partial}{\partial x^{s}}g_{kh}+\frac{\partial\mu}{\partial z^{n+i}}g_{kh}=\frac{\partial\lambda}{\partial z^{i}}g_{kh}\ ,$$

that is

$$\frac{\partial\lambda}{\partial x^{i}}=\frac{\partial\mu}{\partial y^{i}}\ .$$

Now, suppose that the pair (λ,μ) is holomorphic. The part of the system (4) which is not zero is obtained for $\nu=i$, i.e. (4) is equivalent to

$$\frac{\partial\lambda}{\partial y^{i}}g^{c}_{\alpha\beta}=\left(\frac{\partial\lambda}{\partial x^{i}}-\frac{\partial\mu}{\partial y^{i}}\right)g^{v}_{\alpha\beta}.$$

As the pair (λ,μ) is holomorphic, the above result holds. Therefore $G_{\alpha\beta}$ have pure partial derivatives. □

By using Theorem 4 we prove

Corollary 2. *The structure f is covariant constant with respect to the Riemannian connection generated by $G=(G_{\alpha\beta})$.*

Corollary 3. *The Kristoffel symbols generated by G are pure with respect to f. The objects constructed by these symbols are pure, too.*

Corollary 4. *The pair $(\lambda=1,\mu=1)$ is holomorphic. In this case the metrics g^c+g^v and g^c generate the same connection of Levi-Chevita.*

We should note that the metric g^c+g^v is known in [4] as the metric $I+II$.

The previous considerations raise the following question: why do we investigate CH-changes but not conformal ones? Is there a function h such that the metric hg^c generates a connection preserving f?

Theorem 8. *Suppose that $h(z^1,\dots,z^{2n})$ is an arbitrary differentiable function on TM. Then f is covariant constant with respect to the Levi-Chevita connection generated by hg^c if and only if $h=const$.*

Proof. First, assume that $\nabla f=0$, where ∇ is the connection under consideration. By Theorem 4 we have

$$\partial_{\lambda}(hg_{\alpha\beta})f^{\lambda}_{\sigma}=\partial_{\sigma}(hg_{\lambda\beta}f^{\lambda}_{\alpha}).$$

Hence

$$f^{\lambda}_{\sigma}h_{\lambda}\delta^{\beta}_{\alpha}=h_{\sigma}f^{\beta}_{\alpha},\quad\left(h_{\lambda}=\frac{\partial h}{\partial z^{\lambda}}\right).$$

This relation is valid for all values of the indices. In the special case $\beta = n+i$, $\alpha = k$, $f_k^{n+i} = \delta_k^i$ the above equation is reduced to

$$f_\sigma^\lambda h_\lambda \delta_k^{n+i} = h_\sigma \delta_k^i \,,$$

which leads to $h_\sigma = 0$ for all σ.

If $h = const$ the statement is obvious. □

Theorem 9. *The complete lift of the conformal curvature tensor of the base M of TM is CH-invariant under the change*

$$g^c \to \sigma^c g^\nu + \sigma^\nu g^c \ .$$

Proof. Since

$$\sigma^c g^v + \sigma^v g^c = (\sigma g)^c \,,$$

we have a conformal change of the base metric. In this case the tensor of the conformal curvature C is invariant. Thus C^c does not depend on σ. □

We denote by R, $r = \rho(R)$ and $\tau = \tau(R)$ the Riemann curvature tensor on M, Ricci's tensor for R and the corresponding scalar curvature, respectively. Then

$$\begin{aligned} C^c_{\alpha\beta\gamma\sigma} &= R^c_{\alpha\beta\gamma\sigma} + \frac{1}{n-2}\Big[r^c_{\alpha\gamma} g^v_{\beta\sigma} + r^v_{\alpha\gamma} g^c_{\beta\sigma} + r^c_{\beta\sigma} g^v_{\alpha\gamma} + r^v_{\beta\sigma} g^c_{\alpha\gamma} \\ &- r^v_{\alpha\sigma} g^c_{\beta\gamma} - r^v_{\alpha\sigma} g^c_{\beta\sigma} - r^c_{\beta\sigma} g^v_{\alpha\sigma} - r^v_{\beta\gamma} g^c_{\alpha\sigma} \Big] \\ &+ \frac{\tau^c}{(n-1)(n-2)} (g^v_{\beta\gamma} g^v_{\alpha\sigma} - g^v_{\beta\sigma} g^v_{\alpha\gamma}) \\ &+ \frac{\tau^v}{(n-1)(n-2)} (g^c_{\beta\gamma} g^v_{\alpha\sigma} + g^v_{\beta\gamma} g^c_{\alpha\sigma} - g^c_{\beta\sigma} g^v_{\alpha\gamma} - g^v_{\beta\sigma} g^c_{\alpha\gamma}) \ . \end{aligned}$$

Regarding f, $C^c_{\alpha\beta\gamma\sigma}$ is pure with respect to all indices. Hence

$$C^v_{\alpha\beta\gamma\sigma} = C^c_{\lambda\beta\gamma\sigma} f^\lambda_\alpha$$

is also CH-invariant.

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