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ON THE GENERALIZED OF HARMONIC AND BI-HARMONIC MAPS

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Abstract. In this note, we extend the definition of harmonic and biharmonic maps between two Riemannian manifolds, and we present some properties for f-harmonic maps and f-biharmonic maps.

1. f-harmonic maps

Definition 1.1. Consider a smooth map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ between Riemannian manifolds and $f : (x, y) \in M \times N \longrightarrow f(x, y) \in (0, +\infty)$ be a smooth positive function. The f-energy functional of φ is defined by

$$E_f(\varphi) = \frac{1}{2}\int_M f(x, \varphi(x))\, |d\varphi|^2\, v_g \tag{1.1}$$

(or over any compact subset $K \subset M$).

A map is called f-harmonic if it is a critical point of the f-energy functional over any compact subset of M.

Remark 1.1. :
- Definition 1.1, is a natural generalization of harmonic map ([2], [6], [7]) and f-harmonic map ([5], [10]).
- Definition 1.1, is also a generalization of p-harmonic map ([3]) and F-harmonic map [1]), when φ has no critical points.

1.1. The first variation of the f-energy.

Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map, we denote by :

$$\tau(\varphi) = trace_g \nabla d\varphi$$

the tension field of φ, and $e(\varphi) = \frac{1}{2}|d\varphi|^2$ the energy density of φ (for more detail, see [2], [4], [6] and [7]).

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Theorem 1.1. *Let $I = (-\epsilon, \epsilon) \subset \mathbb{R}$ and $\{\varphi_t\}_{t\in I}$ be a smooth variation of φ. Then*

$$\frac{d}{dt}E_f(\varphi_t)\Big|_{t=0} = -\int_M h(\tau_f(\varphi), v)v_g\,, \tag{1.2}$$

where

$$\tau_f(\varphi) = f_\varphi \tau(\varphi) + d\varphi(grad^M f_\varphi) - e(\varphi)(grad^N f)\circ\varphi, \tag{1.3}$$

f_φ is the smooth function $x \in M^m \to f_\varphi(x) = f(x, \varphi(x)) \in (0, +\infty)$,

and $v = \frac{\partial \varphi_t}{\partial t}\Big|_{t=0}$ denotes the variation vector field of $\{\varphi_t\}_{t\in I}$.

Proof:

Let $\phi : I \times M \longrightarrow N$ be a smooth map satisfying for all $t \in I$ and all $x \in M$

$$\phi(t, x) = \varphi_t(x)\,,$$

and

$$\phi(0, x) = \varphi(x)\,.$$

The variation vector field $v \in \Gamma(\varphi^{-1}TN)$ associated to the variation $\{\varphi_t\}_{t\in I}$ is given for all $x \in M$, by

$$v(x) = d_{(0,x)}\phi(\frac{\partial}{\partial t}),$$

We have

$$\begin{aligned}\frac{d}{dt}E_f(\varphi_t)|_{t=0} =& \frac{1}{2}\int_M \frac{\partial}{\partial t}(f(x, \varphi_t(x))|d_x\varphi_t|^2)\Big|_{(0,x)} v_g\\ =& \frac{1}{2}\int_M \Big\{\frac{\partial}{\partial t}f(x, \varphi_t(x))\Big|_{(0,x)}|d_x\varphi|^2\\ &+ f(x, \varphi(x))\frac{\partial}{\partial t}|d_x\varphi_t|^2\Big|_{(0,x)}\Big\}v_g\end{aligned} \tag{1.4}$$

First, note that:

$$\begin{aligned}\frac{\partial}{\partial t}f(x, \varphi_t(x))\Big|_{(0,x)} &= \frac{\partial}{\partial t}f(x, \phi(t, x))\Big|_{(0,x)}\\ &= d_{(x,\varphi(x))}f(0, v(x))\\ &= d_{\varphi(x)}f_x(v(x)).\end{aligned}$$

where f_x is the smooth function $y \in N \longrightarrow f_x(y) = f(x, y) \in (0, +\infty)$. Hence

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}f(x, \varphi_t(x))\Big|_{(0,x)}|d_x\varphi|^2 &= d_{\varphi(x)}f_x(v(x))e(\varphi)_x\\ &= h((grad^N f_x)_{\varphi(x)}, v(x))e(\varphi)_x\\ &= h(e(\varphi)_x(grad^N f_x)_{\varphi(x)}, v(x)).\end{aligned} \tag{1.5}$$

Other hand:

Let $\{e_i\}_{i=1}^m$ be an orthonormal frame with respect to g on M, such that $\nabla^M_{e_i}e_j = 0$, at $x \in M$ for all $i, j = 1, ..., m$. from equality

$$d_x\varphi_t(e_i) = d_{(t,x)}\phi(0, e_i)\,,$$

Summing over the index i, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}|d_x\varphi_t|^2\Big|_{(0,x)} &= \frac{1}{2}\frac{\partial}{\partial t}h(d_x\varphi_t(e_i),d_x\varphi_t(e_i))\Big|_{(0,x)}\\
&= \frac{1}{2}\frac{\partial}{\partial t}h(d\phi(0,e_i),d\phi(0,e_i))\Big|_{(0,x)}\\
&= h(\nabla^{\phi}_{\frac{\partial}{\partial t}}d\phi(0,e_i),d\phi(0,e_i))\Big|_{(0,x)}\\
&= h(\nabla^{\phi}_{(0,e_i)}d\phi(\frac{\partial}{\partial t}),d\phi(0,e_i))\Big|_{(0,x)}\\
&= (0,e_i)(h(d\phi(\frac{\partial}{\partial t}),d\phi(0,e_i)))\Big|_{(0,x)}\\
&\quad -h(d\phi(\frac{\partial}{\partial t}),\nabla^{\phi}_{(0,e_i)}d\phi(0,e_i)))\Big|_{(0,x)}\\
&= e_i(h(v,d\varphi(e_i)))_x - h(v,\tau(\varphi))_x\,.
\end{aligned}$$

Let X be a vector field with compact support on M, such that for any vector field Y on M, we have

$$g(X,Y)=h(v,d\varphi(Y))\,,$$

then

$$\frac{1}{2}\frac{\partial}{\partial t}|d_x\varphi_t|^2\Big|_{(0,x)} = (divX)_x - h(v,\tau(\varphi))_x.$$

So :

$$\begin{aligned}
\frac{1}{2}f(x,\varphi(x))\frac{\partial}{\partial t}|d_x\varphi_t|^2\Big|_{(0,x)} &= \Big[f_\varphi divX - h(v,f_\varphi\tau(\varphi))\Big]_x\\
&= \Big[div(f_\varphi X) - h(v,d\varphi(grad^M f_\varphi)) - h(v,f_\varphi\tau(\varphi))\Big]_x. \qquad (1.6)
\end{aligned}$$

Substituting (1.5)and (1.6) in (1.4), and consider the divergence theorem (see [2]) we obtain

$$\begin{aligned}
\frac{d}{dt}E_f(\varphi_t)\Big|_{t=0} &= \int_M h\Big(e(\varphi)_x(grad^N f_x)_{\varphi(x)} - d\varphi(grad^M f_\varphi)_x\\
&\quad -f(x,\varphi(x))\tau(\varphi)_x, v(x)\Big)v_g.
\end{aligned}$$

Definition 1.2. The field $\tau_f(\varphi)$ defined by :

$$\begin{aligned}
\tau_f(\varphi) &= f_\varphi\tau(\varphi) + d\varphi(grad^M f_\varphi) - e(\varphi)(grad^N f)\circ\varphi \qquad (1.7)\\
&= trace_g\nabla f_\varphi d\varphi - e(\varphi)(grad^N f)\circ\varphi.
\end{aligned}$$

is called the f-tension field of φ.

From Theorem 1.1, we deduce

Theorem 1.2. *Let $\varphi:(M^m,g)\longrightarrow(N^n,h)$ be a smooth map,. Then φ is f-harmonic, if and only if*

$$\tau_f(\varphi)=f_\varphi\tau(\varphi)+d\varphi(grad^M f_\varphi)-e(\varphi)(grad^N f)\circ\varphi=0$$

Particular Cases

(1) If $f = 1$, then $\tau_f(\varphi) = \tau(\varphi)$, is the natural tension field of φ (see [2], [6], [7]).
(2) Let $f_1 : M \to (0, +\infty)$, be a smooth positif function. If $f(x, y) = f_1(x)$ for all $(x, y) \in M \times N$, then $\tau_f(\varphi) = \tau_{f_1}(\varphi)$, and φ is f-harmonic map, if and only if, φ is f_1-harmonic map (see [5], [10]).
(3) Let $f_2 : N \to (0, +\infty)$, be a smooth positif function. If $f(x, y) = f_2(y)$ for all $(x, y) \in M \times N$, then $\tau_f(\varphi) = f_2 \circ \varphi.\widetilde{\tau}(\varphi)$, where $\widetilde{\tau}(\varphi)$ denote the tension field of φ between the Riemannian manifolds (M^m, g) and $(N^n, \widetilde{h})$ equipped with the conform metric $\widetilde{h} = f_2.h$.
So $\varphi : (M^m, g) \to (N^n, h)$ is f-harmonic map if and only if $\varphi : (M^m, g) \to (N^n, \widetilde{h})$ is harmonic map.
(4) Let $f_1 : M \to (0, +\infty)$ and $f_2 : N \to (0, +\infty)$ be smooth positif functions. If $f(x, y) = f_1(x)f_2(y)$ for all $(x, y) \in M \times N$, then

$$\tau_f(\varphi) = f_2 \circ \varphi\big\{f_1.\widetilde{\tau}(\varphi) + d\varphi(grad^M f_1)\big\}$$

.
(5) If $\varphi : (M^m, g) \longrightarrow (N^n, h)$ has no critical points (i.e. $|d_x\varphi| \neq 0$, then harmonic maps, p-harmonic maps and exponential harmonic maps are f-harmonic map with $f = 1$, $f = |d\varphi|^{p-2}$ and $f = exp(\frac{|d\varphi|^2}{2})$ respectively.
(6) If $\varphi : (M^m, g) \longrightarrow (N^n, h)$ has no critical points, then any F-harmonic map is f-harmonic map with $f = F'(\frac{|d\varphi|^2}{2})$.

Example 1.1. Let $M = (\mathbb{R}^*, dx^2)$, $N = (\mathbb{R}, dy^2)$, $\varphi : M \longrightarrow N$ be a smooth function, and let $f : M \times N \longrightarrow \mathbb{R}_+$ be a C^2 function. From Definition 1.2 (formula (1.7)), we have

$$\tau_f(\varphi)\Big|_x = \Big[f(x, \varphi(x))\varphi''(x) + \frac{\partial f}{\partial x}(x, \varphi(x))\varphi'(x) + \frac{1}{2}\varphi'(x)^2\frac{\partial f}{\partial y}(x, \varphi(x))\Big]\frac{d}{dy}\Big|_{\varphi(x)}.$$

If $f(x, y) = e^{xy}$, by (1.8), φ is harmonic if and only if

$$\varphi''(x) + \varphi(x)\varphi'(x) + \frac{1}{2}x\varphi'(x)^2 = 0. \tag{1.8}$$

A local solution of the equation (1.8) is $\varphi(x) = \frac{4}{x}$.

Example 1.2. Let $\varphi = Id : x \in \mathbb{R}^n \to \varphi(x) = x \in \mathbb{R}^n$, then we have $\tau(\varphi) = 0$, $e(\varphi) = \frac{n}{2}$ and from formula (1.7), we obtain :

$$\tau_f(\varphi) = \{\frac{\partial f}{\partial x^i} + \frac{(2-n)}{2}\frac{\partial f}{\partial y^i}\}\frac{\partial}{\partial x^i}$$

1.2. The second variation of the f-energy.

Theorem 1.3. *Let $\varphi : (M^m, g) \to (N^n, h)$ be an f-harmonic map between Riemannian manifolds, and $\varphi_{t,s} : M \longrightarrow N$ $(-\varepsilon < t, s < \varepsilon)$ be a two-parameter variation with compact support, such that $\varphi_{0,0} = \varphi$. Set*

$$v = \frac{\partial\varphi_{t,s}}{\partial t}\Big|_{t,s=0} \quad and \quad w = \frac{\partial\varphi_{t,s}}{\partial s}\Big|_{t,s=0}.$$

Under the notation above we have the following:

$$\frac{\partial^2}{\partial t\partial s}E_f(\varphi_{t,s})\Big|_{t,s=0} = \int_M h(J_{\varphi,f}(v), w)v_g,$$

where

$$
\begin{aligned}
J_{\varphi,f}(v) = &- f_\varphi trace_g R^N(v, d\varphi)d\varphi - trace_g \nabla^\varphi f_\varphi \nabla^\varphi v \\
&+ e(\varphi)(\nabla^N_v grad^N f)\circ\varphi - d\varphi(grad^M v(f)) \\
&- v(f)\tau(\varphi) + < \nabla^\varphi v, d\varphi > (grad^N f)\circ\varphi,
\end{aligned} \tag{1.9}
$$

and

$$
trace_g \nabla^\varphi f_\varphi \nabla^\varphi v = \sum_{i=1}^m \Big(\nabla^\varphi_{e_i} f_\varphi \nabla^\varphi_{e_i} v - f_\varphi \nabla^\varphi_{\nabla^M_{e_i} e_i} v \Big).
$$

for any orthonormal frame $(e_i)_i$ *on* (M, g). *Here* $\langle\, ,\, \rangle$ *denote the inner product on* $T^*M \otimes \varphi^{-1}TN$ *and* R^N *is the curvature tensor on* (N, h).

Definition 1.3. $J_{\varphi,f}$ is called the f-Jacobi operator corresponding to φ.

Proof of Theorem 1.3 :

Let $\phi : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times M \longrightarrow N$ is a map defined by

$$
\phi(t, s, x) = \varphi_{t,s}(x)\,,
$$

where $(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times M$ is equipped with the product metric. If we extend the vector fields $\frac{\partial}{\partial t}$ on $(-\varepsilon, \varepsilon)$ and $\frac{\partial}{\partial s}$ on $(-\varepsilon, \varepsilon)$, then

$$
v = d\phi(\frac{\partial}{\partial t})\Big|_{t,s=0} \qquad and \qquad w = d\phi(\frac{\partial}{\partial s})\Big|_{t,s=0}.
$$

Let $\{e_i\}_{i=1}^m$ be an orthonormal frame with respect to g on M, such that $\nabla^M_{e_i} e_j = 0$, at fixed point $x \in M$ for all $i, j = 1, ..., m$. We compute

$$
\frac{\partial^2}{\partial t \partial s} E_f(\varphi_{t,s}) = \frac{1}{2} \int_M \frac{\partial^2}{\partial t \partial s} \Big[f(x, \varphi_{t,s}(x)) h(d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i)) \Big] v_g,
$$

here summing over the index i. We have

$$
\begin{aligned}
\frac{1}{2}\frac{\partial^2}{\partial t \partial s} \Big[f(x, \varphi_{t,s}(x)) h(d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i)) \Big] = \\
&\frac{1}{2}\frac{\partial^2}{\partial t \partial s} f(x, \varphi_{t,s}(x)).h(d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i)) \\
&+ \frac{\partial}{\partial t} f(x, \varphi_{t,s}(x)).h(\nabla^\phi_{\frac{\partial}{\partial s}} d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i)) \\
&+ \frac{\partial}{\partial s} f(x, \varphi_{t,s}(x)).h(\nabla^\phi_{\frac{\partial}{\partial t}} d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i)) \\
&+ f(x, \varphi_{t,s}(x)).h(\nabla^\phi_{\frac{\partial}{\partial s}} \nabla^\phi_{\frac{\partial}{\partial t}} d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i)) \\
&+ f(x, \varphi_{t,s}(x)).h(\nabla^\phi_{\frac{\partial}{\partial t}} d\varphi_{t,s}(e_i), \nabla^\phi_{\frac{\partial}{\partial s}} d\varphi_{t,s}(e_i)).
\end{aligned} \tag{1.10}
$$

Now we calculate each term of right part in the above equation (1.10):

1.

$$\begin{aligned}\frac{\partial^2}{\partial t\partial s}f(x,\varphi_{t,s}(x)) &= \frac{\partial}{\partial s}\Big[d_{(x,\varphi_{t,s}(x))}f(0,d_{(t,s,x)}\phi(\frac{\partial}{\partial t}))\Big]\\ &= \frac{\partial}{\partial s}\Big[d_{\varphi_{t,s}(x)}f_x(d_{(t,s,x)}\phi(\frac{\partial}{\partial t}))\Big]\\ &= \frac{\partial}{\partial s}\Big[h(grad^N f_x, d_{(t,s,x)}\phi(\frac{\partial}{\partial t}))\Big]\\ &= h(\nabla^\phi_{\frac{\partial}{\partial s}} grad^N f_x\circ\phi, d_{(t,s,x)}\phi(\frac{\partial}{\partial t}))\\ &\quad +h(grad^N f_x\circ\phi, \nabla^\phi_{\frac{\partial}{\partial s}} d_{(t,s,x)}\phi(\frac{\partial}{\partial t})),\end{aligned}$$

then

$$\begin{aligned}&\frac{1}{2}\frac{\partial^2}{\partial t\partial s}f(x,\varphi_{t,s}(x)).h(d\varphi_{t,s}(e_i),d\varphi_{t,s}(e_i))\Big|_{t=s=0} = \\ &h(\nabla^N_w grad^N f_x, v)e(\varphi) + h\Big((grad^N f_x)_{\varphi(x)}, \nabla^\phi_{\frac{\partial}{\partial s}} d\phi(\frac{\partial}{\partial t})\Big|_{t=s=0}\Big)e(\varphi).\end{aligned} \tag{1.11}$$

2.

$$\begin{aligned}&\frac{\partial}{\partial t}f(x,\varphi_{t,s}(x)).h(\nabla^\phi_{\frac{\partial}{\partial s}} d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i))\Big|_{t=s=0}\\ &\qquad = h(grad^N f_x, v)h(\nabla^\phi_{\frac{\partial}{\partial s}} d\phi(e_i), d\phi(e_i))\Big|_{t=s=0}\\ &\qquad = v(f_x)h(\nabla^\phi_{e_i} d\phi(\frac{\partial}{\partial s}), d\phi(e_i))\Big|_{t=s=0}\\ &\qquad = v(f_x)\Big[e_i(h(w,d\varphi(e_i))) - h(w,\tau(\varphi))\Big].\end{aligned}$$

If X is the compactly supported vector field on M such for any vector field Y on M:

$$g(X,Y) = h(w, d\varphi(Y))\,,$$

then

$$\begin{aligned}&\frac{\partial}{\partial t}f(x,\varphi_{t,s}(x)).h(\nabla^\phi_{\frac{\partial}{\partial s}} d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i))\Big|_{t=s=0}\\ &\qquad = v(f_x)divX - h(w, v(f_x)\tau(\varphi))\\ &\qquad = div(v(f_x)X) - h(w, d\varphi(grad^M v(f_x))) - h(w, v(f_x)\tau(\varphi)) \qquad .\end{aligned} \tag{1.12}$$

3.

$$\begin{aligned}&\frac{\partial}{\partial s}f(x,\varphi_{t,s}(x)).h(\nabla^\phi_{\frac{\partial}{\partial t}} d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i))\Big|_{t=s=0}\\ &\qquad = h(grad^N f_x, w). < \nabla^\varphi v, d\varphi >\\ &\qquad = h(< \nabla^\varphi v, d\varphi > grad^N f_x, w).\end{aligned} \tag{1.13}$$

4.

$$f(x,\varphi_{t,s}(x)).h(\nabla^{\phi}_{\frac{\partial}{\partial s}}\nabla^{\phi}_{\frac{\partial}{\partial t}}d\varphi_{t,s}(e_i),d\varphi_{t,s}(e_i))\Big|_{t=s=0}$$
$$\begin{aligned}&= f(x,\varphi(x))h(\nabla^{\phi}_{\frac{\partial}{\partial s}}\nabla^{\phi}_{e_i}d\varphi_{t,s}(\frac{\partial}{\partial t}),d\varphi(e_i))\\ &= f_\varphi h(R^N(d\varphi(\frac{\partial}{\partial s}),d\varphi(e_i))d\varphi(\frac{\partial}{\partial t}),d\varphi(e_i))\\ &\quad +f_\varphi h(\nabla^{\phi}_{e_i}\nabla^{\phi}_{\frac{\partial}{\partial s}}d\varphi_{t,s}(\frac{\partial}{\partial t}),d\varphi(e_i))\\ &= -f_\varphi h(R^N(v,d\varphi(e_i))d\varphi(e_i),w)\\ &\quad +f_\varphi e_i(h(\nabla^{\phi}_{\frac{\partial}{\partial s}}d\varphi_{t,s}(\frac{\partial}{\partial t}),d\varphi(e_i)))\\ &\quad -f_\varphi h(\nabla^{\phi}_{\frac{\partial}{\partial s}}d\varphi_{t,s}(\frac{\partial}{\partial t}),\tau(\varphi))\end{aligned}\tag{1.14}$$

let X_2 be a compactly supported vector field on M such that

$$g(X_2,Y)=h(\nabla^{\phi}_{\frac{\partial}{\partial s}}d\varphi_{t,s}(\frac{\partial}{\partial t}),d\varphi(Y))|_{t=s=0},$$

for any vector field Y on M, then the formula (1.14) becomes

$$f(x,\varphi_{t,s}(x)).h(\nabla^{\phi}_{\frac{\partial}{\partial s}}\nabla^{\phi}_{\frac{\partial}{\partial t}}d\varphi_{t,s}(e_i),d\varphi_{t,s}(e_i))\Big|_{t=s=0}$$
$$\begin{aligned}&= -f_\varphi h(trace_g R^N(v,d\varphi)d\varphi,w)\\ &\quad +f_\varphi divX_2 - f_\varphi h(\nabla^{\phi}_{\frac{\partial}{\partial s}}d\varphi_{t,s}(\frac{\partial}{\partial t}),\tau(\varphi))\\ &= -f_\varphi h(trace_g R^N(v,d\varphi)d\varphi,w)+div(f_\varphi X_2)\\ &\quad -h(\nabla^{\phi}_{\frac{\partial}{\partial s}}d\varphi_{t,s}(\frac{\partial}{\partial t})\Big|_{t=s=0},d\varphi(grad^M f_\varphi))\\ &\quad -f_\varphi h\Big(\nabla^{\phi}_{\frac{\partial}{\partial s}}d\varphi_{t,s}(\frac{\partial}{\partial t})\Big|_{t=s=0},\tau(\varphi)\Big).\end{aligned}\tag{1.15}$$

5.

$$f(x,\varphi_{t,s}(x)).h(\nabla^{\phi}_{\frac{\partial}{\partial t}}d\varphi_{t,s}(e_i),\nabla^{\phi}_{\frac{\partial}{\partial s}}d\varphi_{t,s}(e_i))\Big|_{t=s=0}$$
$$\begin{aligned}&= f_\varphi h(\nabla^{\phi}_{e_i}d\varphi_{t,s}(\frac{\partial}{\partial t}),\nabla^{\phi}_{e_i}d\varphi_{t,s}(\frac{\partial}{\partial s}))\\ &= f_\varphi\Big[e_i(h(\nabla^{\varphi}_{e_i}v,w))-h(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}v,w)\Big].\end{aligned}\tag{1.16}$$

Let X_3 be a compactly supported vector field on M such that

$$g(X_3,Y)=h(\nabla^{\varphi}_{Y}v,w),$$

for any vector field Y on M, then the formula (1.16) becomes

$$f(x,\varphi_{t,s}(x)).h(\nabla^{\phi}_{\frac{\partial}{\partial t}}d\varphi_{t,s}(e_i),\nabla^{\phi}_{\frac{\partial}{\partial s}}d\varphi_{t,s}(e_i))\Big|_{t=s=0}$$
$$\begin{aligned}&= f_\varphi\Big[divX_3 - h(trace_g(\nabla^{\varphi})^2v,w)\Big]\\ &= div(f_\varphi X_3)-h(\nabla^{\varphi}_{grad^M f_\varphi}v,w)\\ &\quad -h(f_\varphi trace_g(\nabla^{\varphi})^2v,w)\\ &= div(f_\varphi X_3)-h(trace_g\nabla^{\varphi}f_\varphi\nabla^{\varphi}v,w).\end{aligned}\tag{1.17}$$

Substituting the formulas (1.11), (1.12), (1.13), (1.15) and (1.17) in (1.10), and integrate it, the Theorem 1.3 follows.

2. f-Biharmonic Maps

A natural generalization of f-harmonic maps is given by integrating the square of the norm of the f-tension field. More precisely, the f-bi-energy functional of a smooth map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is defined by

$$E_{2,f}(\varphi) = \frac{1}{2}\int_M |\tau_f(\varphi)|^2 \, v_g \, .$$

A map φ is called f-biharmonic if it is a critical point of the f-energy functional.

2.1. **First variation of the f-bi-energy.**

Theorem 2.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map and let $\{\varphi_t\}_t$ $(-\varepsilon < t < \varepsilon)$, be a smooth variation of φ. Then*

$$\frac{d}{dt}E_{2,f}(\varphi_t)\Big|_{t=0} = -\int_M h(\tau_{2,f}(\varphi), v)\, v_g \, ,$$

where $v = \frac{\partial\varphi_t}{\partial t}\Big|_{t=0}$ denotes the variation vector field of $\{\varphi_t\}_t$,

$$\begin{aligned}\tau_{2,f}(\varphi) \;\; &= \;\; -f_\varphi trace_g R^N(\tau_f(\varphi), d\varphi)d\varphi - trace_g \nabla^\varphi f_\varphi \nabla^\varphi \tau_f(\varphi) \\ &\quad + e(\varphi)(\nabla^N_{\tau_f(\varphi)} grad^N f) \circ \varphi - d\varphi(grad^M \tau_f(\varphi)(f)) \\ &\quad - \tau_f(\varphi)(f)\tau(\varphi) + < \nabla^\varphi \tau_f(\varphi), d\varphi > (grad^N f) \circ \varphi.\end{aligned}$$

and

$$trace_g \nabla^\varphi f_\varphi \nabla^\varphi \tau_f(\varphi) = \sum_{i=1}^m \Big(\nabla^\varphi_{e_i} f_\varphi \nabla^\varphi_{e_i} \tau_f(\varphi) - f_\varphi \nabla^\varphi_{\nabla^M_{e_i} e_i} \tau_f(\varphi) \Big).$$

for any orthonormal frame $(e_i)_i$ on (M, g)

Definition 2.1. $\tau_{2,f}(\varphi)$ is called the f-bi-tension field of φ.

Proof of Theorem 2.1 :

Let $\phi : (-\varepsilon, \varepsilon) \times M \longrightarrow N$ be defined by $\phi(t, x) = \varphi_t(x)$, where $(-\varepsilon, \varepsilon) \times M$ is equipped with the product metric. We extend the vector fields $\frac{\partial}{\partial t}$ naturally on $(-\varepsilon, \varepsilon) \times M$, then

$$\frac{d}{dt}E_{2,f}(\varphi_t) = \int_M h(\nabla^\phi_{\frac{\partial}{\partial t}} \tau_f(\varphi_t), \tau_f(\varphi_t))v_g, \tag{2.1}$$

choose a local orthonormal frame $\{e_i\}_{1\leq i\leq m}$, such that $\nabla_{e_i} e_j = 0$ for all $i, j = 1, ..., m$ at a fixed point $x \in M$, then by (1.7) we have

$$\nabla^\phi_{\frac{\partial}{\partial t}} \tau_f(\varphi_t) = \nabla^\phi_{\frac{\partial}{\partial t}} \nabla^\phi_{e_i} f_{\varphi_t} d\varphi_t(e_i) - \nabla^\phi_{\frac{\partial}{\partial t}} e(\varphi_t)(grad^N f) \circ \varphi_t,$$

First, from formula of curvature tensor, we obtain

$$\begin{aligned}\nabla^\phi_{\frac{\partial}{\partial t}} \nabla^\phi_{e_i} f_{\varphi_t} d\varphi_t(e_i) \;\; &= \;\; R^N(d\phi(\frac{\partial}{\partial t}), d\phi(e_i)) f_{\varphi_t} d\varphi_t(e_i) \\ &\quad + \nabla^\phi_{e_i} \nabla^\phi_{\frac{\partial}{\partial t}} f_{\varphi_t} d\varphi_t(e_i).\end{aligned} \tag{2.2}$$

We have

$$\begin{aligned} h(\nabla^{\phi}_{e_i}\nabla^{\phi}_{\frac{\partial}{\partial t}} f_{\varphi_t} d\varphi_t(e_i), \tau_f(\varphi_t)) &= e_i(h(\nabla^{\phi}_{\frac{\partial}{\partial t}} f_{\varphi_t} d\varphi_t(e_i), \tau_f(\varphi_t))) \\ &\quad -h(\nabla^{\phi}_{\frac{\partial}{\partial t}} f_{\varphi_t} d\varphi_t(e_i), \nabla^{\phi}_{e_i}\tau_f(\varphi_t))) \\ &= e_i(h(\nabla^{\phi}_{\frac{\partial}{\partial t}} f_{\varphi_t} d\varphi_t(e_i), \tau_f(\varphi_t))) \\ &\quad -\frac{\partial f_{\varphi_t}}{\partial t}.h(d\varphi_t(e_i), \nabla^{\phi}_{e_i}\tau_f(\varphi_t))) \\ &\quad -f_{\varphi_t} h(\nabla^{\phi}_{\frac{\partial}{\partial t}} d\varphi_t(e_i), \nabla^{\phi}_{e_i}\tau_f(\varphi_t))). \end{aligned}$$

let X be the compactly supported vector field on M such that

$$g(X,Y) = h\big((\nabla^{\phi}_{\frac{\partial}{\partial t}} f_{\varphi_t} d\varphi_t)(Y), \tau_f(\varphi_t)\big)\Big|_{t=0}, \qquad \forall Y \in \Gamma(TM),$$

then

$$\begin{aligned} h(\nabla^{\phi}_{e_i}\nabla^{\phi}_{\frac{\partial}{\partial t}} f_{\varphi_t} d\varphi_t(e_i), \tau_f(\varphi_t))\Big|_{t=0} &= divX - h(grad^N f, v). < d\varphi, \nabla^{\varphi}\tau_f(\varphi) > \\ &\quad -f_{\varphi} h(\nabla^{\phi}_{\frac{\partial}{\partial t}} d\varphi_t(e_i), \nabla^{\phi}_{e_i}\tau_f(\varphi_t)))\Big|_{t=0} \\ &= divX - h(grad^N f, v). < d\varphi, \nabla^{\varphi}\tau_f(\varphi) > \\ &\quad -f_{\varphi}\Big[e_i(h(v, \nabla^{\varphi}_{e_i}\tau_f(\varphi))) - h(v, \nabla^{\phi}_{e_i}\nabla^{\varphi}_{e_i}\tau_f(\varphi))\Big]. \\ &= divX - h(grad^N f, v). < d\varphi, \nabla^{\varphi}\tau_f(\varphi) > \\ &\quad -f_{\varphi}\Big[divX_2 - h(v, trace_g(\nabla^{\varphi})^2\tau_f(\varphi))\Big] \\ &= divX - h(grad^N f, v). < d\varphi, \nabla^{\varphi}\tau_f(\varphi) > \\ &\quad -div(f_{\varphi}X_2) + h(v, \nabla^{\varphi}_{grad^M f_{\varphi}}\tau_f(\varphi)) \\ &\quad +h(v, f_{\varphi} trace_g(\nabla^{\varphi})^2\tau_f(\varphi)). \end{aligned} \tag{2.3}$$

where X_2 is the compactly supported vector field on M such that

$$g(X_2,Y) = h(v, \nabla^{\varphi}_Y \tau_f(\varphi)), \qquad \forall Y \in \Gamma(TM),$$

By the formulas (2.2) and (2.3), we have

$$\begin{aligned} h(\nabla^{\phi}_{\frac{\partial}{\partial t}}\nabla^{\phi}_{e_i} f_{\varphi_t} d\varphi_t(e_i), \tau_f(\varphi_t))\Big|_{t=0} &= h(f_{\varphi} trace_g R^N(\tau_f(\varphi), d\varphi)d\varphi, v) \\ &\quad +divX - h(< d\varphi, \nabla^{\varphi}\tau_f(\varphi) > .grad^N f, v) \\ &\quad -div(f_{\varphi}X_2) + h(v, \nabla^{\varphi}_{grad^M f_{\varphi}}\tau_f(\varphi)) \\ &\quad +h(v, f_{\varphi} trace_g(\nabla^{\varphi})^2\tau_f(\varphi)) \\ &= h(f_{\varphi} trace_g R^N(\tau_f(\varphi), d\varphi)d\varphi, v) \\ &\quad +divX - h(< d\varphi, \nabla^{\varphi}\tau_f(\varphi) > .grad^N f, v) \\ &\quad -div(f_{\varphi}X_2) + h(trace_g\nabla^{\varphi} f_{\varphi}\nabla^{\varphi}\tau_f(\varphi), v). \end{aligned} \tag{2.4}$$

On the other hand, we have

$$\nabla^{\phi}_{\frac{\partial}{\partial t}} e(\varphi_t)(grad^N f)\circ\phi = \frac{\partial e(\varphi_t)}{\partial t}(grad^N f)\circ\phi + e(\varphi_t)\nabla^{\phi}_{\frac{\partial}{\partial t}}(grad^N f)\circ\phi, \tag{2.5}$$

since

$$\begin{aligned}\frac{\partial e(\varphi_t)}{\partial t}\Big|_{t=0} &= h(\nabla^{\phi}_{\frac{\partial}{\partial t}} d\phi(e_i), d\phi(e_i))\Big|_{t=0}\\ &= h(\nabla^{\phi}_{e_i} d\phi(\frac{\partial}{\partial t}), d\phi(e_i))\Big|_{t=0}\\ &= e_i(h(v, d\varphi(e_i))) - h(v, \tau(\varphi)).\\ &= divX_3 - h(v, \tau(\varphi)) \qquad (2.6)\end{aligned}$$

where X_3 is the compactly supported vector field on M such that

$$g(X_3, Y) = h(v, d\varphi(Y)), \qquad \forall Y \in \Gamma(TM).$$

By the formulas (2.5) and (2.6), we obtain

$$\begin{aligned} h(\nabla^{\phi}_{\frac{\partial}{\partial t}} e(\varphi_t)(grad^N f) \circ \phi, \tau_f(\varphi_t))\Big|_{t=0} & \\ &= \tau_f(\varphi)(f)\, divX_3 - \tau_f(\varphi)(f)\, h(v, \tau(\varphi))\\ &\quad + e(\varphi) h(\nabla^{\phi}_{\frac{\partial}{\partial t}} grad^N f \circ \phi, \tau_f(\varphi_t))\Big|_{t=0}\\ &= div(\tau_f(\varphi)(f)\, X_3) - h(v, d\varphi(grad^M \tau_f(\varphi)(f)))\\ &\quad -\tau_f(\varphi)(f) h(v, \tau(\varphi)) + e(\varphi) h(\nabla^N_{\tau_f(\varphi)} grad^N f, v) \qquad (2.7)\end{aligned}$$

Substituting the formulas (2.4) and (2.7) in (2.1), the Theorem 2.1 follows.

From Theorem 2.1, we deduce

Theorem 2.2. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map . Then φ is f-biharmonic if and only if we have:*

$$\begin{aligned}\tau_{2,f}(\varphi) &= -f_{\varphi} trace_g R^N(\tau_f(\varphi), d\varphi) d\varphi - trace_g \nabla^{\varphi} f_{\varphi} \nabla^{\varphi} \tau_f(\varphi)\\ &\quad + e(\varphi)(\nabla^N_{\tau_f(\varphi)} grad^N f) \circ \varphi - d\varphi(grad^M \tau_f(\varphi)(f))\\ &\quad -\tau_f(\varphi)(f)\tau(\varphi) + < \nabla^{\varphi} \tau_f(\varphi), d\varphi > (grad^N f) \circ \varphi.\\ &= 0\end{aligned}$$

Particular Cases

(1) If $f = 1$, then $\tau_{2,f}(\varphi) = \tau(\varphi)$, is the natural bi-tension field of φ (see [11], [12]).

(2) Let $f_1 : M \to (0, +\infty)$, be a smooth positif function. If $f(x, y) = f_1(x)$ for all $(x, y) \in M \times N$, then $\tau_{2,f}(\varphi) = \tau_{2,f_1}(\varphi)$, and φ is f-bi-harmonic map, if and only if, φ is f_1-biharmonic map (see [10]).

(3) Let $f_2 : N \to (0, +\infty)$, be a smooth positif function. If $f(x, y) = f_2(y)$ for all $(x, y) \in M \times N$, then $\varphi : (M^m, g) \to (N^n, h)$ is f-bi-harmonic map if and only if $\varphi : (M^m, g) \to (N^n, \widetilde{h})$ is bi-harmonic map, where $(N^n, \widetilde{h})$ equipped with the conform metric $\widetilde{h} = f_2.h$ (see [11]).

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