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SOME QUESTIONS ON PLANE CURVES

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Abstract. We consider some properties of simple plane curves, starting from a unusual metric formulation of the tangent line.

1. Introduction

In previous paper ([12]) a new notion of tangency is formulated in suitable metric spaces and an equivalent metric formulation of the notion of tangency at a point of the graphic of a real function ([12] (Example 2.6) and [11](Teorema 1.1)) is considered.

Then it is possible to reconsider the notion of tangent line at a point of a plane curve (and also of normal line) in a very different way with respect the classical one. This paper is devoted to consider such notion and to investigate some new aspects and questions on plane curves.

It is well know that "... Not all curves are rectificable; some do not have a tangent at any of their points..." (see[15] (pag.46)).

From your study at every point of a plane curve there is, in a some specified meaning, at least a "tangent direction" and it is possible, from a pure theoretical point of view, to have many "‘tangent directions"’ at every point.

In the last section we present some open problems wich seem news and interesting.

Following [12], we consider two abstract operations. Let (X, d) be a metric space and A, B be non-empty, compact (or locally compact) subsets of X. Assume that $x_0 \in A \cap B$ is an accumulation point of A. We define the functions:

$$\underline{D}_{x_0}(A, B) = \liminf_{A\setminus\{x_0\}\ni x\to x_0} \frac{d(x, B)}{d(x, x_0)}; \tag{1.1}$$

$$\overline{D}_{x_0}(A, B) = \limsup_{A\setminus\{x_0\}\ni x\to x_0} \frac{d(x, B)}{d(x, x_0)}; \tag{1.2}$$

where $d(x, B) = \inf\{d(x, y)|\ y \in B\}$.

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When $\underline{D}_{x_0}(A,B) = \overline{D}_{x_0}(A,B)$, we write $D_{x_0}(A,B)$. We remark that: $0 \leq \underline{D}_{x_0}(A,B) \leq \overline{D}_{x_0}(A,B) \leq 1$; hence we have: $\overline{D}_{x_0}(A,B) = 0 \Longrightarrow \underline{D}_{x_0}(A,B) = D_{x_0}(A,B) = 0$.

The previous operations are investigated, early and with other motivations, from different authors (see [6] [16], [17], [7],[8],[9], [13], [14]).

In [11] we prove the following result, establishing the connession with the usual notion of tangent line.

Take as metric space R^2, endowed with the usual euclidean metric, let $f : R \to R$ be a continuous function and $G_f \subset R^2$ its graph. Let $p_0 = (x_0, f(x_0)) \in G_f$, and consider the functions $\underline{D}_{p_0}$, $\overline{D}_{p_0}$. For r a straight line through p_0, the following conditions are equivalent:

(i) r is the tangent line to G_f at p_0;
(ii) $\overline{D}_{p_0}(r, G_f) = 0$ and $\underline{D}_{p_0}(s, G_f) > 0$ for every line $s \neq r$ through p_0;
(iii) $\overline{D}_{p_0}(G_f, r) = 0$ and $\underline{D}_{p_0}(G_f, s) > 0$ for every line $s \neq r$ through p_0.

Then one can agree the following definition:

Definition 1.1. Let A, B be non-empty, compact (or locally compact) sets of the metric space X and let x_0 be an accumulation point of A and B. We say that A *is tangent to* B *in* x_0 if and only if $D_{x_0}(A,B) = 0$.

We say that A, B *are tangent in* x_0 if and only if both $D_{x_0}(A,B)$ and $D_{x_0}(B,A)$ exist and $D_{x_0}(A,B) = D_{x_0}(B,A) = 0$.

We remark that if A is tangent to B or A, B are tangent in x_0 with respect to the metric d, then the same occur with respect every metric d_1 equivalent to d.

Furthermore for $\underline{D}$ and $\overline{D}$ we have the following result:

Proposition 1.1. *Let A, B, C be non-empty, compact (or locally compact) subsets of the metric space X; let x_0 be an accumulation point for A, B and C. We have the following*

$$\overline{D}_{x_0}(A,C) - \overline{D}_{x_0}(A,B) \leq \overline{D}_{x_0}(B,C) \cdot [1 + \overline{D}_{x_0}(A,B)] \tag{1.3}$$

$$\underline{D}_{x_0}(A,C) + \overline{D}_{x_0}(A,B) \geq \underline{D}_{x_0}(B,C) \cdot [1 - \overline{D}_{x_0}(A,B)]. \tag{1.4}$$

$$\underline{D}_{x_0}(A,B) = 0 \qquad \Longleftrightarrow \qquad \underline{D}_{x_0}(B,A) = 0. \tag{1.5}$$

and if $D_{x_0}(A,B)$ and $D_{x_0}(B,A)$ exist, then: $D_{x_0}(A,B) = 0 \Longleftrightarrow D_{x_0}(B,A) = 0$.

For the proof, compare [12](cfr. Propositions 2.1 and 2.3 and remark 2.2).

Now we consider the metric space (X, d), where $X = R^2$ and d is the usual euclidean metric; we denote with $\Re(x)$ the set of half-straight lines through the point x, that is: $r \in \Re(x) \Longleftrightarrow \exists v \in R^2, \quad ||v|| = 1, \quad r = \{x + tv | t \geq 0\}$.

The following condition hold:

$$\forall r \in \Re(x) \Longrightarrow x \in r; \tag{1.6}$$

$$\forall r \in \Re(x), \textit{every bounded sequence in } r \textit{ is compact}; \tag{1.7}$$

$$\forall r, s \in \Re(x) \Longrightarrow \exists D_x(r,s), \exists D_x(s,r) \textit{ and } D_x(r,s) = D_x(s,r); \tag{1.8}$$

$$\forall r, s \in \Re(x): \quad D_x(r,s) = 0 \Longrightarrow r = s; \tag{1.9}$$

$$0 < d(x,y) \Longrightarrow \exists r \in \Re(x), \quad \exists s \in \Re(y) : y \in r \quad \textit{and} \quad x \in s; \tag{1.10}$$

(1.11)
$$\forall x \in R^2, \forall (r_n), r_n \in \Re(x) : \exists (r_{n_k}) \subseteq (r_n),\ \exists s \in \Re(x) \text{ such that } \lim_n D_x(r_{n_k}, s) = 0.$$

We remark that, in (1.10), if $r = \{x + tv | t \geq 0\}$ and $s = \{y + tu | t \geq 0\}$, then $u = -v$; moreover $D_x(r,s) = \frac{d(x,s)}{d(x,x_0)} = \frac{d(y,r)}{d(y,x_0)} \quad \forall x \in r, x \neq x_0, \quad \forall y \in s, y \neq x_0$.

The following propositions hold:

Proposition 1.2. *Let $A \subset R^2$ be a non-empty subset, if there exists $r^* \in \Re(x_0)$ such that $D_{x_0}(A, r^*) = 0$, then r^* is unique.*

Proof. We assume there exist r^*, r^{**} such that $\overline{D}_{x_0}(A, r^*) = \overline{D}_{x_0}(A, r^{**}) = 0$. By (1.4), we have:

$$0 = \overline{D}_{x_0}(A,r^*)+\overline{D}_{x_0}(A,r^{**}) \geq \overline{D}_{x_0}(r^*,r^{**})(1-\overline{D}_{x_0}(A,r^*)) = \overline{D}_{x_0}(r^*,r^{**}) = D_{x_0}(r^*,r^{**}).$$

Then, by (1.9), $r^* = r^{**}$. □

Furthermore, we have the proposition:

Proposition 1.3. *Let $A \subset R^2$ be a non-empty subset, if there exists $r^*, r^{**} \in \Re(x_0)$ such that: $\overline{D}_{x_0}(A, r^{**}) = \overline{D}_{x_0}(r^*, A) = 0$, then $r^* = r^{**}$.*

Proof. From (1.3), we have:

$$\overline{D}_{x_0}(r^*, r^{**}) - \overline{D}_{x_0}(r^*, A) \leq \overline{D}_{x_0}(A, r^{**})(1 + \overline{D}_{x_0}(r^*, A)) = 0.$$

Hence $\overline{D}_{x_0}(r^*, r^{**}) = 0$.

Proposition 1.4. *Let $C \subset R^2$ be a non-empty subset, if we assume that:*

$$\underline{D}_{x_0}(r_1, C) > \overline{D}_{x_0}(r_2, C) \tag{1.12}$$

then

$$r_1 \neq r_2 \quad \textit{and} \quad 0 < \frac{\underline{D}_{x_0}(r_1, C) - \underline{D}_{x_0}(r_2, C)}{1 + \underline{D}_{x_0}(r_1, C)} \leq D_{x_0}(r_1, r_2). \tag{1.13}$$

Proof. Because, by (1.4) with $A = r_2, B = r_1$,

$$\underline{D}_{x_0}(r_1, C)(1 - D_{x_0}(r_1, r_2)) \leq D_{x_0}(r_1, r_2) + \overline{D}_{x_0}(r_2, C),$$

then we receive

$$0 < \frac{\underline{D}_{x_0}(r_1, C) - \underline{D}_{x_0}(r_2, C)}{1 + \underline{D}_{x_0}(r_1, C)} \leq D_{x_0}(r_1, r_2).$$

Hence, by (1.9), $r_1 \neq r_2$. □

2. Some results on simple plane curves

In this section we will restrict the considerations on a given simple plane curve. Now, let $\gamma : [0,1] \to R^2$ be continuous and iniective, denote $\Gamma = \{\gamma(t) | t \in [0,1]\}$ and $x_0 = \gamma(t_0), t_0 \in]0,1[$.

We prove the following proposition:

Proposition 2.1.

$$\exists r_1 \in \Re(x_0) \text{ such that } \underline{D}_{x_0}(r_1, \Gamma) = \min\{\underline{D}_{x_0}(s, \Gamma) | s \in \Re(x_0)\} \tag{2.1}$$

$$\exists r_2 \in \Re(x_0) \text{ such that } \underline{D}_{x_0}(r_2, \Gamma) = \max\{\underline{D}_{x_0}(s, \Gamma) | s \in \Re(x_0)\} \tag{2.2}$$

$$\exists r_3 \in \Re(x_0) \text{ such that } \overline{D}_{x_0}(r_3, \Gamma) = \min\{\overline{D}_{x_0}(s, \Gamma) | s \in \Re(x_0)\} \tag{2.3}$$

$$\exists r_4 \in \Re(x_0) \text{ such that } \overline{D}_{x_0}(r_4,\Gamma) = \max\{\overline{D}_{x_0}(s,\Gamma) | s \in \Re(x_0)\} \tag{2.4}$$

Proof. (2.1). Let (r_n) be a sequence in $\Re(x_0)$ such that:

$$\lim_n \underline{D}_{x_0}(r_n,\Gamma) = \inf\{\underline{D}_{x_0}(s,\Gamma)|s \in \Re(x_0)\}. \tag{2.5}$$

By (1.11) we can assume, without loss of generality:

$$\exists \overline{r} \in \Re(x_0) \text{ such that} \lim_n D_{x_0}(r_n,\overline{r}) = 0.$$

Moreover, by (1.4), with $A = r_n, B = \overline{r}, C = \Gamma$, and by (1.8), we argue:

$$\underline{D}_{x_0}(\overline{r},\Gamma)(1 - D_{x_0}(r_n,\overline{r}) \leq \underline{D}_{x_0}(r_n,\Gamma) + D_{x_0}(r_n,\overline{r}) \qquad \forall n \in N,$$

then we have:

$$\underline{D}_{x_0}(\overline{r},\Gamma) \leq \lim_n \underline{D}_{x_0}(r_n,\Gamma) = \inf\{\underline{D}_{x_0}(s,\Gamma)|s \in \Re(x_0)\},$$

and so the thesis follows.

(2.2). Arguing as in the previous proof, let (r_n) be a sequence in $\Re(x_0)$ and $\overline{r} \in \Re(x_0)$ such that:

$$\lim_n \underline{D}_{x_0}(r_n,\Gamma) = \sup\{\underline{D}_{x_0}(s,\Gamma)|s \in \Re(x_0)\}. \tag{2.6}$$

By (1.4), with $A = \overline{r}, B = r_n, C = \Gamma$, and by (1.8), we obtain:

$$\underline{D}_{x_0}(r_n,\Gamma)(1 - D_{x_0}(r_n,\overline{r}) \leq \underline{D}_{x_0}(\overline{r},\Gamma) + \overline{D}_{x_0}(r_n,r) \qquad \forall n \in N.$$

Then

$$\underline{D}_{x_0}(\overline{r},\Gamma) \geq \lim_n \underline{D}_{x_0}(r_n,\Gamma) = \sup\{\underline{D}_{x_0}(s,\Gamma)|s \in \Re(x_0)\}.$$

The proof of (2.3) (and (2.4)) is similar. □

The elements of $\Re(x_0)$ in (2.1), (2.2) can be considered as different types of "'tangent direction"' to the curve γ at the point x_0 and the elements in (2.4) as "'normal direction"'.

In the classical theory of tangency, the tangent line is strictly related to the limit of secant lines to Γ; this can be a justification for the following proposition:

Proposition 2.2.

$$\exists s_1 \in \Re(x_0) \text{ such that } \underline{D}_{x_0}(s_1,\Gamma) = \liminf_{\Gamma\setminus x_0 \ni x \to x_0} \underline{D}_{x_0}(r_{xx_0},\Gamma) \tag{2.7}$$

$$\exists s_2 \in \Re(x_0) \text{ such that } \underline{D}_{x_0}(s_2,\Gamma) = \limsup_{\Gamma\setminus x_0 \ni x \to x_0} \underline{D}_{x_0}(r_{xx_0},\Gamma) \tag{2.8}$$

$$\exists s_3 \in \Re(x_0) \text{ such that } \overline{D}_{x_0}(s_3,\Gamma) = \liminf_{\Gamma\setminus x_0 \ni x \to x_0} \overline{D}_{x_0}(r_{xx_0},\Gamma) \tag{2.9}$$

$$\exists s_4 \in \Re(x_0) \text{ such that } \overline{D}_{x_0}(s_4,\Gamma) = \limsup_{\Gamma\setminus x_0 \ni x \to x_0} \overline{D}_{x_0}(r_{xx_0},\Gamma). \tag{2.10}$$

where r_{xx_0} is the half-straight line with $v = \frac{x-x_0}{||x-x_0||}$

Proof. (2.7). We can assume that:

$$\exists (x_n) \text{ with } x_n \in \Gamma \setminus \{x_0\} \text{ such that : } \lim_n \underline{D}_{x_0}(r_n,\Gamma) = \liminf_{\Gamma\setminus\{x_0\} \ni x \to x_0} \underline{D}(r_{xx_0},\Gamma),$$

where $r_n = r_{x_n x_0}$. Without loss of generality, we assume

$\exists \overline{r} \in \Re(x_0)$ for which $\lim_n D_{x_0}(r_n,\overline{r}) = 0$.

Then we have: $\lim_n \underline{D}_{x_0}(r_n,\Gamma) = \underline{D}_{x_0}(\overline{r},\Gamma)$.

In fact, we consider (1.4) twice; first with $A = r_n, B = \overline{r}, C = \Gamma$ and next changing A and B. Then

$$\underline{D}_{x_0}(\overline{r},\Gamma)(1 - D_{x_0}(r_n,\overline{r})) \leq \underline{D}_{x_0}(r_n,\Gamma) + D_{x_0}(r_n,\overline{r});$$

$$\underline{D}_{x_0}(r_n,\Gamma)(1 - \overline{D}_{x_0}(r_n,\overline{r})) \leq \underline{D}_{x_0}(\overline{r},\Gamma) + \underline{D}_{x_0}(r_n,\overline{r});$$

hence

$$\underline{D}_{x_0}(\overline{r},\Gamma) \leq \lim_n \underline{D}_{x_0}(r_n,\Gamma) \leq \underline{D}_{x_0}(\overline{r},\Gamma).$$

In a similar way we prove the other equalities. □

We remark that the following general inequalities are true:

$$\min\{\underline{D}_{x_0}(r,\Gamma)|r \in \Re(x_0)\} \leq \inf\{\underline{D}_{x_0}(r_{xx_0},\Gamma)|x \in \Gamma\backslash x_0\} \leq \liminf_{\Gamma\backslash\{x_0\}\ni x\to x_0} \underline{D}_{x_0}(r_{xx_0},\Gamma) \leq$$

$$\leq \limsup_{\Gamma\backslash\{x_0\}\ni x\to x_0} \underline{D}_{x_0}(r_{xx_0},\Gamma) \leq \sup\{\underline{D}_{x_0}(r_{xx_0},\Gamma)|x \in \Gamma\backslash x_0\} \leq \max\{\underline{D}_{x_0}(r,\Gamma)|r \in \Re(x_0)\};$$

$$\min\{\overline{D}_{x_0}(r,\Gamma)|r \in \Re(x_0)\} \leq \inf\{\overline{D}_{x_0}(r_{xx_0},\Gamma)|x \in \Gamma\backslash x_0\} \leq \liminf_{\Gamma\backslash\{x_0\}\ni x\to x_0} \overline{D}_{x_0}(r_{xx_0},\Gamma) \leq$$

$$\leq \limsup_{\Gamma\backslash\{x_0\}\ni x\to x_0} \overline{D}_{x_0}(r_{xx_0},\Gamma) \leq \sup\{\overline{D}_{x_0}(r_{xx_0},\Gamma)|x \in \Gamma\backslash x_0\} \leq \max\{\overline{D}_{x_0}(r,\Gamma)|r \in \Re(x_0)\}.$$

We can prove the following proposition:

Proposition 2.3. *If*

$$\liminf_{\Gamma\backslash\{x_0\}\ni x\to x_0} \overline{D}_{x_0}(r_{xx_0},\Gamma) = 0, \tag{2.11}$$

then

$$\exists r^* \in \Re(x_0) \text{ for which } D_{x_0}(r^*,\Gamma) = 0. \tag{2.12}$$

Proof. By the previous inequalities, we deduce:

$$\min\{\overline{D}_{x_0}(s,\Gamma)|s \in \Re(x_0)\} = 0.$$

Then, from Proposition 2.1, $\exists r^* \in \Re(x_0)$ such that $0 \leq \overline{D}_{x_0}(r^*,\Gamma) = 0$; hence the thesis follows. □

Furthemore, we have the following theorem:

Theorem 2.1.

$$\min\{\overline{D}_{x_0}(s,\Gamma)|s \in \Re(x_0)\} = 0. \tag{2.13}$$

Proof. We can assume: $\min\{\overline{D}_{x_0}(s,\Gamma)|s \in \Re(x_0)\} = \lambda > 0$; then, for every $s \in \Re(x_0)$ we have: $\overline{D}_{x_0}(s,\Gamma) > \frac{\lambda}{2}$. Now, we consider two directions s_1 and s_2; then there exists a ball B about the centre x_0 for which, in such ball, Γ has the half-lines about directions s_1 and s_2 . We can assume: $D(s_1,s_2) < \frac{\lambda}{2}$, we consider, in B, the convex (open) cone C given by s_1 and s_2. We have $C \cap \Gamma = \emptyset$. In effect, if this fact is not true, eventually with a change of the radius of the ball B, we have, for $x \in s_1$:

$$\frac{d(x,s_2)}{d(x,x_0)} = \frac{d(x,x_2)}{d(x,x_0)} \geq \frac{d(x,x_*)}{d(x,x_0)} \geq \frac{d(x,\Gamma)}{d(x,x_0)},$$

where x_2 is a point of s_2 and x_* is a point of Γ.

After a finite steps, we can consider a ball B^* of x_0 such that: $B^* \cap \Gamma = \{x_0\}$. Then a contradiction.

Now the following proposition holds:

Proposition 2.4.

$$\exists r^* \in \Re(x_0) \text{ such that } D_{x_0}(r^*, \Gamma) = 0. \tag{2.14}$$

By proposition 1.4 , we deduce easily the following result.

Proposition 2.5. *If*

$$\exists r^* \in \Re(x_0) \text{ such that } D_{x_0}(\Gamma, r^*) = 0 \tag{2.15}$$

then

$$D_{x_0}(\Gamma, r^*) = D_{x_0}(r^*, \Gamma) = 0.$$

Hence the direction r^* and Γ are tangent in x_0.

The following theorem is true:

Theorem 2.2.

$$\liminf_{\Gamma\setminus\{x_0\}\ni x\to x_0} \overline{D}_{x_0}(r_{xx_0}, \Gamma) = 0. \tag{2.16}$$

Proof. Let $s^* \in \Re(x_0)$ such that: $\overline{D}_{x_0}(s*, \Gamma) = 0$; then there exists a sequence (x_n) for which: $x_n \in s^*$; $x_n \neq x_0$ and $\lim_n \frac{d(x_n,\Gamma)}{d(x_n,x_0)} = 0$. Moreover we can assume there exists a sequence $(y_n) \subset \Gamma$ (and $y_n \neq x_0$) such that we have: $\frac{d(x_n,y_n)}{d(x_n,x_0)} = \frac{d(x_n,\Gamma)}{d(x_n,x_0)}$ and $\lim_n \frac{d(x_n,y_n)}{d(x_n,x_0)} = 0$. From the following inequality: $\frac{d(x_n,y_n)}{d(x_n,x_0)} \geq |\frac{d(x_n,x_0)-d(x_0,y_n)}{d(x_n,x_0)}| = |1 - \frac{d(x_0,y_n)}{d(x_n,x_0)}|$, we have: $\lim_n d(y_n, x_0) = 0$.

We denote $r_n = r_{x_0 y_n}$; we can assume there exists $r^* \in \Re(x_0)$ such that $\lim_n D_{x_0}(r_n, r^*) = 0$.

Then: $\frac{d(x_n,\Gamma)}{d(x_n,x_0)} = \frac{d(x_n,y_n)}{d(x_n,x_0)} \geq \frac{d(x_n,r_n)}{d(x_n,x_0)} = D_{x_0}(s^*, r_n)$.

Hence: $0 \leq \lim_n D_{x_0}(s^*, r_n) \leq \lim_n \frac{d(x_n,\Gamma)}{d(x_n,x_0)} = 0$.

Moreover, from: $D_{x_0}(s^*, r^*) - D_{x_0}(s^*, r_n) \leq D_{x_0}(r_n, s^*)(1 + D_{x_0}(s^*, r_n))$, we deduce $D_{x_0}(s^*, r^*) = 0$. Hence: $s^* = r^*$.

Now, because $D_{x_0}(r_n, \Gamma) - D_{x_0}(r_n, s^*) \leq D_{x_0}(s^*, \Gamma)(1 + D_{x_0}(r_n, s^*)) = 0$; the following condition holds: $0 \leq D_{x_0}(r_n, \Gamma) \leq D_{x_0}(r_n, s^*)$. Hence: $\lim_n D_{x_0}(r_n, \Gamma) = 0$.

3. Open problems

From a theoretical point of view some questions arise in consideration of the obtained results.

-The first is the question to reconsider this point of view with respect to other metric in R^2 not equivalent to euclidean metric, for example with a non homogeneous metric. In such situation it is possible to need a change of the sets $\Re(x)$ with some appropriate "'geodetic"' directions and that the theorem 2.2 is not true.

-In the framework of the euclidean metric we can consider one of the following non-empty sets (see proposition 2.1))

$$T_1 = \left\{r \in \Re(x) | \underline{D}_{x_0}(r, \Gamma) = \min\left\{\underline{D}_{x_0}(s, \Gamma)\right\}\right\}$$
$$T_2 = \left\{r \in \Re(x) | \underline{D}_{x_0}(r, \Gamma) = \max\left\{\underline{D}_{x_0}(s, \Gamma)\right\}\right\}$$
$$N_1 = \left\{r \in \Re(x) | \overline{D}_{x_0}(r, \Gamma) = \min\left\{\overline{D}_{x_0}(s, \Gamma)\right\}\right\}$$

$$N_2 = \left\{ r \in \Re(x) | \overline{D}_{x_0}(r, \Gamma) = \max \left\{ \overline{D}_{x_0}(s, \Gamma) \right\} \right\}.$$

Then we can consider the more interesting question of the existence of plane simple curve for wich one of the sets have, for every point x a number, fixed and greater than 2, of elements.

Of course, starting from proposition 2.2, analogous set can be defined and for it the same question considered.

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