International Electronic Journal of Geometry
Volume 5 No.1 pp. 108–119 (2012) ©IEJG

GENERIC LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE SASAKIAN MANIFOLD

K.L. DUGGAL AND D. J. JIN

(Communicated by H. Hilmi HACISALİHOĞLU)

Abstract. In this paper, we define generic lightlike submanifolds M of an indefinite Sasakian manifold $\bar{M}$. We provide several new results on a class of generic lightlike submanifolds of $\bar{M}$ subject to the conditions: (1) M is totally umbilical, or (2) its screen distribution S(TM) is totally umbilical in M.

1. Introduction

In the classical theory of spacetime, the Riemannian curvature tensor will affect the rate of change of separation of null and timelike curves (see Section 4.1 and 4.2 in [7]). Null curves can represent the histories of photons, the effect of the Riemannian curvature tensor will be to distort or focus small bundles of light rays. While the rest spaces of timelike curves are spacelike subspaces of the tangent spaces, the rest spaces of null curves are lightlike subspaces of the tangent spaces [13]. To investigate this, Hawking and Ellis introduced the notion of so-called screen spaces in section 4.2 of their book [7]. Since for any semi-Riemannian manifold there is a natural existence of lightlike subspaces, in a 1996 book [3] Duggal-Bejancu published their work on the general theory of degenerate (lightlike) submanifolds to fill a gap in the study of submanifolds. Since then there has been very active study on lightlike geometry of submanifolds (see up-to date results in two books [4, 5]). The geometry of lightlike submanifolds is used in mathematical physics, in particular, in general relativity since lightlike submanifolds can be models of different types of horizons (event horizons, Cauchy's horizons, Kruskal's horizons). Although now we have lightlike version of a large variety of Riemannian submanifolds, unfortunately, a general notion of generic lightlike submanifolds (see definition 3.1) has not been introduced as yet. Only there are some limited papers on particular subcases recently studied by Jin [8, 9, 10].

Motivated by the rich existing Riemannian geometry (see two 1980 papers of Yano-Kon [14, 15]) of generic submanifolds, the objective of this paper is to study generic lightlike submanifolds M of an indefinite Sasakian manifold $\bar{M}$ subject to the conditions: (1) M is totally umbilical, or (2) its screen distribution S(TM) is

Date: Received: December 14, 2011 and Accepted: January 06, 2012.
2000 *Mathematics Subject Classification.* 53C25, 53C40, 53C50.
Key words and phrases. Totally umbilical, locally symmetric, indefinite Sasakian manifold.

totally umbilical in M. In section 2, we recall some of fundamental formulas in the theory of r-lightlike submanifolds. In section 3, we define generic lightlike submanifolds. Using this definition, we prove some theorems on totally umbilical generic r-lightlike submanifolds of $\bar{M}$. In section 4, we study generic r-lightlike submanifolds of $\bar{M}$ such that $S(TM)$ is totally umbilical in M. In section 5, we investigate generic r-lightlike submanifolds of an indefinite Sasakian space form $\bar{M}(c)$ satisfy the conditions (1) or (2).

2. Lightlike submanifolds

An odd dimensional semi-Riemannian manifold $\bar{M}$ is called an *indefinite Sasakian manifold* [2] if there exists a contact metric structure $(J,\, \theta,\, \zeta,\, \bar{g})$, where J is a $(1,1)$-type tensor field, θ is a 1-form, ζ is a vector field, called the *characteristic vector field* of $\bar{M}$, and $\bar{g}$ is a semi-Riemannian metric of index $\mu(>0)$ satisfying

$$\begin{aligned} &J^2X = -X + \theta(X)\zeta,\; J\zeta = 0,\; \theta\circ J = 0,\; \theta(\zeta) = 1,\\ &\bar{g}(\zeta,\zeta) = \epsilon, \quad \bar{g}(JX, JY) = \bar{g}(X, Y) - \epsilon\,\theta(X)\theta(Y), \\ &\theta(X) = \epsilon\bar{g}(\zeta, X), \quad d\theta(X, Y) = \bar{g}(JX, Y),\; \epsilon = \pm 1, \end{aligned} \tag{2.1}$$

$$\bar{\nabla}_X\zeta = JX, \tag{2.2}$$

$$(\bar{\nabla}_X J)Y = \epsilon\,\theta(Y)X - \bar{g}(X, Y)\zeta, \tag{2.3}$$

for any vector fields X and Y of $\bar{M}$, where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$. Without any loss of generality, in this paper we assume that $\epsilon = 1$, that is, the characteristic vector field ζ of $\bar{M}$ is spacelike.

Let (M, g) be an m-dimensional lightlike submanifold of an $(m+n)$-dimensional indefinite Sasakian manifold $(\bar{M}, \bar{g})$. Then the radical distribution $Rad(TM) = TM\cap TM^\perp$ is a vector subbundle of the tangent bundle TM and the normal bundle $TM^\perp$, of rank $r\,(1 \le r \le \min\{m,\, n\})$. In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^\perp)$ of $Rad(TM)$ in TM and $TM^\perp$ respectively, called the *screen* and *co-screen distributions* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM)\,, \quad TM^\perp = Rad(TM) \oplus_{orth} S(TM^\perp), \tag{2.4}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike submanifold by $(M, g, S(TM), S(TM^\perp))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. We use the same notation for any other vector bundle. We use the following range of indices:

$$i,\, j,\, k,\, ... \in \{1,\, ...\,,\, r\}, \qquad \alpha,\, \beta,\, \gamma,\, ... \in \{r\,+\,1,\, ...\,,\, n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}_{|M}$ and $TM^\perp$ in $S(TM)^\perp$ respectively and let $\{N_1,\, \cdots,\, N_r\}$ be a lightlike basis of $ltr(TM)_{|\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,$$

where $\{\xi_1,\, \cdots,\, \xi_r\}$ is a lightlike basis of $Rad(TM)_{|\mathcal{U}}$. Then we have

$$\begin{aligned} T\bar{M} = \;& TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ =\;& \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^\perp). \end{aligned} \tag{2.5}$$

We say that a lightlike submanifold $(M, g, S(TM), S(TM^\perp))$ of $\bar{M}$ is

(1) *r-lightlike* if $1 \le r < \min\{m,\, n\}$;

(2) *co-isotropic* if $1 \le r = n < m$;
(3) *isotropic* if $1 \le r = m < n$;
(4) *totally lightlike* if $1 \le r = m = n$.

The above three classes (2)∼(4) are particular cases of the class (1) as follows: $S(TM^{\perp}) = \{0\}$, $S(TM) = \{0\}$ and $S(TM) = S(TM^{\perp}) = \{0\}$ respectively. The geometry of r-lightlike submanifolds is more general than that of the other three type submanifolds. For this reason, we consider only r-lightlike submanifolds $M \equiv (M, g, S(TM), S(TM^{\perp}))$, with following local quasi-orthonormal field of frames of $\bar{M}$:

$$\{\xi_1, \cdots, \xi_r\,,\ N_1, \cdots, N_r\,,\ F_{r+1}, \cdots, F_m\,,\ W_{r+1}, \cdots, W_n\},$$

where $\{F_{r+1}, \cdots, F_m\}$ and $\{W_{r+1}, \cdots, W_n\}$ are orthonormal bases of $S(TM)$ and $S(TM^{\perp})$, respectively. Now we set $\epsilon_\alpha = \bar{g}(W_\alpha, W_\alpha)(= \pm 1)$ is the sign of W_α.

Let P be the projection morphism of TM on $S(TM)$ with respect to the first equation of the decomposition (2.4). For any r-lightlike submanifold, the local Gauss-Weingartan formulas of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X, Y) N_i + \sum_{\alpha=r+1}^{n} h_\alpha^s(X, Y) W_\alpha, \tag{2.6}$$

$$\bar{\nabla}_X N_i = -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X) N_j + \sum_{\alpha=r+1}^{n} \rho_{i\alpha}(X) W_\alpha, \tag{2.7}$$

$$\bar{\nabla}_X W_\alpha = -A_{W_\alpha} X + \sum_{i=1}^{r} \phi_{\alpha i}(X) N_i + \sum_{\beta=r+1}^{n} \sigma_{\alpha\beta}(X) W_\beta, \tag{2.8}$$

$$\nabla_X PY = \nabla^*_X PY + \sum_{i=1}^{r} h_i^*(X, PY)\xi_i, \tag{2.9}$$

$$\nabla_X \xi_i = -A^*_{\xi_i} X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j, \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$, where ∇ and ∇^* are induced linear connections on TM and $S(TM)$ respectively, the bilinear forms h_i^{ℓ} and h_α^s are called the *local lightlike* and *screen second fundamental forms* on TM respectively, h_i^* are called the *local second fundamental forms* on $S(TM)$. A_{N_i}, $A^*_{\xi_i}$ and A_{W_α} are linear operators on TM and τ_{ij}, $\rho_{i\alpha}$, $\phi_{\alpha i}$ and $\sigma_{\alpha\beta}$ are 1-forms on TM. Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and both h_i^{ℓ} and h_α^s are symmetric. From the fact $h_i^{\ell}(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi_i)$, we know that each h_i^{ℓ} are independent of the choice of $S(TM)$. We say that

$$h(X, Y) = \sum_{i=1}^{r} h_i^{\ell}(X, Y) N_i + \sum_{\alpha=r+1}^{n} h_\alpha^s(X, Y) W_\alpha$$

is the *second fundamental tensor* of M. The induced connection ∇ on TM is not metric and satisfies

$$(\nabla_X g)(Y, Z) = \sum_{i=1}^{r} \{h_i^{\ell}(X, Y)\,\eta_i(Z) + h_i^{\ell}(X, Z)\,\eta_i(Y)\}, \tag{2.11}$$

for all $X, Y \in \Gamma(TM)$, where η_is are the 1-forms such that

$$\eta_i(X) = \bar{g}(X, N_i),\ \forall\, X \in \Gamma(TM).$$

But the connection ∇^* on $S(TM)$ is metric. The above three local second fundamental forms are related to their shape operators by

$$(2.12)\ g(A^*_{\xi_i}X,Y) = h^\ell_i(X,Y) + \sum_{j=1}^{r} h^\ell_j(X,\xi_i)\eta_j(Y),\ \bar{g}(A^*_{\xi_i}X,N_j) = 0,$$

$$(2.13)\ g(A_{W_\alpha}X,Y) = \epsilon_\alpha h^s_\alpha(X,Y) + \sum_{i=1}^{r} \phi_{\alpha i}(X)\eta_i(Y),\ \bar{g}(A_{W_\alpha}X,N_i) = \epsilon_\alpha\rho_{i\alpha}(X),$$

$$(2.14)\ g(A_{N_i}X,PY) = h^*_i(X,PY), \qquad \eta_j(A_{N_i}X) + \eta_i(A_{N_j}X) = 0,$$

where X, $Y \in \Gamma(TM)$. Replacing Y by ξ_j to $(2.12)_1$, we have

$$(2.15) \qquad h^\ell_i(X,\xi_j) + h^\ell_j(X,\xi_i) = 0, \quad h^\ell_i(X,\xi_i) = 0, \quad h^\ell_i(\xi_j,\xi_k) = 0,$$

for all $X \in \Gamma(TM)$. Also, replacing X by ξ_j to $(2.12)_1$ and using (2.15), we have

$$(2.16) \qquad h^\ell_i(X,\xi_j) = g(X, A^*_{\xi_i}\xi_j), \quad A^*_{\xi_i}\xi_j + A^*_{\xi_j}\xi_i = 0, \quad A^*_{\xi_i}\xi_i = 0.$$

For any r-lightlike submanifold, replacing Y by ξ_i to (2.13), we have

$$(2.17) \qquad h^s_\alpha(X,\xi_i) = -\epsilon_\alpha\phi_{\alpha i}(X),\ \forall\, X \in \Gamma(TM).$$

3. Generic lightlike submanifolds of $\bar{M}$

In case g is non-degenerate, there exists a class of submanifolds of an almost complex manifold $\bar{M}$. We say that M is a generic (anti-holomorphic) submanifold of $\bar{M}$ if the normal bundle $TM^\perp$ of M is mapped into the tangent bundle TM by action of the structure tensor J of $\bar{M}$, i.e., $J(TM^\perp) \subset TM$ [14, 15]. The purpose of this section is to extend and study the concept of generic submanifold in case M is a lightlike submanifold of an almost complex manifold $\bar{M}$.

Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^* = TM/Rad(TM)$ considered by Kupeli [12]. Thus all screen distributions $S(TM)$ are mutually isomorphic. Moreover, while TM is lightlike, all $S(TM)$ are non-degenerate. Due to these reasons, we define generic lightlike submanifolds of an almost complex manifold $\bar{M}$ as follow:

Definition 3.1. We say that M is *generic lightlike submanifold* [11] of an almost complex manifold $\bar{M}$ if there exists a screen distribution $S(TM)$ of M such that

$$(3.1) \qquad J(S(TM)^\perp) \subset S(TM).$$

Example 3.1. Any lightlike hypersurface M of an indefinite Sasakian manifold $\bar{M}$ is a generic lightlike submanifold of $\bar{M}$ [8]. Also, any 1-lightlike submanifold M of codimension 2 of an indefinite Sasakian manifold $\bar{M}$ is a generic lightlike submanifold of $\bar{M}$ [9, 10].

From the decomposition (2.5) of $T\bar{M}$, the vector field ζ is decomposed by

$$(3.2) \qquad \zeta = \omega + \sum_{i=1}^{r} a_i\,\xi_i + \sum_{i=1}^{r} b_i\,N_i + \sum_{\alpha=r+1}^{n} e_\alpha\,W_\alpha,$$

where ω is a smooth vector field of $S(TM)$ and $a_i = \theta(N_i)$, $b_i = \theta(\xi_i)$ and $e_\alpha = \epsilon_\alpha\theta(W_\alpha)$ are smooth functions on $\bar{M}$.

Theorem 3.1. *Let M be a generic r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then the characteristic vector field ζ is not normal to M.*

Proof. Assume that the vector field ζ is normal to M. Then (3.2) reduce to

$$\zeta = \sum_{i=1}^{r} a_i \xi_i + \sum_{\alpha=r+1}^{n} e_\alpha W_\alpha.$$

Applying $\bar{\nabla}_X$ to this and using (2.2), (2.6), (2.8), (2.10) and (2.17), we have

$$\begin{aligned} JX &= \sum_{i=1}^{r} (Xa_i)\xi_i + \sum_{\alpha=r+1}^{n} (Xe_\alpha) W_\alpha \\ &+ \sum_{i=1}^{r} a_i \{ -A^*_{\xi_i} X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j + \sum_{j=1}^{r} h^\ell_j(X, \xi_i) N_j - \sum_{\alpha=r+1}^{n} \epsilon_\alpha \phi_{\alpha i}(X) W_\alpha \} \\ &+ \sum_{\alpha=r+1}^{n} e_\alpha \{ -A_{W_\alpha} X + \sum_{j=1}^{r} \phi_{\alpha j}(X) N_j + \sum_{\beta=r+1}^{n} \sigma_{\alpha\beta}(X) W_\beta \}, \end{aligned}$$

for any $X \in \Gamma(TM)$. Taking the scalar product with ξ_k and $JN_k \in \Gamma(TM)$ in this equation by turns and using (2.1), $(2.15)_1$ and (3.1), we have respectively

$$\sum_{\alpha=r+1}^{n} e_\alpha \phi_{\alpha k}(X) - \sum_{i=1}^{r} a_i h^\ell_i(X, \xi_k) = -g(X, J\xi_k),$$

$$\eta_k(X) = a_k \theta(X) - \sum_{i=1}^{r} a_i h^\ell_i(X, JN_k) - \sum_{\alpha=r+1}^{n} \epsilon_\alpha e_\alpha h^s_\alpha(X, JN_k).$$

Replacing X by ξ_k to the second equation of the above relations and using the first equation with $X = JN_k$ and $b_k = 0$, we have the following impossible result:

$$1 = \sum_{\alpha=r+1}^{n} e_\alpha \phi_{\alpha k}(JN_k) - \sum_{i=1}^{r} a_i h^\ell_i(\xi_k, JN_k) = -g(J\xi_k, JN_k) = -1.$$

Thus the characteristic vector field ζ of $\bar{M}$ is not normal to M. □

Definition 3.2. An r-lightlike submanifold M of $\bar{M}$ is said to be *totally umbilical* [6] if there is a smooth vector field $\mathcal{H} \in \Gamma(tr(TM))$ such that

$$h(X, Y) = \mathcal{H}\, g(X, Y), \ \forall\, X,\, Y \in \Gamma(TM).$$

In case $\mathcal{H} = 0$, we say that M is *totally geodesic.*

It is easy to see [6] that M is totally umbilical if and only if, on each coordinate neighborhood $\mathcal{U}$, there exist smooth functions A_i and B_α such that

$$h^\ell_i(X, Y) = A_i\, g(X, Y), \ h^s_\alpha(X, Y) = B_\alpha\, g(X, Y), \ \forall\, X,\, Y \in \Gamma(TM). \tag{3.3}$$

Theorem 3.2. *Let M be a totally umbilical generic r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then the vector field ζ is not tangent to M.*

Proof. Assume that ζ is tangent to M. Using (2.2) and (2.6), we obtain

$$JX = \nabla_X \zeta + \sum_{j=1}^{r} h^\ell_j(X, \zeta) N_j + \sum_{\beta=r+1}^{n} h^s_\beta(X, \zeta) W_\beta.$$

Taking the scalar product with ξ_i and W_α in this equation by turns, we have

$$h^\ell_i(X, \zeta) = -g(X, J\xi_i), \quad h^s_\alpha(X, \zeta) = -\epsilon_\alpha g(X, JW_\alpha),$$

respectively. Since M is totally umbilical, from this equations and (3.3), we obtain

$$A_i\, g(X,\zeta) = -g(X, J\xi_i), \quad B_\alpha\, g(X,\zeta) = -\epsilon_\alpha g(X, JW_\alpha).$$

Replacing X by JN_i to the first equation of the above relations and X by JW_α to the second equation and using $b_i = \bar{g}(\zeta, \xi_i) = 0$ for all i and $e_\alpha = \epsilon_\alpha \bar{g}(\zeta, W_\alpha) = 0$ for all α, we deduce the following two impossible results:

$$0 = A_i\, 0 = A_i\, g(JN_i, \zeta) = -g(JN_i, J\xi_i) = -1,$$
$$0 = B_\alpha\, 0 = B_\alpha\, g(JW_\alpha, \zeta) = -\epsilon_\alpha g(JW_\alpha, JW_\alpha) = -1,$$

respectively. Thus the characteristic vector field ζ is not tangent to M. □

Combining Theorem 3.1 and 3.2, we have the following theorem.

Theorem 3.3. *Let M be a totally umbilical generic r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then ζ is neither tangent nor normal to M.*

Theorem 3.4. *Let M be a totally umbilical generic r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then M is totally geodesic and there exists $k \in \{1, \ldots, r\}$ such that $b_k = \theta(\xi_k) = \bar{g}(\zeta, \xi_k) \neq 0$. Moreover, all the induced connections of M are metric connections.*

Proof. First of all, we prove that there exists $k \in \{1, \ldots, r\}$ such that $b_k = \theta(\xi_k) \neq 0$. Assume that $b_i = \theta(\xi_i) = 0$ for all i. If M is totally umbilical, we obtain

$$(3.4) \qquad h_j^\ell(X, \xi_i) = h_\alpha^s(X, \xi_i) = \phi_{\alpha i}(X) = 0, \quad \bar{\nabla}_X \xi_i = \nabla_X \xi_i, \quad \forall\, X \in \Gamma(TM).$$

Applying $\bar{\nabla}_X$ to $\bar{g}(\zeta, \xi_i) = 0$ and using (2.2), (2.6) and (2.10), we have

$$g(X, J\xi_i) + \bar{g}(A^*_{\xi_i} X, \zeta) = 0, \quad \forall\, X \in \Gamma(TM).$$

As M is totally umbilical, we have $A^*_{\xi_i} X = A_i PX$ by $(2.12)_1$. Thus we have

$$A_i\, \bar{g}(X, \zeta) + g(X, J\xi_i) = 0, \quad \forall\, X \in \Gamma(TM).$$

Replacing X by JN_i in this and using (2.1) and $g(J\xi_i, JN_i) = 1$, we have

$$0 = A_i\, \bar{g}(\zeta, JN_i) + g(J\xi_i, JN_i) = 1.$$

It is a contradiction. Thus we have $b_k = \bar{g}(\zeta, \xi_k) \neq 0$ for some k.

Next, applying $\bar{\nabla}_X$ to $\bar{g}(J\xi_i, W_\alpha) = 0$ for all i and using (2.1), (2.3), (2.8), (2.10), (2.12), (2.13), (3.1), $(3.4)_3$ and the fact M is totally umbilical, we have

$$A_i\, g(X, JW_\alpha) = \epsilon_\alpha B_\alpha\, g(X, J\xi_i), \ \forall\, X \in \Gamma(TM) \text{ and } \forall\, i.$$

Replacing X by $J\xi_k$ for some k such that $b_k \neq 0$, X by JN_i and X by JW_α to this equation by turns and using the fact $b_k \neq 0$, we have respectively

$$A_i e_\alpha = \epsilon_\alpha B_\alpha b_i, \ A_i a_i e_\alpha = \epsilon_\alpha B_\alpha (a_i b_i - 1), \ A_i (e_\alpha^2 - 1) = \epsilon_\alpha B_\alpha b_i e_\alpha.$$

Substituting the first equation of the above relations into the second and third equations by turns, we get $A_i = 0$ for all i and $B_\alpha = 0$ for all α. Thus we have

$$\mathcal{H} = \sum_{i=1}^{r} A_i N_i + \sum_{\alpha=r+1}^{n} B_\alpha W_\alpha = 0.$$

Thus M is totally geodesic. Final statement on the induced metric connections follows from [3, page 166], which completes the proof. □

The local Weingarten formula (2.7) of M becomes

$$\bar{\nabla}_X N = -A_N X + \nabla^{\ell}_X N + D^s(X, N), \quad \forall\, X \in \Gamma(TM), \;\; N \in \Gamma(ltr(TM)),$$

where ∇^{ℓ} is a linear connection on $ltr(TM)$. We call ∇^{ℓ} the *lightlike transversal connection* of M. In this case, for any $N_i \in \Gamma(ltr(TM))$ and $X \in \Gamma(TM)$, we have

$$\nabla^{\ell}_X N_i = \sum_{j=1}^{r} \tau_{ij}(X) N_j, \quad D^s(X, N_i) = \sum_{\alpha=r+1}^{n} \rho_{i\alpha}(X) W_{\alpha}.$$

Definition 3.3. We define the curvature tensor R^{ℓ} of the lightlike transversal connection ∇^{ℓ} on the lightlike transversal vector bundle $ltr(TM)$ by

$$R^{\ell}(X, Y)N = \nabla^{\ell}_X \nabla^{\ell}_Y N - \nabla^{\ell}_Y \nabla^{\ell}_X N - \nabla^{\ell}_{[X,Y]} N, \tag{3.5}$$

for all X, $Y \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$. We say that the lightlike transversal connection ∇^{ℓ} of M is *flat* [8] if R^{ℓ} vanishes identically on M.

Proposition 3.1. *Let M be a lightlike submanifold of a semi-Riemannian $(\bar{M}, \bar{g})$. Then the lightlike transversal connection ∇^{ℓ} of M is flat if and only if each 1-forms τ_{ij} is a solution of the following equations*

$$2d\tau_{ij}(X, Y) + \sum_{k=1}^{r} \{\tau_{ik}(Y)\tau_{kj}(X) - \tau_{ik}(X)\tau_{kj}(Y)\} = 0, \tag{3.6}$$

for all X, $Y \in \Gamma(TM)$ and i, $j \in \{1, \cdots, r\}$.

Proof. Applying the operator ∇^{ℓ}_X to $\nabla^{\ell}_Y N_i = \sum_{j=1}^{r} \tau_{ij}(Y) N_j$, we have

$$\nabla^{\ell}_X \nabla^{\ell}_Y N_i = \sum_{j=1}^{r} \{X(\tau_{ij}(Y)) + \sum_{k=1}^{r} \tau_{ik}(Y)\tau_{kj}(X)\} N_j.$$

By straightforward calculations from this equation and (2.5), we have

$$R^{\ell}(X, Y)N_i = \sum_{j=1}^{r} \{2d\tau_{ij}(X, Y) + \sum_{k=1}^{r} [\tau_{ik}(Y)\tau_{kj}(X) - \tau_{ik}(X)\tau_{kj}(Y)]\} N_j,$$

for all X, $Y \in \Gamma(TM)$ and i. From this result we deduce our assertion. □

Definition 3.4. We say that M is *locally symmetric* [8] if its curvature tensor R be parallel, i.e., have vanishing covariant differential, $\nabla R = 0$.

Theorem 3.5. *Let M be a totally umbilical generic r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with flat lightlike transversal connection. If M is locally symmetric, then M is a space of constant positive curvature* 1.

Proof. Consider locally vector fields V_i on M and its 1-forms v_i given by

$$V_i = -J\xi_i, \qquad v_i(X) = -g(X, V_i), \tag{3.7}$$

for all $X \in \Gamma(TM)$ and $i \in \{1, \ldots, r\}$. As M is totally umbilical, by Theorem 3.4, we get $b_k \neq 0$ for some k, $h^{\ell}_i = h^s_{\alpha} = 0$ and $A^*_{\xi_i} = \phi_{\alpha i} = 0$ for all i and α. Applying $\bar{\nabla}_X$ to $\bar{g}(\zeta, \xi_i) = b_i$ for all i and using (2.2), (2.10) and $(3.4)_3$, we obtain

$$Xb_i + \sum_{j=1}^{r} b_j \tau_{ji}(X) = -v_i(X), \quad \forall\, X \in \Gamma(TM). \tag{3.8}$$

Applying $\bar{\nabla}_X$ to $V_i = -J\xi_i$ and using (2.3), (2.6), (2.10), $(3.4)_3$ and (3.7), we get

$$\nabla_X V_i = -b_i X - \sum_{j=1}^{r} \tau_{ji}(X) V_j. \tag{3.9}$$

Substituting (3.9) into the right term of the equation

$$R(X, Y)V_i = \nabla_X \nabla_Y V_i - \nabla_Y \nabla_X V_i - \nabla_{[X, Y]} V_i,$$

and using (3.6), (3.8) and the fact ∇ is torsion free connection, we have

$$R(X, Y)V_i = v_i(X)Y - v_i(Y)X. \tag{3.10}$$

Applying ∇_Y to $(3.7)_2$ and using (3.7) and (3.9), we have

$$(\nabla_X v_i)(Y) = b_i g(X, Y) - \sum_{j=1}^{r} \tau_{ji}(X) v_j(Y). \tag{3.11}$$

Applying ∇_Z to (3.10) and using (3.10) and the fact $\nabla_Z R = 0$, we have

$$R(X, Y)\nabla_Z V_i = (\nabla_Z v_i)(X)Y - (\nabla_Z v_i)(Y)X, \ \forall\, i. \tag{3.12}$$

Taking $i = k$ in (3.12) such that $b_k \neq 0$ and Substituting (3.10) and (3.11) in (3.12) with $i = k$ and using (3.10), we obtain

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \quad \forall\, X,\, Y \in \Gamma(TM),$$

due to $b_k \neq 0$. Thus M is a space of constant positive curvature 1. $\square$

4. Totally umbilical screen distributions in M

Definition 4.1. A screen distribution $S(TM)$ of M is said to be *totally umbilical* [6] in M if, for each locally second fundamental form h^*_i, there exist smooth functions C_i on any coordinate neighborhood $\mathcal{U}$ in M such that

$$h^*_i(X, PY) = C_i\, g(X, Y), \ \forall\, X,\, Y \in \Gamma(TM). \tag{4.1}$$

In case $h^*_i = 0$ for all i, we say that $S(TM)$ is *totally geodesic* in M.

Due to $(2.14)_1$ and (4.1), we know that $S(TM)$ is totally umbilical in M if and only if each shape operators A_{N_i} of $S(TM)$ satisfies

$$g(A_{N_i} X, PY) = C_i\, g(X, PY), \ \forall\, X,\, Y \in \Gamma(TM), \tag{4.2}$$

for some smooth functions C_i on $\mathcal{U} \subseteq M$.

Theorem 4.1. *Let M be a generic r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. If $S(TM)$ is totally umbilical in M, then $S(TM)$ is totally geodesic in M and $a_i = \theta(N_i) = \bar{g}(\zeta, N_i) = 0$ for all $i \in \{1, \cdots, r\}$.*

Proof. Applying $\bar{\nabla}_X$ to $\bar{g}(JN_k, N_j) = 0$ for some k and j such that $k \neq j$, and using (2.3), (2.7), (2.14), (3.1) and the fact $S(TM)$ is totally umbilical, we have

$$a_k \eta_j(X) + C_k\, g(X, JN_j) = a_j \eta_k(X) + C_j\, g(X, JN_k).$$

Replacing X by ξ_j to this and using the facts $\eta_i(\xi_j) = \delta_{ij}$ and $JN_i \in \Gamma(S(TM))$ for all i, we have $a_k = 0$. If we take $(k,\, j) = (1,\, 2),\ (2,\, 3),\ \cdots,\ (r-1,\, r)$ and $(r,\, 1)$ by turns and using the above method, we have $a_i = 0$ for all $i \in \{1, \cdots, r\}$ and

$$C_k\, g(X, JN_j) = C_j\, g(X, JN_k), \ \forall\, X \in \Gamma(TM),$$

for some k and j such that $k \neq j$. Replacing X by $J\xi_j$ in this equation and using the fact $a_i = 0$ for all i, we have $C_k = 0$. By the above method for (k, j), we have $C_i = 0$ for all $i \in \{1, \cdots, r\}$. Thus we have our assertion. □

Definition 4.2. A lightlike submanifold M is said to be *irrotational*[12] if $\bar{\nabla}_X \xi_i \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi_i \in \Gamma(Rad(TM))$ for all i.

For any r-lightlike submanifold M, the above definition is equivalent to

$$h^\ell_j(X, \xi_i) = 0, \; h^s_\alpha(X, \xi_i) = \phi_{\alpha i}(X) = 0, \; \forall\, X \in \Gamma(TM). \tag{4.3}$$

Theorem 4.2. *Let M be a generic r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ such that $S(TM)$ is totally umbilical in M. Then M is not irrotational.*

Proof. Applying $\bar{\nabla}_X$ to $\bar{g}(J\xi_i, N_j) = 0$ for all i and j, and using (2.3), (2.6), (2.7), (2.10), (2.12), (2.14), (3.1) and the fact $h^*_j = 0$ due to Theorem 4.1, we have

$$b_i \eta_j(X) + h^\ell_i(X, JN_j) = h^*_j(X, J\xi_i) = 0, \; \forall\, X \in \Gamma(TM). \tag{4.4}$$

Assume that M is irrotational. Replacing X by ξ_j to (4.4) and using $(3.3)_1$, we have $b_i = 0$ for all i. Thus the equation (4.4) deduce to

$$h^\ell_i(X, JN_j) = 0, \; \forall\, X \in \Gamma(TM). \tag{4.5}$$

Applying $\bar{\nabla}_X$ to $\bar{g}(\zeta, \xi_i) = 0$ and using (2.2), (2.6), (2.10), (2.12), (3.2) and (4.3), we have

$$h^\ell_i(X, \omega) = -g(X, J\xi_i), \; \forall\, X \in \Gamma(TM). \tag{4.6}$$

Replacing X by JN_i in this equation and using (4.5) with $X = \omega$, we have

$$0 = h^\ell_i(JN_i, \omega) = -g(JN_i, J\xi_i) = -1.$$

It is a contradiction. Thus M is not irrotational. □

Since any totally umbilical r-lightlike submanifold of $\bar{M}$ is irrotational due to (3.4), by Theorem 4.2, we have the following result:

Corollary 4.1. *There exist no totally umbilical generic r-lightlike submanifolds M of an indefinite Sasakian manifold $\bar{M}$ such that $S(TM)$ is totally umbilical in M.*

Theorem 4.3. *Let M be a generic r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ such that $S(TM)$ is totally umbilical in M and each shape operator $A^*_{\xi_i}$ is self-adjoint. Then M is locally a product manifold $M_r \times M_s$ where M_r and M_s are leaves of $Rad(TM)$ and $S(TM)$ respectively and $r + s = m$.*

Proof. Using (2.12) and the fact that h^ℓ_i are symmetric, we have

$$g(A^*_{\xi_i}X, Y) - g(X, A^*_{\xi_i}Y) = \sum_{k=1}^{r} \{h^\ell_k(X, \xi_i)\eta_k(Y) - h^\ell_k(Y, \xi_i)\eta_k(X)\}.$$

From this, (2.15) and (2.16), we show that $A^*_{\xi_i}$ are self-adjoint on $\Gamma(TM)$ with respect to g if and only if $h^\ell_i(X, \xi_j) = 0$ for all $X \in \Gamma(TM)$, i and j if and only if $A^*_{\xi_i}\xi_j = 0$ for all i, j. It follows from (2.10) that the radical distribution $Rad(TM)$ of a lightlike submanifold M with the self-adjoint shape operators $A^*_{\xi_i}$ is an auto-parallel distribution.

As $S(TM)$ is totally geodesic in M by Theorem 4.1, we show that $S(TM)$ is also an auto-parallel distribution on M due to (2.9). Using the decomposition (2.4) and the decomposition theorem of De Rham [1], we have $M = M_r \times M_s$, where M_r and M_s are the leaves of $Rad(TM)$ and $S(TM)$ respectively. □

5. Generic lightlike submanifolds of $\bar{M}(c)$

An indefinite Sasakian manifold $\bar{M}$ is called an *indefinite Sasakian space form*, denoted by $\bar{M}(c)$, if it has the constant J-sectional curvature c [8]. The curvature tensor $\bar{R}$ of the indefinite Sasakian space form $\bar{M}(c)$ is given by

$$\begin{aligned}(5.1)\qquad 4\bar{R}(X,Y)Z &= (c+3)\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\}\\ &+(c-1)\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X + \bar{g}(X,Z)\theta(Y)\zeta\\ &-\bar{g}(Y,Z)\theta(X)\zeta + \bar{g}(JY,Z)JX + \bar{g}(JZ,X)JY - 2\bar{g}(JX,Y)JZ\},\end{aligned}$$

for any vector fields X, Y and Z in $\bar{M}$.

Theorem 5.1. *Let M be a totally umbilical generic r-lightlike submanifold of an indefinite Sasakian space form $\bar{M}(c)$. Then we have $c = 1$.*

Proof. As M is totally umbilical, M is totally geodesic due to Theorem 3.4. Thus, using (2.6) we have

$$(5.2)\qquad \bar{R}(X,Y)Z = R(X,Y)Z, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM).$$

In case $b_k \neq 0$. Taking the scalar product with ξ_k to (5.1) and using (5.2), we get

$$\begin{aligned}&(c-1)\{b_k g(X,Z)\theta(Y) - b_k g(Y,Z)\theta(X) - \bar{g}(JY,Z)g(X,J\xi_k)\\ &\quad - \bar{g}(JZ,X)g(Y,J\xi_k) + 2\bar{g}(JX,Y)g(Z,J\xi_k)\} = 0,\ \forall\, X,\, Y,\, Z \in \Gamma(TM).\end{aligned}$$

Replacing Z by $J\xi_k$ and Y by ξ_k in this equation and using (2.1) and (3.1), we have

$$4b_k^2(c-1)g(X,J\xi_k) = 0,\ \forall\, X \in \Gamma(TM).$$

Replace X by $J\xi_k$ in this equation, we obtain $b_k^4(c-1) = 0$. Since $b_k \neq 0$ and $(c-1)$ is a constant, we have $c = 1$. In case $b_k = 0$. Taking the scalar product with ξ_k to (5.1) and using (5.2), we obtain

$$\begin{aligned}&(c-1)\{-\bar{g}(JY,Z)g(X,J\xi_k) - \bar{g}(JZ,X)g(Y,J\xi_k)\\ &\quad + 2\bar{g}(JX,Y)g(Z,J\xi_k)\} = 0,\ \forall\, X,\, Y,\, Z \in \Gamma(TM).\end{aligned}$$

Replace Z by JN_k and Y by ξ_k in this equation and use (2.1), we have

$$3(c-1)g(X,J\xi_k) = 0,\ \forall\, X \in \Gamma(TM),$$

because $\eta_i(JN_k) = 0$ by (3.1) and $g(J\xi_k, JN_k) = 1$. Replace X by JN_k in this equation, we get $c = 1$. □

Corollary 5.1. *There exist no totally umbilical generic r-lightlike submanifolds M of an indefinite Sasakian space form $\bar{M}(c)$ with $c \neq 1$.*

Theorem 5.2. *Let M be a totally umbilical generic r-lightlike submanifold of an indefinite Sasakian space form $\bar{M}(c)$. Then M is a locally symmetric space of constant positive curvature* 1 *and the lightlike transversal connection ∇^ℓ is flat.*

Proof. Using the Gauss-Weingarten formulas (2.6)∼(2.8) for M, for all $X, Y \in \Gamma(TM)$, we obtain the following Codazzi equation for M:

$$(5.3)\qquad \bar{g}(\bar{R}(X,Y)N_i,\, \xi_j) = 2d\tau_{ij}(X,Y) + \sum_{k=1}^{r}\{\tau_{ik}(Y)\tau_{kj}(X) - \tau_{ik}(X)\tau_{kj}(Y)\}.$$

On the other hand, from (5.1) and the fact $c = 1$ by Theorem 5.1, we have

$$\bar{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \quad \forall\, X,\, Y,\, Z \in \Gamma(T\bar{M}).$$

Using this, we show that $\bar{g}(\bar{R}(X,Y)N_i,\,\xi_j) = -\bar{g}(\bar{R}(X,Y)\xi_j,\,N_i) = 0$. From this equation and (5.3), we have (3.6). Thus, by Proposition 3.1, the lightlike transversal connection ∇^ℓ of M is flat. From (5.2) and the last equation, we have

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM). \tag{5.4}$$

Applying ∇_U to (5.4) and using Theorem 3.4, we have $(\nabla_U R)(X,Y)Z = 0$, i.e., $\nabla_U R = 0$ for all $U \in \Gamma(TM)$. Thus M is a locally symmetric space of constant positive curvature 1. □

Theorem 5.3. *Let M be a generic r-lightlike submanifold of an indefinite Sasakian space form $\bar{M}(c)$ such that $S(TM)$ is totally umbilical in M. Then we have $c = -3$.*

Proof. Using the local Gauss-Weingarten formulas for M, we obtain

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z,\,N_i) &= \bar{g}(R(X,Y)Z,\,N_i) \\ &+ \sum_{j=1}^{r} \{h_j^\ell(X,Z)\eta_i(A_{N_j}Y) - h_j^\ell(Y,Z)\eta_i(A_{N_j}X)\}, \\ &+ \sum_{\alpha=r+1}^{n} \epsilon_\alpha\{h_\alpha^s(X,Z)\rho_{i\alpha}(Y) - h_\alpha^s(Y,Z)\rho_{i\alpha}(X)\}, \end{aligned} \tag{5.5}$$

for all $X,\, Y,\, Z \in \Gamma(TM)$. By using (2.9) and (2.10), we have

$$\begin{aligned} g(R(X,Y)PZ,\,N_i) &= (\nabla_X h_i^*)(Y,PZ) - (\nabla_Y h_i^*)(X,PZ) \\ &+ \sum_{j=1}^{r} \{h_j^*(X,PZ)\tau_{ij}(Y) - h_j^*(Y,PZ)\tau_{ij}(X)\}. \end{aligned} \tag{5.6}$$

Consider locally lightlike vector fields U_i on M and its 1-forms u_i given by

$$U_i = -JN_i, \quad u_i(X) = -g(X,U_i),\ \forall\, X \in \Gamma(TM),\ \ i. \tag{5.7}$$

Then we have $g(U_i,U_j) = 0$ and $g(U_i,V_j) = \delta_{ij}$ due to $a_i = 0$. Assume that $S(TM)$ is totally umbilical in M. Then, from Theorem 4.1, we have $h_i^* = a_i = 0$ for all i. Thus we get $g(R(X,Y)PZ,\,N_i) = 0$. From this, (5.1) and (5.5), we have

$$\begin{aligned} &(c+3)\{g(Y,PZ)\eta_i(X) - g(X,PZ)\eta_i(Y)\} \\ &\quad + (c-1)\{\theta(X)\theta(PZ)\eta_i(Y) - \theta(Y)\theta(PZ)\eta_i(X) \\ &\qquad - u_i(X)\bar{g}(JY,PZ) + u_i(Y)\bar{g}(JX,PZ) + 2u_i(PZ)\bar{g}(JX,Y)\} \\ &= 4\sum_{j=1}^{r} \{h_j^\ell(X,PZ)\bar{g}(A_{N_j}Y,N_i) - h_j^\ell(Y,PZ)\bar{g}(A_{N_j}X,N_i)\} \\ &\quad + 4\sum_{\alpha=r+1}^{n} \epsilon_\alpha\{h_\alpha^s(X,PZ)\rho_{i\alpha}(Y) - h_\alpha^s(Y,PZ)\rho_{i\alpha}(X)\}. \end{aligned} \tag{5.8}$$

Applying $\bar{\nabla}_X$ to $\bar{g}(JN_i,W_\alpha) = 0$ and using (2.3), (2.7), (2.8) and (2.14), we have

$$e_\alpha\eta_i(X) + h_\alpha^s(X,JN_i) = \epsilon_\alpha h_i^*(X,JW_\alpha) = 0,\ \forall\, X \in \Gamma(TM),\ i,\ \alpha. \tag{5.9}$$

The equations (4.4) and (5.9) reduce to the following equations:

$$h_j^\ell(X,U_i) = b_j\eta_i(X), \quad h_\alpha^s(X,U_i) = e_\alpha\eta_i(X), \quad \forall\, X \in \Gamma(TM). \tag{5.10}$$

Applying $\bar{\nabla}_X$ to $\bar{g}(\zeta, N_i) = 0$ and using (2.7) and (2.2), we have

$$\bar{g}(X, JN_i) + \bar{g}(A_{N_i}X, \zeta) - \sum_{\alpha=r+1}^{n} \epsilon_\alpha e_\alpha \rho_{i\alpha}(X) = 0.$$

Substituting (3.2) into the last equation and using (5.1), we have

$$\sum_{\alpha=r+1}^{n} \epsilon_\alpha e_\alpha \rho_{i\alpha}(X) + \sum_{j=1}^{r} b_j \bar{g}(A_{N_j}X, N_i) = u_i(X). \tag{5.11}$$

Replacing PZ by U_k in (5.8) with $i \neq k$ and using (5.7), (5.10), (5.11) and the facts $\theta(U_k) = -\theta(JN_k) = 0$ and $u_i(U_k) = g(U_i, U_k) = 0$, we have

$$(c+3)\{u_k(X)\eta_i(Y) - u_k(Y)\eta_i(X) + u_i(X)\eta_k(Y) - u_i(Y)\eta_k(X)\} = 0.$$

Replacing X by V_k and Y by ξ_i to this equation, we have $c = -3$. □

Corollary 5.2. *There exist no generic r-lightlike submanifolds M of an indefinite Sasakian space form $\bar{M}(c)(c \neq -3)$ such that $S(TM)$ is totally umbilical in M.*

References

[1] De Rham, G. Sur la réductibilité d'un espace de Riemannian, Comm. Math. Helv. 26(1952), 328–344.

[2] Duggal, K. L. Spacetime manifolds and contact structures, Int. J. Math. & Math. Sci., 13,(1990), 545–554.

[3] Duggal, K. L. and Bejancu, A. *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Acad. Publishers, Dordrecht, 1996.

[4] Duggal, K. L. and Jin, D. H . *Null Curves and Hypersurfaces of Semi-Riemannian Manifolds*, World Scientific, 2007.

[5] Duggal, K. L. and Sahin, B. *Differential geometry of lightlike submanifolds*, Frontiers in Mathematics, Birkhäuser, 2010.

[6] Duggal, K. L. and Jin, D. H., Totally umbilical lightlike submanifolds, Kodai Math. J. vol. 26(2003), 49-68.

[7] Hawking, S.W. and Ellis, G.F.R . *The large scale structure of space-time,* Cambridge University Press, Cambridge, 1973.

[8] Jin, D. H. Geometry of lightlike hypersurfaces of an indefinite Sasakian manifold, Indian J. of Pure and Applied Math., 41(1)(2010), 569–581.

[9] Jin, D. H. Transversal half lightlike submanifolds of an indefinite Sasakian manifold, J. Korean Soc Math. Edu. Ser. B: Pure Appl. Math., 18(1)(2011), 51–61.

[10] Jin, D. H. Half lightlike submanifold of an indefinite Sasakian manifold, J. Korean Soc Math. Edu. Ser. B: Pure Appl. Math., 18(2)(2011), 173–183.

[11] Jin, D. H. and Lee, J. W. Generic lightlike submanifolds of an indefinite cosymplectic manifold, Mathematical Problems in Engineering, Article ID 610986,(2011), 16 pages.

[12] Kupeli, D. N. *Singular Semi-Riemannian Geometry*, Mathematics and Its Applications, vol. 366, Kluwer Acad. Publishers, Dordrecht, 1996.

[13] O'Neill, B. *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, 1983.

[14] Yano, K. and Kon, M. Generic submanifolds, Ann. di Math. pura Appl., 123(1980), 59–92.

[15] Yano, K. and Kon, M. Generic submanifolds of Sasakian manifolds, Kodai Math. J., 3(1980), 163–196.

Department of Mathematics and Statistics, University of Windsor, Windsor, Ontario, N9B3P4, CANADA
E-mail address: `yq8@uwindsor.ca`

Department of Mathematics, Dongguk University, Gyeongju 780-714, KOREA
E-mail address: `jindh@dongguk.ac.kr`