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TANGENCY AND ORTHOGONALITY OF ARCS IN METRIC SPACES

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Abstract. In this paper the problems of tangency and orthogonality of rectifiable arcs in metric spaces are considered. Some connections between tangency and orthogonality relations of rectifiable arcs have been given here.

1. Introduction

Let E be an arbitrary non-empty set and ρ a metric of this set. Let I_p denotes the class of all arcs A with origin at point $p \in E$ in the metric space (E, ρ). For A belonging to I_p we understand here homeomorphic image of closed interval $[0, 1]$. Let $\ell_A(\breve{px})$ denotes the length of arc $\breve{px}$ of A with origin at point $p \in E$ and end at point $x \in E$. By $\widetilde{A}_p$ we shall denote the class of all arcs $A \in I_p$ fulfilling the condition:

$$\lim_{A \ni x \to p} \frac{\ell_A(\breve{px})}{\rho(p, x)} = g < \infty. \tag{1.1}$$

The arc $A \in I_p$ fulfilling the condition (1.1) we call the rectifiable arc at point $p \in E$ of metric space (E, ρ). If the arc $A \in I_p$ is rectifiable in each its point, then we call it a rectifiable arc.

If $g = 1$, then we say that the arc $A \in \widetilde{A}_p$ has Archimedean property at point $p \in E$. It is easily to prove that every regular arc at point $p \in E$ of the Cartesian space (E, ρ) has Archimedean property in this point.

In the book [1] A.D. Aleksandrov defines the angle between arcs $A, B \in I_p$ as an angle $\alpha \in [0, \pi]$ fulfilling the following equality:

$$\cos \alpha = \lim_{(x,y) \to (p,p)} \frac{\rho^2(p, x) + \rho^2(p, y) - \rho^2(x, y)}{2\rho(p, x)\rho(p, y)} \quad \text{for } x \in A \text{ and } y \in B. \tag{1.2}$$

Before we will move to the definition of the tangency of arcs in metric space (E, ρ) based on the definition of Aleksandrov′s angle between these arcs, we will talk over in short the so called Riemannian angle for regular arcs.

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Let us assume now that E is a Riemannian manifold with symmetric tensor field g about the valence $(0, 2)$. Using the metric tensor we may define in manifold E among others such notions as: distance of points, length of arc, length of tangency vector and scalar product of vectors.

By ρ we shall denote the metric of manifold E generated by its metric tensor. Let A, B be regular arcs defined respectively by the vector equations: $\overrightarrow{r_1} = \overrightarrow{r_1}(t)$, $\overrightarrow{r_2} = \overrightarrow{r_2}(t)$ for $t \in [0, 1]$. Let′s accept moreover, that $p = \overrightarrow{r_1}(0) = \overrightarrow{r_2}(0)$.

The Riemannian angle between these arcs is defined as an angle $\gamma \in [0, \pi]$ between vectors tangent to these arcs at the point p, by formula:

$$\cos\gamma = \frac{(\overrightarrow{r}_1(0) \mid \overrightarrow{r}_2(0))}{|\overrightarrow{r}_1(0)|\,|\overrightarrow{r}_2(0)|}\,, \tag{1.3}$$

where $(\overrightarrow{r}_1(0) \mid \overrightarrow{r}_2(0))$ denotes the scalar product of vectors $\overrightarrow{r}_1(t)$, $\overrightarrow{r}_2(t)$ at the point $t = 0$.

Two regular arcs $A, B \in I_p$ are tangent at the point $p \in E$ corresponding to the parameter $t = 0$ if they have at this point equal tangent vectors or ones differing at most in a positive factor, i.e. $\overrightarrow{r}_1(0) = \lambda\,\overrightarrow{r}_2(0)$ for $\lambda > 0$.

Hence and from (1.3) it follows that the regular arcs $A, B \in I_p$ are tangent at the point $p \in E$ if $\cos\gamma = 1$, where γ denotes the Riemannian angle between these arcs.

M.R. Bridson and A. Haefliger in the book [2] proved that the Riemannian angle γ between the regular arcs in Riemannian manifold is equal to the Alexandrov′s angle α between them (see Figure 1).

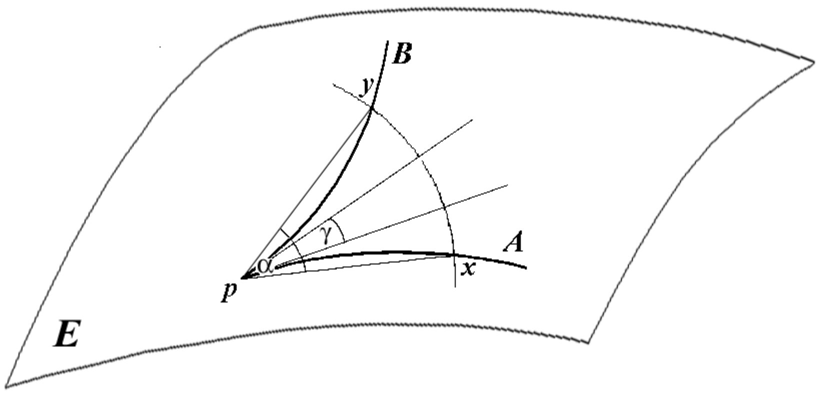

FIGURE 1

Based on the definition of Aleksandrov′s angle between arcs $A, B \in I_p$ S. Midura gives in the paper [5] the following definition of the tangency of arcs in the metric space (E, ρ):

Definition 1.1. The arc A is tangent to the arc B at the point $p \in E$, if for $x \in A$ and $y \in B$

$$\lim_{(x,y)\to(p,p)} \frac{\rho^2(p, x) + \rho^2(p, y) - \rho^2(x, y)}{2\rho(p, x)\rho(p, y)} = 1. \tag{1.4}$$

If the arc A is tangent to the arc B at the point $p \in E$, then we write that $(A, B) \in T_p$. The set T_p of the form:

$$(1.5)\quad T_p = \left\{(A,B)\colon\ A, B \in I_p \text{ and } \lim_{A\times B\ni(x,y)\to(p,p)} \frac{\rho^2(p,x)+\rho^2(p,y)-\rho^2(x,y)}{2\rho(p,x)\rho(p,y)} = 1\right\}.$$

we call the tangency relation of arcs of the class I_p at point $p \in E$.

From the definition (1.5) it results immediately that the tangency relation T_p is symmetric in the class of arcs I_p. This relation is moreover transitive in the class I_p what shows Theorem 4 of the paper [5]. The set T_p is not reflexive relation in the class I_p. If we assume however that $(A, B) \in T_p$ for $A, B \in I_p$, then $(A, A) \in T_p$ and $(B, B) \in T_p$, what results from symmetric and transitivity of this relation in the class of arcs I_p.

2. Tangency and orthogonality of rectifiable arcs

In this Section we shall consider the tangency and the orthogonality of rectifiable arcs of the class $\widetilde{A}_p$ and their mutual relationship. Similarly as in the definition of tangency of arcs, we can use the angle of Alexandrov′s to the definition of orthogonality of rectifiable arcs.

Definition 2.1.

$$(2.1)\quad O_p = \left\{(A,B)\colon A, B \in \widetilde{A}_p \text{ and } \lim_{A\times B\ni(x,y)\to(p,p)} \frac{\rho^2(p,x)+\rho^2(p,y)-\rho^2(x,y)}{\rho(p,x)\rho(p,y)} = 0\right\}.$$

The set O_p we call the orthogonality relation of arcs of the class $\widetilde{A}_p$ at the point p of the metric space (E, ρ).

If $(A, B) \in O_p$ for $A, B \in \widetilde{A}_p$, then we say that the arc A is orthogonal to the arc B at the point $p \in E$.

It results directly from the definition (2.1) that the orthogonality relation O_p is symmetric in the class of rectifiable arcs $\widetilde{A}_p$.

Before we give some theorem related to the tangency and orthogonality relations of arcs of the class $\widetilde{A}_p$, we will formulate two lemmas necessary to prove this theorem.

Lemma 2.1. *If arcs $A, B \in \widetilde{A}_p$ are tangent at point $p \in E$ and*

$$(2.2)\qquad \rho(p,x) = \rho(p,y) = r \quad \text{for} \quad x \in A,\ \ y \in B,$$

then

$$(2.3)\qquad \lim_{r\to 0^+} \frac{\rho(x,y)}{r} = 0.$$

Proof. Using the definition (1.5) of the tangency of arcs and the assumption (2.2) of this lemma we have

$$\lim_{r\to 0^+} \frac{2r^2 - \rho^2(x,y)}{2r^2} = 1.$$

Hence we get the equality

$$\lim_{r\to 0^+} \frac{\rho^2(x,y)}{r^2} = 0,$$

from where the thesis (2.3) of this lemma results immediately.

Lemma 2.2. *If arcs* $A, B \in \widetilde{A}_p$ *are orthogonal at point* $p \in E$ *and the condition* (2.2) *is fulfilled, then*

$$\lim_{r\to 0^+} \frac{\rho(x,y)}{r} = \sqrt{2}. \tag{2.4}$$

Proof. From the definition (2.1) of the orthogonality of arcs and from the condition (2.2) we have

$$\lim_{r\to 0^+} \frac{2r^2 - \rho^2(x,y)}{2r^2} = 0.$$

Hence follows

$$\lim_{r\to 0^+} \frac{\rho^2(x,y)}{r^2} = 2,$$

from where we get the thesis (2.4) of this lemma.

Using the above lemmas we will prove the following theorem:

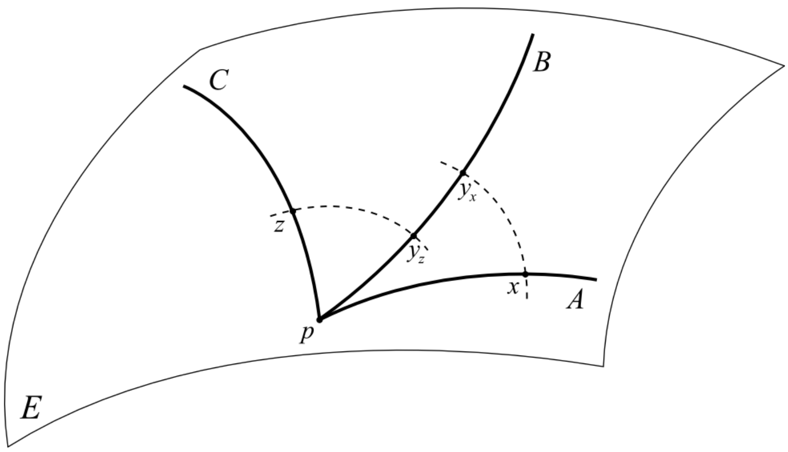

FIGURE 2

Theorem 2.1. *If* $(A, B) \in T_p$ *and* $(B, C) \in O_p$, *then* $(A, C) \in O_p$ *for arbitrary arcs* $A, B, C \in \widetilde{A}_p$.

Proof. Let accordingly x and z be a points of rectifiable arcs $A, C \in \widetilde{A}_p$. Let′s suppose that $\rho(p, z) \le \rho(p, x)$ (see Figure 2). For any point $x \in A$ there exists a point $y_x \in B$ such that $\rho(p, x) = \rho(p, y_x)$. Hence and from assumption that $(A, B) \in T_p$ and from Lemma 2.1 equality follows

$$\lim_{x\to p} \frac{\rho(x, y_x)}{\rho(p, x)} = 0. \tag{2.5}$$

Similarly for any point $z \in C$ there exists a point $y_z \in B$ such that $\rho(p, z) = \rho(p, y_z)$. Hence, from the fact that $(B, C) \in O_p$ and from Lemma 2.2 we get

$$\lim_{z\to p} \frac{\rho(z, y_z)}{\rho(p, z)} = \sqrt{2}. \tag{2.6}$$

Moreover from the assumption that $(A,B)\in T_p,\ (B,C)\in O_p$ and from the definition of tangency and ortogonality of arcs equalities result

$$\lim_{(x,y_z)\to(p,p)}\frac{\rho^2(p,x)+\rho^2(p,y_z)-\rho^2(x,y_z)}{2\rho(p,x)\rho(p,y_z)}=1, \tag{2.7}$$

and

$$\lim_{(z,y_x)\to(p,p)}\frac{\rho^2(p,z)+\rho^2(p,y_x)-\rho^2(z,y_x)}{\rho(p,z)\rho(p,y_x)}=0. \tag{2.8}$$

From the triangle inequality for metric ρ and the points x,z,y_z we have

$$|\rho(x,y_z)-\rho(y_z,z)|\le\rho(x,z)\le\rho(x,y_z)+\rho(y_z,z). \tag{2.9}$$

Hence it follows

$$\begin{aligned}&-\rho^2(x,y_z)-\rho^2(y_z,z)-2\rho(x,y_z)\rho(y_z,z)\le-\rho^2(x,z)\\&\le-\rho^2(x,y_z)-\rho^2(y_z,z)+2\rho(x,y_z)\rho(y_z,z).\end{aligned} \tag{2.10}$$

Therefore, remembering that $\rho(p,z)=\rho(p,y_z)$, we receive

$$\begin{aligned}&\frac{\rho^2(p,x)+\rho^2(p,y_z)-\rho^2(x,y_z)-\rho^2(y_z,z)-2\rho(x,y_z)\rho(y_z,z)}{\rho(p,x)\rho(p,z)}\\&\le\frac{\rho^2(p,x)+\rho^2(p,z)-\rho^2(x,z)}{\rho(p,x)\rho(p,z)}\\&\le\frac{\rho^2(p,x)+\rho^2(p,y_z)-\rho^2(x,y_z)-\rho^2(y_z,z)+2\rho(x,y_z)\rho(y_z,z)}{\rho(p,x)\rho(p,z)},\end{aligned}$$

that is

$$\begin{aligned}&\frac{\rho^2(p,x)+\rho^2(p,y_z)-\rho^2(x,y_z)}{\rho(p,x)\rho(p,y_z)}-\frac{\rho^2(y_z,z)}{\rho(p,x)\rho(p,y_z)}-2\frac{\rho(x,y_z)\rho(y_z,z)}{\rho(p,x)\rho(p,y_z)}\\&\le\frac{\rho^2(p,x)+\rho^2(p,z)-\rho^2(x,z)}{\rho(p,x)\rho(p,z)}\\&\le\frac{\rho^2(p,x)+\rho^2(p,y_z)-\rho^2(x,y_z)}{\rho(p,x)\rho(p,y_z)}-\frac{\rho^2(y_z,z)}{\rho(p,x)\rho(p,y_z)}+2\frac{\rho(x,y_z)\rho(y_z,z)}{\rho(p,x)\rho(p,y_z)}.\end{aligned} \tag{2.11}$$

If $x\to p$, then from assumption that $\rho(p,y_z)=\rho(p,z)\le\rho(p,x)$ and from (2.6) results

$$\lim_{(x,y_z)\to(p,p)}\frac{\rho^2(y_z,z)}{\rho(p,x)\rho(p,y_z)}=\lim_{z\to p}\frac{\rho^2(y_z,z)}{\rho^2(p,z)}=2. \tag{2.12}$$

Moreover from (2.5) by assumption that $\rho(p,y_z)=\rho(p,z)\le\rho(p,x)=\rho(x,y_x)$ we get

$$0=\lim_{x\to p}\frac{\rho(x,y_x)}{\rho(p,x)}=\lim_{x\to p}\frac{\rho(x,y_z)}{\rho(p,x)}. \tag{2.13}$$

Hence, from (2.6), (2.7), (2.12) and from the inequality (2.11) follows

$$\lim_{(x,z)\to(p,p)}\frac{\rho^2(p,x)+\rho^2(p,z)-\rho^2(x,z)}{\rho(p,x)\rho(p,z)}=0, \tag{2.14}$$

that is to say that $(A,C)\in O_p$, when $\rho(p,z)\le\rho(p,x)$.

If $\rho(p,x)<\rho(p,z)$, then in a similar way we get the equality (2.14). This ends the proof.

The proof of this theorem gets considerably simpler when x, y, z are respectively the points of arcs $A, B, C \in \widetilde{A}_p$ such that

$$\rho(p,x) = \rho(p,y) = \rho(p,z) = r, \tag{2.15}$$

this means that they are the points created from the intersection of these arcs with the sphere $S_\rho(p,r)$ about the centre at the point $p \in E$ and the radius r in the metric space (E, ρ).

3. Tangency of higher order for rectifiable arcs

In this Section we shall define the tangency relation of higher order ($k \geq 1$) for rectifiable arcs of the class $\widetilde{A}_p$ and we will give its certain properties. Using Lemma 2.1 we may give the following definition of the tangency of rectifiable arcs:

Definition 3.1. If

$$\lim_{r \to 0^+} \frac{\rho(x,y)}{r^k} = 0 \tag{3.1}$$

for $x \in A$, $y \in B$ and $r = \rho(p,x) = \rho(p,y)$, then we say that the arcs $A, B \in \widetilde{A}_p$ are tangent (have the tangency) of order $k \geq 1$ at the point $p \in E$.

If the rectifiable arcs $A, B \in \widetilde{A}_p$ are tangent of order k at the point $p \in E$, then we shall write that $(A, B) \in T_{p,k}$. The set $T_{p,k}$ we call the tangency relation of order k at the point p for rectifiable arcs of the class $\widetilde{A}_p$.
Therefore

$$T_{p,k} = \{(A,B) : A, B \in \widetilde{A}_p \text{ and for } x \in A,\ y \in B,\ r = \rho(p,x) = \rho(p,y)$$
$$\lim_{r \to 0^+} \frac{\rho(x,y)}{r^k} = 0\}. \tag{3.2}$$

From given here definitions it follows immediately that $(A, B) \in T_{p,1}$, if and only if $(A, B) \in T_p$ for $A, B \in \widetilde{A}_p$.

In the paper [6] W.Waliszewski gave the following definition of the tangency relation of sets (more exactly: (a, b)-tagency) of order k at the point $p \in E$ in generalized metric space (E, l):

$$T_l(a,b,k,p) = \{(A,B) :\ A, B \in E_0, \text{ the pair of sets } (A,B) \text{ is } (a,b)\text{-clustered}$$
$$\text{at point } p \text{ of the space } (E,l) \text{ and}$$
$$\lim_{r \to 0^+} \frac{1}{r^k} l(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) = 0\}. \tag{3.3}$$

In the formula (3.3) l denotes any non-negative real function defined on the Cartesian product $E_0 \times E_0$ of the family E_0 of all non-empty subsets of the set E, however a, b are non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$\lim_{r \to 0^+} a(r) = 0 \quad \text{and} \quad \lim_{r \to 0^+} b(r) = 0. \tag{3.4}$$

If in the definition of tangency of sets W.Waliszewski′s we will assume that $a(r) = b(r) = 0$ for $r \geq 0$ and

$$l(\{x\}, \{y\}) = \rho(x,y) \quad \text{for} \quad x, y \in E, \tag{3.5}$$

then we will receive definition 3.1 of the tangency of rectifiable arcs of the class $\widetilde{A}_p$ in metric space (E, ρ).

We shall give now some basic properties of the tangency relation (3.2) of rectifiable arcs. Using properties of the metric we can easily prove the following theorem:

Theorem 3.1. *In the class of rectifiable arcs $\widetilde{A}_p$ the tangency relation $T_{p,k}$ is equivalence, i.e.*

1^0 $(A, A) \in T_{p,k}$ *(reflexivity)*,

2^0 $(A, B) \in T_{p,k} \Longrightarrow (B, A) \in T_{p,k}$ *(symmetry)*,

3^0 $(A, B) \in T_{p,k} \ \wedge \ (B, C) \in T_{p,k} \Longrightarrow (A, C) \in T_{p,k}$ *(transitivity)*

for any arcs $A, B, C \in \widetilde{A}_p$.

Lemma 3.1. *If $(A, B) \in T_{p,n}$, then $(A, B) \in T_{p,k}$ for $A, B \in \widetilde{A}_p$ and $n \geq k$.*

Proof. We will assume that $(A, B) \in T_{p,n}$ for $A, B \in \widetilde{A}_p$. Therefore

$$\lim_{r \to 0^+} \frac{\rho(x, y)}{r^n} = 0$$

for $x \in A$, $y \in B$ and $r = \rho(p, x) = \rho(p, y)$. Hence and from the assumption that $n - k \geq 0$ we get

$$\lim_{r \to 0^+} \frac{\rho(x, y)}{r^k} = \lim_{r \to 0^+} r^{n-k} \lim_{r \to 0^+} \frac{\rho(x, y)}{r^n} = 0,$$

what marks that $(A, B) \in T_{p,k}$.

Theorem 3.2. *If $(A, B) \in T_{p,n}$ and $(B, C) \in T_{p,k}$, then $(A, C) \in T_{p,m}$ for any arcs $A, B, C \in \widetilde{A}_p$ and $m = \min\{n, k\}$.*

Proof. We shall assume that $(A, B) \in T_{p,n}$ and $(B, C) \in T_{p,k}$ for $A, B, C \in \widetilde{A}_p$. Hence it follows

$$\lim_{r \to 0^+} \frac{\rho(x, y)}{r^n} = 0 \quad \text{and} \quad \lim_{r \to 0^+} \frac{\rho(y, z)}{r^n} = 0 \tag{3.6}$$

for $x \in A$, $y \in B$, $z \in C$ and $r = \rho(p, x) = \rho(p, y) = \rho(p, z)$.
From Lemma 3.1, from the assumption of this theorem and from (3.6) we get

$$\lim_{r \to 0^+} \frac{\rho(x, y)}{r^m} = 0 \quad \text{and} \quad \lim_{r \to 0^+} \frac{\rho(y, z)}{r^m} = 0.$$

Hence and from the triangle inequality for the metric ρ results

$$\begin{aligned} 0 \ \leq \ & \lim_{r \to 0^+} \frac{\rho(x, z)}{r^m} \leq \lim_{r \to 0^+} \frac{\rho(x, y) + \rho(y, z)}{r^m} \\ = \ & \lim_{r \to 0^+} \frac{\rho(x, y)}{r^m} + \lim_{r \to 0^+} \frac{\rho(y, z)}{r^m} = 0, \end{aligned}$$

what means that $(A, C) \in T_{p,m}$.

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