

ON THE LIFTING PROBLEM IN \mathbb{P}^4 IN CHARACTERISTIC p

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ABSTRACT. Given \mathbb{P}_k^4 , with k algebraically closed field of characteristic $p > 0$, and $X \subset \mathbb{P}_k^4$ integral surface of degree d , let $Y = X \cap H$ be the general hyperplane section of X . In this paper we study the problem of lifting, i.e. extending, a surface of degree s in H containing Y to a hypersurface of same degree s in \mathbb{P}^4 containing X . In the case in which this extension does not exist we get upper bounds for d depending on s and p .

1. INTRODUCTION

Let $X \subset \mathbb{P}_k^4$, with k algebraically closed field of positive characteristic p , be an integral surface of degree d and let $Y = X \cap H$ be the general hyperplane section of X . In this paper we study the problem of lifting a surface of H of degree s containing Y to a hypersurface in \mathbb{P}^4 of degree s containing X . In the main results of the paper, Theorem 4.1 and Theorem 4.2, under the hypotheses that $h^0 \mathcal{I}_Y(s) \neq 0$ and $h^0 \mathcal{I}_X(s) = 0$ for some $s > 0$, we give sharp upper bounds of d in terms of s and p .

In the case that $\text{char } k = 0$ the problem has been studied and solved by Mezzetti and Raspanti in [15] and in [17], showing that $d \leq s^2 - s + 2$ and that this bound is sharp, and in [16] Mezzetti classifies the border case $d = s^2 - s + 2$. Other results concerning the lifting problem have been obtained in characteristic 0 for curves in \mathbb{P}^3 (see [10, Corollary p. 147], [5] and [24, Corollario 2]) and for integral varieties of codimension 2 in \mathbb{P}^n (see, for example, [17] for $n = 5$, [21] for $n = 6$ and [25] and [20] for the general case). In the case that $\text{char } p > 0$ the lifting problem has been studied for curves in \mathbb{P}^3 in [1].

In this paper we study for the first time the lifting problem in \mathbb{P}^4 in the characteristic p case. The starting point is that the non lifting section of $H^0 \mathcal{I}_Y(s)$ determines a nonzero element $\alpha \in H^1 \mathcal{I}_X(s)$ such that $\alpha \cdot H = 0$ in $H^1 \mathcal{I}_X(s)$. α is called a sporadic zero. The order of α is the maximum integer $m \in \mathbb{N}$ such that $\alpha = \beta \cdot H^m$ for some $\beta \in H^1 \mathcal{I}_X(s - m - 1)$. For $p < s$ we need to relate s and m . In particular, taken p^n such that $p^n \leq m + 1$ and $p^{n+1} > m + 1$, in Theorem 4.1 we suppose that $p < s$ and we show that:

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- (1) $d \leq s^2 - s + 2 + p^n$, if $s \geq 2m + 3$;
- (2) $d \leq s^2$ if $s \leq 2m + 2$.

As a consequence, we see that for $p < s$ it must be $d \leq s^2$. In the case that $p \geq s$ we see in Theorem 4.2 that $d \leq s^2 - s + 2$, i.e. we find the same bound as in the characteristic 0 case. In Example 4.1 we see that the bounds given in Theorem 4.1 and in Theorem 4.2 are sharp.

Let us now give a sketch of the proof of Theorem 4.1 (the proof of Theorem 4.2 is analogous). We follow the idea of the proof of Theorem 1.1 given by Gruson and Peskine in [5] and used also in [1]. So we take $V \subset \mathbb{P}^4 \times \mathbb{P}^4$ such that the fibre over the generic point in \mathbb{P}^4 is a hypersurface of degree s containing Y . Then using Theorem 3.1 we factor the projection $V \rightarrow \mathbb{P}^4$ through a generically smooth morphism $V_r \rightarrow \mathbb{P}^4$, with $V_r = V \times_{\mathbb{P}^4, F^r} \mathbb{P}^4$ and F^r some r -th power of the absolute Frobenius of \mathbb{P}^4 . Proceeding as in [5] we see that the inequalities given in the statement of the Theorem 4.1 are proved once we prove that the second Chern class of a certain reflexive sheaf of rank 3 on the general hyperplane H is non negative. This is proved by using the concept of semistability for reflexive sheaves and the Bogomolov inequality.

We need to remark that the proves of the main results of this paper are similar to the proof of the main result given in [1], which deals with curves in \mathbb{P}^3 over a field of characteristic p . The difference between the proves lies in the rank of the reflexive sheaf whose second Chern class must be non negative: in the \mathbb{P}^3 case it has rank 2, while in this paper it has rank 3.

2. HILBERT FUNCTION OF POINTS IN \mathbb{P}^2

Let us denote by X a zero-dimensional scheme in \mathbb{P}_k^2 , where k is an algebraically closed field of any characteristic. Let $H_X : \mathbb{N} \rightarrow \mathbb{N}$ be the Hilbert function of X and let us consider the first difference of H_X :

$$\Delta H_X(i) = H_X(i) - H_X(i - 1).$$

It is known [4] that there exist $a_1 \leq a_2 \leq t$ such that:

$$\Delta H_X(i) = \begin{cases} i + 1 & \text{for } i = 0, \dots, a_1 - 1 \\ a_1 & \text{for } i = a_1, \dots, a_2 - 1 \\ < a_1 & \text{for } i = a_2 \\ \text{non increasing} & \text{for } i = a_2 + 1, \dots, t \\ 0 & \text{for } i > t. \end{cases}$$

Definition 2.1. We say that X has the *Hilbert function of decreasing type* if for $a_2 \leq i < j < t$ we have $\Delta H_X(i) > \Delta H_X(j)$.

The following theorem is well known in characteristic 0 (see [6] and [11, Corollary 2]) and proved in any characteristic in [2, Corollary 4.3].

Theorem 2.1. *Let $C \subset \mathbb{P}^3$ be an integral curve and let X be its general plane section. Then H_X is of decreasing type.*

The following proposition will be useful in the proof of the main results of the paper.

Proposition 2.1. *Let $X \subset \mathbb{P}^2$ be a 0-dimensional scheme whose Hilbert function is of decreasing type. Let us suppose that $h^0 \mathcal{I}_X(s-1) = 0$ for some $s > 0$ and that one of the following conditions, incompatible with one another, holds:*

- (1) $h^0 \mathcal{I}_X(s) \geq 3$;
- (2) $h^0 \mathcal{I}_X(s) = 2$ and there exists $i \in \mathbb{N}$ such that $\Delta H_X(s+i) \leq s-i-2$.

Then $\deg X \leq s^2 - s + i + 1$.

Proof. The proof is a straightforward computation and follows by the fact that the Hilbert function of X is of decreasing type.

If $h^0 \mathcal{I}_X(s) \geq 3$, then we see that:

$$\deg X \leq \frac{s(s+1)}{2} + \frac{(s-2)(s-1)}{2} = s^2 - s + 1 < s^2 - s + i + 1.$$

Let $h^0 \mathcal{I}_X(s) = 2$ and let us suppose that $i = \min\{k \in \mathbb{N} \mid \Delta H_X(s+k) \leq s-k-2\}$. Since H_X is of decreasing type, $\Delta H_X(s+k) = s-k-1$ for $k \leq i-1$ and $\Delta H_X(s+k) \leq s-k-2$ for $k \geq i$. Then:

$$\deg X \leq \frac{s(s+1)}{2} + \sum_{k=0}^{i-1} (s-k-1) + \sum_{k=i}^{s-3} (s-k-2) = s^2 - s + i + 1.$$

□

3. FROBENIUS MORPHISM AND INCIDENCE VARIETIES

In this section we show some results about incidence varieties and Frobenius morphism on \mathbb{P}^n for any n . First let us recall the definition of absolute and relative Frobenius morphism:

Definition 3.1. The absolute Frobenius morphism of a scheme X of characteristic $p > 0$ is $F_X: X \rightarrow X$, where F_X is the identity as a map of topological spaces and on each U open set $F_X^\#: \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(U)$ is given by $f \mapsto f^p$ for each $f \in \mathcal{O}_X(U)$. Given $X \rightarrow S$ for some scheme S and $X^{p/S} = X \times_{S, F_S} S$, the absolute Frobenius morphisms on X and S induce a morphism $F_{X/S}: X \rightarrow X^{p/S}$, called the Frobenius morphism of X relative to S .

Given \mathbb{P}^n for some $n \in \mathbb{N}$, let us consider the bi-projective space $\check{\mathbb{P}}^n \times \mathbb{P}^n$ and let $r \in \mathbb{N}$ be a non negative integer. Let $k[t]$ and $k[\underline{x}]$ be the coordinate rings of $\check{\mathbb{P}}^n$ and \mathbb{P}^n , respectively. Let $M_r \subset \check{\mathbb{P}}^n \times \mathbb{P}^n$ be the hypersurface of equation:

$$h_r := \sum_{i=0}^n t_i x_i^{p^r} = 0.$$

Note that in the case $r = 0$ M_r is the usual incidence variety M of equation $\sum t_i x_i = 0$. If $r \geq 1$, M_r is determined by the following fibred product:

$$(3.1) \quad \begin{array}{ccc} M & \xrightarrow{(F_M)^r} & M \\ \text{---} F_{M_r} \text{---} \searrow & & \downarrow \pi \\ & M_r & \xrightarrow{\pi} & M \\ \downarrow p & \downarrow p_{M_r} & & \downarrow p \\ & \mathbb{P}^n & \xrightarrow{F^r} & \mathbb{P}^n \end{array}$$

where $F: \mathbb{P}^n \rightarrow \mathbb{P}^n$ is the absolute Frobenius.

Remark 3.1. Since $M = \mathbb{P}(\mathcal{T}_{\mathbb{P}^n}(-1))$, by [3, Lemma 1.5] we get that $M_r = \mathbb{P}(F^{r*}(\mathcal{T}_{\mathbb{P}^n}(-1)))$ and by [7, Ch.II, ex. 7.9] that $\text{Pic}(M_r) = \mathbb{Z} \times \mathbb{Z}$ for any $r \geq 0$. Moreover, since we have:

$$0 \rightarrow \mathcal{O}_{\check{\mathbb{P}}^n \times \mathbb{P}^n}(-1, -p^r) \rightarrow \mathcal{O}_{\check{\mathbb{P}}^n \times \mathbb{P}^n} \rightarrow \mathcal{O}_{M_r} \rightarrow 0$$

and $H^1 \mathcal{O}_{\check{\mathbb{P}}^n \times \mathbb{P}^n}(m, n) = 0$ for any $m, n \in \mathbb{Z}$ by the Künneth formula [18, Ch.VI, Corollary 8.13], then any hypersurface $V \subset M_r$ is the complete intersection given by $g = h_r = 0$ for some bi-homogeneous $g \in k[\underline{t}, \underline{x}]$.

Let $\eta \in \check{\mathbb{P}}^n$ be the generic point and consider $g_{M_r}: M_r \rightarrow \check{\mathbb{P}}^n$. Then $g_{M_r}^{-1}(\eta)$ is isomorphic to the hypersurface H_r of \mathbb{P}^n of degree p^r such that, over the algebraic closure $\overline{k(\eta)}$ of $k(\eta)$, $(H_r)_{\text{red}}$ is the generic hyperplane H of \mathbb{P}^n . By abuse of notation we identify $(g_{M_r}^{-1}(\eta))_{\text{red}}$ with H .

Proposition 3.1. $\Omega_{M_r/\mathbb{P}^n}|_H \cong F^{r*} \mathcal{T}_H(-p^r)$.

Proof. The sheaf $\mathcal{E} = F^{r*}(\mathcal{T}_{\mathbb{P}^n}(-1)) = F^{r*} \mathcal{T}_{\mathbb{P}^n}(-p^r)$ is determined by the exact sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-p^r) \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \rightarrow \mathcal{E} \rightarrow 0$ and, since $M_r = \mathbb{P}(F^{r*}(\mathcal{T}_{\mathbb{P}^n}(-1)))$, by [7, Ch.III, Ex. 8.4(b)] we have also:

$$0 \rightarrow \Omega_{M_r/\mathbb{P}^n} \rightarrow p_{M_r}^*(F^{r*} \mathcal{T}_{\mathbb{P}^n}(-p^r)) \otimes_{\mathcal{O}_{M_r}} \mathcal{O}_{M_r}(-1, 0) \rightarrow \mathcal{O}_{M_r} \rightarrow 0.$$

When we restrict to H , by the fact that the sequence locally splits it follows that the following sequence is exact:

$$0 \rightarrow \Omega_{M_r/\mathbb{P}^n}|_H \rightarrow p_{M_r}^*(F^{r*} \mathcal{T}_{\mathbb{P}^n}(-p^r)) \otimes_{\mathcal{O}_{M_r}} \mathcal{O}_{M_r}(-1, 0)|_H \rightarrow \mathcal{O}_H \rightarrow 0.$$

By the identification of H with $(g_{M_r}^{-1}(\eta))_{\text{red}}$ we get:

$$(3.2) \quad 0 \rightarrow \Omega_{M_r/\mathbb{P}^n}|_H \rightarrow F^{r*} \mathcal{T}_{\mathbb{P}^n}(-p^r)|_H \rightarrow \mathcal{O}_H \rightarrow 0.$$

By the fact that $\mathcal{T}_{\mathbb{P}^n}(-1)|_H \cong \mathcal{T}_H(-1) \oplus \mathcal{O}_H$, we get:

$$F^{r*} \mathcal{T}_{\mathbb{P}^n}(-p^r)|_H \cong F^{r*} \mathcal{T}_H(-p^r) \oplus \mathcal{O}_H.$$

Since $F^{r*} \mathcal{T}_H$ is stable ([19, Ch.II, Theorem 1.3.2] and [14, Theorem 2.1]) and $\mu(F^{r*} \mathcal{T}_H(-p^r)) > 0$, we have $\text{Hom}(F^{r*} \mathcal{T}_H(-p^r), \mathcal{O}_H) = 0$ and so by (3.2) also that $\Omega_{M_r/\mathbb{P}^n}|_H \cong F^{r*} \mathcal{T}_H(-p^r)$. \square

Now we prove some results about the projection from a hypersurface in M_r to \mathbb{P}^n .

Theorem 3.1. *Let $V \subset \check{\mathbb{P}}^n \times \mathbb{P}^n$ be an integral hypersurface in M such that the projection $\pi: V \rightarrow \mathbb{P}^n$ is dominant and not generically smooth. Then there exist $r \geq 1$ and $V_r \subset M_r$ integral hypersurface such that π can be factored in the following way:*

$$\begin{array}{ccc} V & \xrightarrow{\pi} & \mathbb{P}^n \\ & \searrow F_r & \nearrow \pi_r \\ & & V_r \end{array}$$

where the projection π_r is dominant and generically smooth and F_r is induced by the commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{F_r} & V_r \\ \downarrow j & & \downarrow i \\ M & \xrightarrow{F_{M_r}} & M_r \end{array}$$

Proof. The proof works as in [1, Theorem 3.3]. Indeed, the proofs of Lemma 3.1 and Proposition 3.2 in [1] work also for $\mathbb{P}^n \times \mathbb{P}^n$. \square

Proposition 3.2. *Let $V_r \subset M_r$ be an integral hypersurface given by:*

$$\begin{cases} q(t, \underline{x}) = 0 \\ \sum_{i=0}^n t_i x_i^{p^r} = 0 \end{cases}$$

such that the projection $\pi_r: V_r \rightarrow \mathbb{P}^n$ is generically smooth. Then π_r is not smooth exactly on the following closed subset of V_r :

$$V_r \cap V \left(x_i^{p^r} \frac{\partial q}{\partial t_j} - x_j^{p^r} \frac{\partial q}{\partial t_i} \mid i, j = 0, \dots, n \right).$$

Proof. Let $P_0 = (\underline{a}, \underline{b}) \in V_r$ be such that V_r is not smooth in P_0 . Then there exists $\lambda \in k$ such that:

$$(3.3) \quad \frac{\partial q}{\partial t_i}(P_0) = \lambda b_i^{p^r}$$

for any $i = 0, \dots, n$.

If P_0 is a regular point, then the projective tangent space T_{V_r, P_0} at $P_0 \in V_r$ is given by the equations:

$$\sum_{i=0}^n \frac{\partial q}{\partial x_i}(P_0) x_i + \sum_{i=0}^n \frac{\partial q}{\partial t_i}(P_0) t_i = \sum_{i=0}^n (a_i x_i + b_i t_i) = 0$$

if $r = 0$ and by the equations:

$$\sum_{i=0}^n \frac{\partial q}{\partial x_i}(P_0) x_i + \sum_{i=0}^n \frac{\partial q}{\partial t_i}(P_0) t_i = \sum_{i=0}^n b_i^{p^r} t_i = 0$$

if $r \geq 1$. In both cases the projection on $\overline{T_{\mathbb{P}^n, \pi(P_0)}}$ is not surjective if and only if there exists $\lambda \in k$ such that:

$$\frac{\partial q}{\partial t_i}(P_0) = \lambda b_i^{p^r} \quad \forall i = 0, \dots, n.$$

This together with (3.3) proves the statement. \square

4. LIFTING PROBLEM

Let $X \subset \mathbb{P}^4$ be a scheme and, following the previous notation, consider the projections $p_{M_r}: M_r \rightarrow \mathbb{P}^4$ and $g_{M_r}: M_r \rightarrow \mathbb{P}^4$. Let $T_r = p_{M_r}^{-1}(X)$ and:

$$\mathcal{I}_r(m, n) = g_{M_r}^* (\mathcal{O}_{\mathbb{P}^4}(m)) \otimes_{\mathcal{O}_{M_r}} p_{M_r}^* (\mathcal{I}_X(n))$$

for every $m, n \in \mathbb{Z}$.

Proposition 4.1. *If $\mathcal{I}_r = \mathcal{I}_r(0, 0)$ and \mathcal{I}_{T_r} is the ideal sheaf of T_r in M_r , then $\mathcal{I}_r = \mathcal{I}_{T_r}$.*

Proof. The proof works as in [1, Proposition 4.1]. \square

Let $X \subset \mathbb{P}^4$ be an integral surface of degree d . Let $Y = X \cap H$ be the generic hyperplane section of X and let $Z = Y \cap K$ be the generic plane section of Y . Let \mathcal{I}_X be the ideal sheaf of X in $\mathcal{O}_{\mathbb{P}^4}$, \mathcal{I}_Y the ideal sheaf of Y in \mathcal{O}_H , with $H \cong \mathbb{P}^3$, and \mathcal{I}_Z the ideal sheaf of Z in \mathcal{O}_K , with $K \cong \mathbb{P}^2$. Let us consider for any $s \in \mathbb{N}$ the following maps:

$$\pi_s: H^0 \mathcal{I}_X(s) \rightarrow H^0 \mathcal{I}_Y(s) \quad \text{and} \quad \phi_s: H^1 \mathcal{I}_X(s-1) \rightarrow H^1 \mathcal{I}_X(s)$$

obtained by the cohomology associated to the exact sequence:

$$0 \rightarrow \mathcal{I}_X(s-1) \rightarrow \mathcal{I}_X(s) \rightarrow \mathcal{I}_Y(s) \rightarrow 0.$$

A *sporadic zero* of degree s is an element $\alpha \in \text{coker}(\pi_s) = \ker(\phi_s)$.

Definition 4.1. The *order* of a sporadic zero α is the maximum integer m such that $\alpha = \beta \cdot H^m$, for some $\beta \in H^1 \mathcal{I}_X(s-m-1)$, i.e. such that α is in the image of the map $H^1 \mathcal{I}_X(s-m-1) \rightarrow \mathcal{I}_X(s-1)$ induced by the injective morphism $\mathcal{I}_X(s-m-1) \rightarrow \mathcal{I}_X(s-1)$ defined by the multiplication for H^m .

Proposition 4.2. *Let α be a sporadic zero of degree s and let $h^0 \mathcal{I}_X(s) = 0$. Then one of the following conditions holds:*

- (1) $\deg X \leq s^2 - s + 1$;
- (2) $h^0 \mathcal{I}_Y(s) = 1$ and $h^0 \mathcal{I}_Z(s) = 2$.

Proof. Let $q = \min\{i \mid h^0 \mathcal{I}_Y(i) \neq 0\}$. So $q \leq s$ and by hypothesis there is an integral surface of degree q containing Y that does not lift to an integral surface of degree q containing X . In particular we have a sporadic zero of degree q for X and by [22, Theorem 2.1] we get a sporadic zero for Y of degree $s' \leq q$. By [2, Theorem 4.1] this means that there is an integral curve of degree s' in K containing Z that does not lift to a surface in H of degree s' containing Y . However, by restricting the integral surface of degree q containing Y to K we see that $d \leq qs'$.

If $s' < s$, then we see that $\deg X = \deg Z \leq s^2 - s$.

So we can suppose that $q = s' = s$, which implies that $h^0 \mathcal{I}_Z(s) \geq 1 + h^0 \mathcal{I}_Y(s) \geq 2$. If $h^0 \mathcal{I}_Z(s) \geq 3$, then by Theorem 2.1 and by Proposition 2.1 we get $\deg X = \deg Z \leq s^2 - s + 1$. So we can suppose that $h^0 \mathcal{I}_Z(s) = 2$, which implies also that $h^0 \mathcal{I}_Y(s) = 1$. \square

The following lemma, together with Proposition 4.2, provides us with the tools for the proof of the main results of this paper.

Lemma 4.1. *Let α be a sporadic zero of degree s and order m . Suppose that α determines a non-liftable integral surface R in H of degree s containing Y and that $I_R = (f)$ for some $f \in H^0 \mathcal{O}_H(s)$. Then for some $r \in \mathbb{N}$ such that $p^r \leq m+1$ there exist:*

- (1) $f_i \in H^0 \mathcal{O}_H(s)$ for $i = 0, \dots, 4$ such that the subscheme of H associate to the ideal $(f, x_i^{p^r} f_j - x_j^{p^r} f_i)_{|H}, i, j = 0, \dots, 4$ is a 1-dimensional scheme E (which can have isolated or embedded 0-dimensional subschemes) such that $Y \subset E \subset R$;
- (2) a reflexive sheaf \mathcal{N} of rank 3 such that we have the exact sequence:

$$(4.1) \quad 0 \rightarrow \mathcal{N} \rightarrow F^{r*} \Omega_H(p^r) \rightarrow \mathcal{I}_{E|R}(s) \rightarrow 0,$$

being $\mathcal{I}_{E|R} \subset \mathcal{O}_R$ the ideal sheaf of E .

Proof. Let $r \geq 0$ and let $\mathcal{S}_r = p_{M_r}^* \mathcal{S}_X$. Given the generic point $\eta \in \check{\mathbb{P}}^4$ and $g_{M_r}: M_r \rightarrow \check{\mathbb{P}}^4$, we have seen that $\overline{g_{M_r}^{-1}(\eta)}$ is isomorphic to the hypersurface H_r of \mathbb{P}^4 of degree p^r such that, over $\overline{k(\eta)}$, $(H_r)_{\text{red}} = H$.

By proceeding as in [1, Theorem 1.2, Step 1 and Step 2] and by Theorem 3.1 we see that there exist $r \geq 0$ and $V_r \subset M_r$ hypersurface given by:

$$\begin{cases} q(t, \underline{x}) = 0 \\ \sum_{i=0}^4 t_i x_i^{p^r} = 0 \end{cases}$$

such that the projection $p_{V_r}: V_r \rightarrow \mathbb{P}^4$ is generically smooth and, given $g_{V_r}: V_r \rightarrow \check{\mathbb{P}}^4$, $g_{V_r}^{-1}(\eta)$ is the complete intersection of H_r with a hypersurface of \mathbb{P}^4 of degree s and it is such that $g_{V_r}^{-1}(\eta)_{\text{red}} \cong R$ over $\overline{k(\eta)}$. This means that $m \geq p^r - 1$.

Let $U \subset V_r$ be the subscheme where p_{V_r} is not smooth. Then by Proposition 3.2 we see that:

$$(4.2) \quad U = V_r \cap V \left(x_i^{p^r} \frac{\partial q}{\partial t_j} - x_j^{p^r} \frac{\partial q}{\partial t_i} \mid i, j = 0, \dots, 4 \right).$$

By proceeding as in [1, Theorem 1.2, Step 3] we see that $U \supseteq T_r$, $\dim U = 5$ and we have for some $b > 0$:

$$(4.3) \quad 0 \rightarrow \Omega_{V_r/\mathbb{P}^4}^\vee \rightarrow \Omega_{M_r/\mathbb{P}^4}^\vee \otimes_{\mathcal{O}_{M_r}} \mathcal{O}_{V_r} \rightarrow \mathcal{S}_{U|V_r}(b, s) \rightarrow 0,$$

with $\mathcal{S}_{U|V_r} \subset \mathcal{O}_{V_r}$ ideal sheaf of U .

Restricting (4.3) to H and using Proposition 3.1 we get a surjective map

$$F^{r*} \Omega_H(p^r) \otimes_{\mathcal{O}_H} \mathcal{O}_R \rightarrow \mathcal{S}_{E|R}(s),$$

with $\mathcal{S}_{E|R} \subset \mathcal{O}_R$ ideal sheaf of the 1-dimensional scheme $E = U \cap g_{M_r}^{-1}(\eta)_{\text{red}}$. Note that $E \supseteq T_r \cap g_{M_r}^{-1}(\eta)_{\text{red}} \cong Y$. The kernel of the map $F^{r*} \Omega_H(p^r) \rightarrow \mathcal{S}_{E|R}(s)$ is the sheaf \mathcal{N} that determines the exact sequence (4.1) and it is torsion free and normal and so it is reflexive. Moreover, by (4.2) we get

$$E = V \left(q|_H, x_i^{p^r} \frac{\partial q}{\partial t_i} - x_j^{p^r} \frac{\partial q}{\partial t_i} |_H \mid i, j = 0, \dots, 4 \right),$$

where $q|_H = f$, and so the statement is proved by taking $f_i = \frac{\partial q}{\partial t_i}|_H$ for any $i = 0, \dots, 4$. \square

Now we can prove the first main theorem of the paper.

Theorem 4.1. *Let α be a sporadic zero of degree s and order m and let $p < s$. Let p^n be such that $p^n \leq m + 1$ and $p^{n+1} > m + 1$. Suppose that $h^0 \mathcal{S}_X(s) = 0$. Then:*

- (1) *if $s \geq 2m + 3$, we have $d \leq s^2 - s + p^n + 1$;*
- (2) *if $s \leq 2m + 2$, we have $d \leq s^2$.*

Proof. By Proposition 4.2 we can suppose that $h^0 \mathcal{S}_Y(s) = 1$ and $h^0 \mathcal{S}_Z(s) = 2$. In particular, if $s \leq 2m + 2$, we get the conclusion. So we suppose that $s \geq 2m + 3$ and we also see that the surface R of degree s containing Y that can not be lifted to a hypersurface of degree s containing X is integral. Let $I_R = (f)$ in H be the ideal of R .

By Lemma 4.1 we see that there exist $r \in \mathbb{N}$ with $p^r \leq m + 1$ and $f_i \in H^0 \mathcal{O}_H(s)$ for $i = 0, \dots, 4$ such that the subscheme of H associate to the ideal $(f, x_i^{p^r} f_j -$

$x_j^{p^r} f_i|_H, i, j = 0, \dots, 4$) is a 1-dimensional scheme E , which can have isolated or embedded 0-dimensional schemes, such that $Y \subset E \subset R$. Moreover, there exists a reflexive sheaf \mathcal{N} of rank 3 such that we have the exact sequence:

$$(4.4) \quad 0 \rightarrow \mathcal{N} \rightarrow F^{r*}\Omega_H(p^r) \rightarrow \mathcal{I}_{E|R}(s) \rightarrow 0,$$

being $\mathcal{I}_{E|R} \subset \mathcal{O}_R$ the ideal sheaf of E . We want to prove that $d \leq s^2 - s + 1 + p^r$.

Note that $c_1(\mathcal{N}) = -p^r - s$ and

$$(4.5) \quad c_2(\mathcal{N}) = s^2 + p^r s + p^{2r} - \deg E.$$

Let us recall that by [23, Proposition 1] and [9, Theorem 3.2] (see also Langer's remark in [9] after Corollary 6.3) the Bogomolov inequality holds also in positive characteristic for semistable reflexive sheaves in \mathbb{P}^n . So we see that if \mathcal{N} is semistable, by the Bogomolov inequality and by the fact that $\deg E \geq \deg Y = \deg X$ we get the statement. So we can suppose that \mathcal{N} is unstable. Moreover by Theorem 2.1 and by Proposition 2.1 we can suppose that $\Delta H_Z(s+i) = s-i-1$ for any $i \leq p^r$. Given $g \in H^0 \mathcal{O}_K(s)$ such that $f|_K$ and g are generators of I_Z in degree s , by [12, Proposition 1.4] we see that $f|_K$ and g are the only generators of I_Z in degree $\leq s + p^r$. By this remark we will get a contradiction.

Restricting (4.4) to K we get:

$$(4.6) \quad 0 \rightarrow \mathcal{N}|_K \rightarrow F^{r*}\Omega_K(p^r) \oplus \mathcal{O}_K \rightarrow \mathcal{I}_{E \cap K|R \cap K}(s) \rightarrow 0,$$

where $\mathcal{I}_{E \cap K|R \cap K} \subset \mathcal{O}_{R \cap K}$ is the ideal sheaf of $E \cap K$ in $R \cap K$. Since \mathcal{N} is unstable of rank 3, $F^{r*}\Omega_H(p^r)$ is stable and $c_1(F^{r*}\Omega_H(p^r)) = -p^r < 0$, the maximal destabilizing subsheaf \mathcal{F} of \mathcal{N} has rank at most 2 and $c_1(\mathcal{F}) < 0$. By [13, Theorem 3.1] we see that $\mathcal{F}|_K$ is still semistable and so it must be $h^0 \mathcal{N}|_K = 0$. By (4.6) we see that $h^0 \mathcal{I}_{E \cap K|R \cap K}(s) \geq 1$, which implies that $h^0 \mathcal{I}_{E \cap K}(s) \geq 2$ and, since $E \cap K \supseteq Z$ and $h^0 \mathcal{I}_Z(s) = 2$, we get that $h^0 \mathcal{I}_{E \cap K}(s) = 2$. Since $R \cap K$ is integral of degree s and $R \cap K \supset E \cap K$, we see that $\deg(E \cap K) \leq s^2$.

Recall that for any $i, j = 0, \dots, 4$:

$$x_i^{p^r} f_j - x_j^{p^r} f_i|_H \in H^0 \mathcal{I}_E(s+p^r) \Rightarrow x_i^{p^r} f_j - x_j^{p^r} f_i|_K \in H^0 \mathcal{I}_Z(s+p^r)$$

where $p^r \leq m+1$. By the assumption that $f|_K$ and g generate I_Z in degree $\leq s+p^r$ we can say that:

$$x_i^{p^r} f_j - x_j^{p^r} f_i|_K = h_{ij} f|_K + l_{ij} g,$$

for some $h_{ij}, l_{ij} \in H^0 \mathcal{O}_K(p^r)$. So:

$$(4.7) \quad E \cap K = V(f|_K, l_{ij} g \mid i, j = 0, \dots, 4).$$

So $E \cap K$ contains the complete intersection of two curves of degree s $V(f|_K, g)$, but we have seen that $\deg(E \cap K) \leq s^2$. This implies that $E \cap K$ is the complete intersection $V(f|_K, g)$ and so $\mathcal{I}_{E \cap K|R \cap K} \cong \mathcal{O}_{R \cap K}(-s)$. So by (4.6) we have:

$$(4.8) \quad 0 \rightarrow \mathcal{N}|_K \rightarrow F^{r*}\Omega_K(p^r) \oplus \mathcal{O}_K \rightarrow \mathcal{O}_{R \cap K} \rightarrow 0.$$

By the fact that $h^0 \mathcal{N}|_K = 0$, that $R \cap K$ is integral and by the commutative diagram:

$$\begin{array}{ccccccc}
& & & 0 & & & \\
& & & \downarrow & & & \\
& & & \mathcal{O}_K & \longrightarrow & \mathcal{O}_K & \longrightarrow 0 \\
& & 0 \longrightarrow & \mathcal{N}|_K & \longrightarrow & F^{r*}\Omega_K(p^r) \oplus \mathcal{O}_K & \longrightarrow \mathcal{O}_{R \cap K} \longrightarrow 0 \\
& & & \downarrow & & \downarrow & \\
& & & \mathcal{N}|_K & \longrightarrow & F^{r*}\Omega_K(p^r) & \\
& & & \downarrow & & \downarrow & \\
& & & 0 & & 0 &
\end{array}$$

we get the exact sequence:

$$(4.9) \quad 0 \rightarrow \mathcal{O}_K(-s) \rightarrow \mathcal{N}|_K \rightarrow F^{r*}\Omega_K(p^r) \rightarrow 0.$$

By the exact sequence:

$$0 \rightarrow F^{r*}\Omega_K(p^r) \rightarrow \mathcal{O}_K^{\oplus 3} \rightarrow \mathcal{O}_K(p^r) \rightarrow 0$$

and by the fact that $p^r \leq m+1 < \frac{s}{2}$ we see that $\text{Ext}^1(F^{r*}\Omega_K(p^r), \mathcal{O}_K(-s)) = 0$ and so $\mathcal{N}|_K \cong F^{r*}\Omega_K(p^r) \oplus \mathcal{O}_K(-s)$. Since $F^{r*}\Omega_K(p^r)$ is stable and:

$$\mu(F^{r*}\Omega_K(p^r)) = -\frac{p^r}{2} > \mu(\mathcal{O}_K(-s)) = -s,$$

we see that the maximal destabilizing subsheaf of $\mathcal{N}|_K$ is $F^{r*}\Omega_K(p^r)$. So, since \mathcal{N} is unstable of rank 3, by [13, Theorem 3.1] the maximal destabilizing subsheaf of \mathcal{N} must be a reflexive sheaf \mathcal{F} of rank 2 such that:

$$(4.10) \quad \mathcal{F}|_K \cong F^{r*}\Omega_K(p^r).$$

So, being \mathcal{F} the maximal destabilizing sheaf of \mathcal{N} , we have the following commutative diagram:

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathcal{N} & \longrightarrow & F^{r*}\Omega_H(p^r) & \longrightarrow & \mathcal{I}_{E|R}(s) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& \mathcal{I}_T(-s) & & \mathcal{L} & & \mathcal{I}_{E|R}(s) & \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

where \mathcal{I}_T is the ideal sheaf in \mathcal{O}_H of a zero-dimensional scheme T and \mathcal{Q} is a rank 1 sheaf such that $c_1(\mathcal{Q}) = 0$. Since $Q|_K \cong \mathcal{O}_K$, \mathcal{Q} must be torsion free and so $\mathcal{Q} = \mathcal{I}_W$ for some zero-dimensional scheme W . So we get:

$$0 \rightarrow \mathcal{I}_T(-s) \rightarrow \mathcal{I}_W \rightarrow \mathcal{I}_{E|R}(s) \rightarrow 0,$$

by which we get that $W \neq \emptyset$, because $h^0 \mathcal{I}_Y(s) = 1$. Moreover:

$$(4.11) \quad h^1 \mathcal{I}_E(n) = h^1 \mathcal{I}_{E|R}(n) = \deg W - \deg T$$

for any $n < s$ and:

$$(4.12) \quad h^1 \mathcal{I}_E(s) = h^1 \mathcal{I}_{E|R}(s) = \deg W - \deg T - 1,$$

because $h^0 \mathcal{I}_{E|R}(s) = 0$.

Let $F \subset E$ be the equidimensional component of dimension 1. Then there exists a sheaf \mathcal{K} of finite length determining the following exact sequence:

$$0 \rightarrow \mathcal{I}_E \rightarrow \mathcal{I}_F \rightarrow \mathcal{K} \rightarrow 0.$$

Then we see that $h^1 \mathcal{I}_E(n) = h^0 \mathcal{K}$ for $n \ll 0$, so that by (4.11) we see that $h^0 \mathcal{K} = \deg W - \deg T$. Moreover:

$$h^0 \mathcal{I}_E(s) - h^0 \mathcal{I}_F(s) + h^0 \mathcal{K} - h^1 \mathcal{I}_E(s) + h^1 \mathcal{I}_F(s) = 0$$

and so, since $Y \subset F \subset E$, $h^0 \mathcal{I}_E(s) = h^0 \mathcal{I}_F(s) = 1$ and by (4.12) we get:

$$h^1 \mathcal{I}_F(s) = h^1 \mathcal{I}_E(s) - h^0 \mathcal{K} = -1.$$

This is impossible and so we get a contradiction. \square

Corollary 4.1. *Let $h^0 \mathcal{I}_Y(s) \neq 0$ and let $p < s$. If $\deg X > s^2$, then $h^0 \mathcal{I}_X(s) \neq 0$.*

In the following theorem we see that for $p \geq s$ the bound for d is independent of the order of the sporadic zero α and coincides with the bound of the characteristic zero case (see [15] and [17]).

Theorem 4.2. *Let $h^0 \mathcal{I}_Y(s) \neq 0$, $h^0 \mathcal{I}_X(s) = 0$ and let $p \geq s$. Then $\deg X \leq s^2 - s + 2$.*

Proof. The proof works as in Theorem 4.1. We just need to remark that in the case $p \geq s$ it must be $r = 0$, which means $p^r = 1$. Indeed, proceeding as in Lemma 4.1 we see that we get an exact sequence:

$$0 \rightarrow \mathcal{I}_X(s - p^r) \rightarrow \mathcal{I}_X(s) \rightarrow \mathcal{I}_{X \cap H_r | H_r}(s) \rightarrow 0,$$

where $\mathcal{I}_{X \cap H_r | H_r} \subset \mathcal{O}_{H_r}$ is the ideal sheaf of $X \cap H_r$. Since $h^0 \mathcal{I}_{X \cap H_r | H_r}(s) \neq 0$ and $h^0 \mathcal{I}_X(s) = 0$, it must be $h^1 \mathcal{I}_X(s - p^r) \neq 0$. By the fact that X is integral we see that it must be $p^r < s$ and so $r = 0$ and $p^r = 1$. \square

Now we show that the bounds given in Theorem 4.1 and Theorem 4.2 are sharp.

Example 4.1. Let $r, p, s \in \mathbb{N}$ such that $s \geq 2p^r$. Let us consider $\mathcal{E} = \mathcal{O}_{\mathbb{P}^4}(p^r - 2s) \oplus \mathcal{O}_{\mathbb{P}^4}(-p^r - s)^{\oplus 2}$ and $\mathcal{F} = F^{r*} \Omega_{\mathbb{P}^4}(p^r - s)$. Then, since $\mathcal{E}^\vee \otimes \mathcal{F}$ is generated by global sections, by [8] the dependency locus of a general morphism $\varphi \in \text{Hom}(\mathcal{E}, \mathcal{F})$ is a smooth surface $X \subset \mathbb{P}^4$ and it is determined by the sequence:

$$(4.13) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^4}(p^r - 2s) \oplus \mathcal{O}_{\mathbb{P}^4}^{\oplus 2}(-p^r - s) \rightarrow F^{r*} \Omega_{\mathbb{P}^4}(p^r - s) \rightarrow \mathcal{I}_X \rightarrow 0.$$

Together with:

$$(4.14) \quad 0 \rightarrow F^{r*} \Omega_{\mathbb{P}^4}(p^r) \rightarrow \mathcal{O}_{\mathbb{P}^4} \rightarrow \mathcal{O}_{\mathbb{P}^4}(p^r) \rightarrow 0$$

this implies that $h^1\mathcal{S}_X = 0$, so that $h^0\mathcal{O}_X = 1$ and X is connected and, being smooth, X is integral. Moreover, $h^0\mathcal{S}_X(s) = 0$ and by a computation with Chern classes we see that $\deg X = s^2 - p^r s + 2p^{2r}$.

Let $H \subset \mathbb{P}^4$ be a general hyperplane and let $H_r \subset \mathbb{P}^4$ be the nonreduced hypersurface of degree p^r such that $H_r|_{\text{red}} = H$. Then, $(F^r)^{-1}(H) = H_r$. This shows that we have a commutative diagram:

$$\begin{array}{ccc} H_r & \xrightarrow{\pi} & H \\ \downarrow i & & \downarrow j \\ \mathbb{P}^4 & \xrightarrow{F^r} & \mathbb{P}^4 \end{array}$$

So we have:

$$i^*(F^{r*}\Omega_{\mathbb{P}^4}(p^r)) = i^*(F^{r*}(\Omega_{\mathbb{P}^4}(1))) = \pi^*(j^*(\Omega_{\mathbb{P}^4}(1))) \cong \pi^*(\Omega_H(1)) \oplus \mathcal{O}_{H_r}.$$

This implies that $h^0(F^{r*}\Omega_{\mathbb{P}^4}(p^r)|_{H_r}) \geq 1$. In particular, by (4.13) we see that $h^0\mathcal{S}_{X \cap H_r|_{H_r}}(s) \neq 0$, so that $h^0\mathcal{S}_Y(s) \neq 0$. Moreover, by (4.13) and by (4.14) we see that $h^1\mathcal{S}_X(s - p^r - 1) = 0$. This shows that X has a sporadic zero of degree s and order $m = p^r - 1$. So:

- (1) if $r = 0$ and $s \geq 2$, then $p^r = 1$, $m = 0$ and $\deg X = s^2 - s + 2$;
- (2) if $s = 2p^r + 1$, then $s = 2m + 3$ and $\deg X = s^2 - \frac{s-1}{2} = s^2 - s + p^r + 1$;
- (3) if $s = 2p^r$, then $s = 2m + 2$ and $\deg X = s^2$.

This shows that the bounds in Theorem 4.1 and Theorem 4.2 are sharp.

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