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SEMI-SYMMETRIC AND RICCI SEMI-SYMMETRIC LIGHTLIKE HYPERSURFACES OF AN INDEFINITE GENERALIZED SASAKIAN SPACE FORM

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Abstract. In this paper, we study semi-symmetric, Ricci semi-symmetric lightlike hypersurfaces of a indefinite generalized Sasakian space form with structure vector field tangent to hypersurface. We obtain the condition for Ricci tensor of lightlike hypersurface of indefinite generalized Sasakian space form to be symmetric and parallel.

1. Introduction

In the theory of hypersurfaces of semi-Riemannian manifolds it is interesting to study the geometry of lightlike hypersurfaces due to the fact that the intersection of normal vector bundle and the tangent bundle is non-trivial. Thus, the study becomes more interesting and remarkably different from the study of non-degenerate hypersurfaces. The geometry of lightlike hypersurfaces of semi-Riemannian manifolds was studied in [6].

A semi-Riemannian manifold is called semi-symmetric if $R(X, Y) \cdot R = 0$, where $R(X, Y)$ is the curvature operator act as a derivative on R. Semi-symmetric hypersurfaces of Euclidean spaces were classified by Nomizu [10] and a general study of semi-symmetric Riemannian manifolds was made by Szabo [13]. A semi-Riemannian manifold is said to be Ricci semi-symmetric [4], if the following condition is satisfied: $R(X, Y) \cdot Ric = 0$.

It is clear that every semi-symmetric manifold is Ricci semi-symmetric; however the converse is not true in general. P. J. Ryan [11] raised the following question for hypersurfaces of Euclidean spaces in 1972; "Are the conditions $R(X, Y) \cdot R = 0$ and $R(X, Y) \cdot Ric = 0$ equivalent for hypersurfaces of Euclidean spaces?". The explicit example of Ricci-symmetric but not semi-symmetric hypersurfaces in Euclidean

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space $E^{n+1}(n \geq 4)$ is given in [1, 4]. The lightlike hypersurfaces of semi-Euclidean spaces satisfying curvature conditions of semi-symmetry type was studied in [12].

The purpose of the present paper is to study the semi-symmetric and Ricci semi-symmetric lightlike hypersurface of indefinite generalized Sasakian space form with structure vector field ξ tangent to hypersurface.

In Section 2, we have collected the formulae and information which are useful in our subsequent sections. In Section 3, we study the semi-symmetric, Ricci semi-symmetric lightlike hypersurfaces of an indefinite generalized Sasakian space form.

2. Preliminaries

An odd-dimensional semi-Riemannian manifold $\overline{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $\{\phi, \xi, \eta, \overline{g}\}$, where ϕ is a (1,1) tensor field, ξ a vector field, η a 1-form and $\overline{g}$ is the semi-Riemannian metric on $\overline{M}$ satisfying

$$\left\{\begin{array}{ll} \phi^2 X = -X + \eta(X)\xi, \quad \eta \circ \phi = 0, & \phi\xi = 0, \quad \eta(\xi) = 1 \\ \overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \eta(X)\eta(Y), & \overline{g}(X, \xi) = \eta(X) \end{array}\right. \tag{2.1}$$

for any $X, Y \in \Gamma(T\overline{M})$, where $\Gamma(T\overline{M})$ denotes the Lie algebra of vector fields on $\overline{M}$.

An indefinite almost contact metric manifold $\overline{M}$ is called an indefinite Cosymplectic manifold if [7],

$$(\overline{\nabla}_X \phi)Y = 0, \quad \textit{and} \quad \overline{\nabla}_X \xi = 0 \tag{2.2}$$

for any $X, Y \in T\overline{M}$, where $\overline{\nabla}$ denote the Levi-Civita connection on $\overline{M}$.

An indefinite almost contact metric manifold $\{\overline{M}, \phi, \xi, \eta, \overline{g}\}$ is called an indefinite generalized Sasakian space form if there exist three functions f_1, f_2 and f_3 on $\overline{M}$ such that [2]

$$\begin{aligned} \overline{R}(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X \\ +2g(X, \phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ +g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \end{aligned} \tag{2.3}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$.

We write as follows:

$$\overline{R}(X, Y, Z, W) = \overline{g}(\overline{R}(X, Y)Z, W) \tag{2.4}$$

$$Ric(X, Y) = trace\{Z \to \overline{R}(X, Z)Y\} \tag{2.5}$$

where Ric denotes the Ricci tensor on $\overline{M}$ for $X, Y, Z, W \in \Gamma(T\overline{M})$.

For a (0, k)-tensor field T on $\overline{M}$, $k \geq 1$, the (0, k + 2) tensor field $\overline{R} \cdot T = 0$ is called curvature conditions of semi-symmetry type [4] and given by

$$\begin{aligned} (\overline{R}.T)(X_1, ..., X_k, X, Y) = -T(\overline{R}(X, Y)X_1, X_2, ..., X_k) \\ -... - T(X_1, ..., X_{k-1}, \overline{R}(X, Y)X_k) \end{aligned} \tag{2.6}$$

for $X, Y, X_1, X_k \in \Gamma(T\overline{M})$.

A semi-Riemannian space form $\overline{M}$ is said to be semi-symmetric if $\overline{R} \cdot \overline{R} = 0$. Thus, from (2.6) and properties of curvature tensor, we have

$$\begin{aligned} (\overline{R}(X, Y).\overline{R})(U, V)W = \overline{R}(X, Y)\overline{R}(U, V)W - \overline{R}(U, V)\overline{R}(X, Y)W \\ -\overline{R}(\overline{R}(X, Y)U, V)W - \overline{R}(U, \overline{R}(X, Y)V)W = 0, \end{aligned} \tag{2.7}$$

for any $X, Y, U, V, W \in \Gamma(T\overline{M})$.

A semi-Riemannian space form $\overline{M}$ is said to be Ricci semi-symmetric if $\overline{R}.Ric = 0$, i.e.,

$$(\overline{R}(X,Y).Ric)(X_1,X_2) = -Ric(\overline{R}(X,Y)X_1,X_2) - Ric(X_1,\overline{R}(X,Y)X_2) = 0 \tag{2.8}$$

for any $X, Y, X_1, X_2 \in \Gamma(T\overline{M})$.

Let (M, g) be a hypersurface of a $(2m+1)$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ with index s, $0 < s < 2m+1$ and $g = \overline{g}_{|M}$. Then M is lightlike hypersurface of $\overline{M}$ if g is of constant rank $(2m-1)$ and the normal bundle $TM^{\perp}$ is a distribution of rank 1 on M [6]. A non-degenerate complementary distribution $S(TM)$ of rank $(2m-1)$ to $TM^{\perp}$ in TM, that is, $TM = TM^{\perp} \perp S(TM)$, is called screen distribution. The following result (cf. [6], Theorem 1.1, page 79) has an important role in studying the geometry of lightlike hypersurface.

Theorem 2.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $\overline{M}$. Then, there exists a unique vector bundle $tr(TM)$ of rank 1 over M such that for any non-zero section E of $TM^{\perp}$ on a coordinate neighbourhood $U \subset M$, there exists a unique section N of $tr(TM)$ on U satisfying $\overline{g}(N, E) = 1$ and $\overline{g}(N, N) = \overline{g}(N, W) = 0$, $\forall\, W \in \Gamma(S(TM)|_u)$.*
Then, we have the following decomposition:

$$TM = S(TM) \perp TM^{\perp}, \qquad T\overline{M} = S(TM) \perp (TM^{\perp} \oplus tr(TM)). \tag{2.9}$$

Throughout this paper, all manifolds are supposed to be paracompact and smooth. We denote by $\Gamma(E)$ the smooth sections of the vector bundle E, by $\perp$ and $\oplus$ the orthogonal and the non-orthogonal direct sum of two vector bundles, respectively.

Let $\overline{\nabla}$, ∇ and ∇^t denote the linear connections on $\overline{M}$, M and vector bundle $tr(TM)$, respectively. Then, the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \;\; \forall X, Y \in \Gamma(TM) \tag{2.10}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \;\; \forall V \in \Gamma(tr(TM)) \tag{2.11}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X,Y), \nabla^t_X V\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively and A_V is the shape operator of M with respect to V. Moreover, in view of decomposition (2.9), equations (2.10) and (2.11) take the form

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N \tag{2.12}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N \tag{2.13}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(tr(TM))$, where $B(X,Y)$ and $\tau(X)$ are local second fundamental form and a 1-form on U, respectively. It follows that

$$B(X,Y) = \overline{g}(\overline{\nabla}_X Y, E) = \overline{g}(h(X,Y), E), B(X,E) = 0, \text{ and}$$
$$\tau(X) = \overline{g}(\nabla^t_X N, E).$$

Let P denote the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ and ∇^*, ∇^{*t} denote the linear connections on $S(TM)$ and $STM^{\perp}$, respectively. Then from the decomposition of tangent bundle of lightlike hypersurface, we have

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY) \tag{2.14}$$

$$\nabla_X E = -A^*_E X + \nabla^{*t}_X E \tag{2.15}$$

for any $X, Y \in \Gamma(TM)$ *and* $E \in \Gamma(TM^{\perp})$, *where* h^*, A^* *are the second fundamental form and the shape operator of distribution* $S(TM)$ *respectively.*

By direct calculations using Gauss-Weingarten formulae, (2.14) and (2.15), we find

$$g(A_N Y, PW) = \overline{g}(N, h^*(Y, PW)); \qquad \overline{g}(A_N Y, N) = 0, \tag{2.16}$$

$$g(A^*_E X, PY) = \overline{g}(E, h(X, PY); \qquad \overline{g}(A^*_E X, N) = 0, \tag{2.17}$$

for any $X, Y, W \in \Gamma(TM)$, $E \in \Gamma(TM^{\perp})$ *and* $N \in \Gamma(tr(TM))$.

Locally, we define on U

$$C(X, PY) = \overline{g}(h^*(X, PY), N), \qquad \lambda(X) = \overline{g}(\nabla^{*t}_X E, N). \tag{2.18}$$

Hence,

$$h^*(X, PY) = C(X, PY)E, \qquad \nabla^{*t}_X E = \lambda(X)E. \tag{2.19}$$

On the other hand, by using (2.12), (2.13), (2.15) and (2.18), we obtain

$$\lambda(X) = \overline{g}(\nabla_X E, N) = \overline{g}(\overline{\nabla}_X E, N) = -\overline{g}(E, \overline{\nabla}_X N) = -\tau(X).$$

Thus, locally (2.14) and (2.15) become

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)E, \qquad \nabla_X E = -A^*_E X - \tau(X)E. \tag{2.20}$$

Finally, (2.16) and (2.17), locally become

$$g(A_N Y, PW) = C(Y, PW); \qquad \overline{g}(A_N Y, N) = 0, \tag{2.21}$$

$$g(A^*_E X, PY) = B(X, PY); \qquad \overline{g}(A^*_E X, N) = 0. \tag{2.22}$$

We note that second equation of (2.21) implies that $A_N X \in \Gamma(S(TM))$ *for* $X \in \Gamma(TM)$, *i.e.* A_N *is* $\Gamma(S(TM))$ *valued. On the other hand, from* $\overline{g}(\overline{\nabla}_X E, E) = 0$, *we have*

$$B(X, E) = 0. \tag{2.23}$$

In general, the induced connection ∇ *on* M *is not a metric connection. Since* $\overline{\nabla}$ *is a metric connection, we have*

$$0 = (\overline{\nabla}_X \overline{g})(Y, Z) = X(\overline{g}(Y, Z)) - \overline{g}(\overline{\nabla}_X Y, Z) - \overline{g}(Y, \overline{\nabla}_X Z).$$

By using (2.12) in this equation, we obtain

$$(\nabla_X g)(Y, Z) = B(X, Y)\theta(Z) + B(X, Z)\theta(Y), \quad X, Y \in \Gamma(S(TM)|_u), \tag{2.24}$$

where θ *is a differential 1-form locally defined on* M *by* $\theta(\cdot) = \overline{g}(N, \cdot)$.

If $\overline{R}$ *and* R *are the curvature tensors of* $\overline{M}$ *and* M, *then using (2.12) in the equation* $\overline{R}(X, Y)Z = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z$, *we obtain*

$$\begin{aligned} \overline{R}(X, Y)Z &= R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X \\ &+\{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z)\}N \end{aligned} \tag{2.25}$$

$$(\nabla_X B)(Y, Z) = XB(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \tag{2.26}$$

3. Semi-symmetric and Ricci semi-symmetric Lightlike Hypersurfaces in Indefinite Generalized Sasakian Space Form

In this section, we consider semi-symmetric and Ricci semi-symmetric lightlike hypersurfaces M in an indefinite generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$.

For $X \in \Gamma(TM)$, we write

$$\phi X = tX + \beta(X)N \tag{3.1}$$

where tX is the tangential parts of ϕX and β is the one form on M.

Definition 3.1. Let M be a lightlike hypersurface of a $(2m+1)$-dimensional indefinite generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$. We say that M is semi-symmetric if the following condition is satisfied

$$(R(X,Y)\cdot R)(X_1, X_2, X_3, X_4) = 0 \tag{3.2}$$

for $X, Y, X_1, X_2, X_3, X_4 \in \Gamma(TM)$. We note that $(R(X,Y)\cdot R)(X_1, X_2, X_3, E) = 0$ for $E \in \Gamma(TM^{\perp})$, therefore equation (3.2) reduces to

$$(R(X,Y)\cdot R)(X_1, X_2, X_3, PX_4) = 0. \tag{3.3}$$

We have following :

Lemma 3.1. *Let M be a lightlike hypersurface of a $(2m+1)$-dimensional indefinite generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$. Then the Gauss equation of M is given by*

$$\begin{aligned} R(X,Y)Z = B(Y,Z)A_NX - B(X,Z)A_NY + f_1\{g(Y,Z)X - g(X,Z)Y\} \\ +f_2\{g(X,\phi Z)tY - g(Y,\phi Z)tX + 2g(X,\phi Y)tZ\} \\ +f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ -g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{3.4}$$

Proof: *From (2.3), (2.25), (3.1) and comparing the tangential part, we obtain (3.4).*

Theorem 3.1. *Let M be a lightlike hypersurface of a $(2m+1)$-dimensional indefinite generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$. Then, we have*

$$\begin{aligned} Ric(X,Y) = -(2m-1)f_1 g(X,Y) - f_2\{g(tX,\phi Y) + 2g(\phi X, tY) \\ +g(E,\phi Y)g(tX,N) - 2g(X,\phi E)g(tY,N)\} + f_3\{g(X,Y) \\ +(2m-2)\eta(X)\eta(Y)\} + \textstyle\sum_{i=1}^{2m-1}\epsilon_i B(w_i,Y)C(X,w_i) \\ -\alpha B(X,Y) \end{aligned} \tag{3.5}$$

where $\{(w_i), i = 1, 2, ..., (2m-1)\}$ is the orthogonal basis of $S(TM)$ and $\alpha = \sum_{i=1}^{2m-1}\epsilon_i C(W_i, W_i)$.

Proof: *By the definition of Ricci curvature*

$$Ric(X,Y) = \textstyle\sum_{i=1}^{2m-1}\epsilon_i g(R(X,w_i)Y, w_i) + \overline{g}(R(X,E)Y,N).$$

From (3.4), we have

$$\begin{aligned} Ric(X,Y) = -(2m-1)f_1 g(X,Y) - f_2\{g(tX,\phi Y) + 2g(\phi X, tY) \\ +g(E,\phi Y)g(tX,N) - 2g(X,\phi E)g(tY,N)\} + f_3\{g(X,Y) \\ +(2m-2)\eta(X)\eta(Y)\} + \textstyle\sum_{i=1}^{2m-1} \epsilon_i\{B(W_i,Y)C(X,W_i) \\ -B(X,Y)C(W_i,W_i)\}. \end{aligned} \tag{3.6}$$

Since $\sum_{i=1}^{2m-1} \epsilon_i C(W_i,W_i) = \alpha$, *hence (3.5) follows from (3.6).*

From theorem 3.1, we have:

Proposition 3.1. *The Ricci tensor of a lightlike hypersurface in a* $(2m+1)$*-dimensional indefinite generalized Sasakian space form* $\overline{M}(f_1,f_2,f_3)$ *is degenerate if* $f_2 = 0$.

Proposition 3.2. *The Ricci tensor of a lightlike hypersurface in a* $(2m+1)$*-dimensional indefinite generalized Sasakian space form* $\overline{M}(f_1,f_2,f_3)$ *is symmetric if* $f_2 = 0$ *and the shape operator of a lightlike hypersurface of* $\overline{M}(f_1,f_2,f_3)$ *is symmetric with respect to the second fundamental form of* M.
Proof. *From (3.5), we have*

$$\begin{aligned} Ric(X,Y) - Ric(Y,X) = -f_2\{g(tX,\phi Y) - g(tY,\phi X) + 2g(\phi X,tY) \\ -2g(\phi Y,tX) + g(E,\phi Y)g(tX,N) - g(E,\phi X)g(tY,N) \\ -2g(X,\phi E)g(tY,N) + 2g(Y,\phi E)g(tX,N)\} \\ +\textstyle\sum_{i=1}^{2m-1} \epsilon_i\{B(W_i,Y)C(X,W_i) - B(W_i,X)C(Y,W_i)\}. \end{aligned} \tag{3.7}$$

On the other hand, using equation (2.21) and (2.22), we get

$$\textstyle\sum_{i=1}^{m} \epsilon_i B(W_i,Y)C(X,W_i) = B(Y,A_N X). \tag{3.8}$$

From (3.7) and (3.8), we find

$$\begin{aligned} Ric(X,Y) - Ric(Y,X) = -f_2\{g(tX,\phi Y) - g(tY,\phi X) \\ +2g(\phi X,tY) - 2g(\phi Y,tX) + g(E,\phi Y)g(tX,N) \\ -g(E,\phi X)g(tY,N) - 2g(X,\phi E)g(tY,N) \\ +2g(Y,\phi E)g(tX,N)\} + B(Y,A_N X) - B(X,A_N Y). \end{aligned} \tag{3.9}$$

If $f_2 = 0$ *and the shape operator is symmetric with respect to the second fundamental form of* M, *then the result follows from (3.9).*

Corollary 3.1. *The Ricci tensor of a lightlike hypersurface in a* $(2m+1)$*-dimensional indefinite generalized Sasakian space form* $\overline{M}(f_1,f_2,f_3)$ *is symmetric if* $f_2 = 0$ *and* $C(X, A^*_\xi Y) = C(Y, A^*_\xi X)$.

Theorem 3.2. *Let* M *be a totally geodesic lightlike hypersurface of a cosymplectic indefinite generalized Sasakian space form* $\overline{M}(f_1,f_2,f_3)$. *Then, the Ricci tensor of* M *is parallel with respect to* ∇ *if* f_1, f_3 *are constants and* $f_2 = 0$.
Proof: *The derivative of Ricci tensor is given by*

$$(\nabla_Z Ric)(X,Y) = \nabla_Z Ric(X,Y) - Ric(\nabla_Z X, Y) - Ric(X, \nabla_Z Y).$$

Then, from (2.24) and (3.5), we have

$$
\begin{aligned}
(\nabla_Z Ric)(X,Y) &= -(2m-1)[Z(f_1)g(X,Y) + f_1\{B(Z,X)\theta(Y) \\
&+B(Z,Y)\theta(X)\}] - Z(f_2)\{g(tX,\phi Y) + 2g(\phi X,tY) \\
&+g(E,\phi Y)g(tX,N) - 2g(X,\phi E)g(tY,N)\} - f_2\{B(Z,tX)\theta(\phi Y) \\
&+B(Z,\phi Y)\theta(tX) + g((\nabla_Z t)X,\phi Y) + g(tX,(\nabla_Z\phi)Y) \\
&+B(Z,\phi Y)g(tX,N) + g(\nabla_Z E,\phi Y)g(tX,N) \\
&+g(E,(\nabla_Z\phi)Y)g(tX,N) + g(E,\phi Y)g((\nabla_Z t)X,N) \\
&+g(E,\phi Y)g(tX,\nabla_Z N) - 2B(Z,X)\theta(\phi E)g(tY,N) \\
&-2B(Z,\phi E)\theta(X)g(tY,N) - 2g(X,\phi E)g((\nabla_Z t)Y,N) \\
&-2g(X,\phi E)g(tY,\nabla_Z N) - 2g(X,\nabla_Z\phi E)g(tY,N)\} \\
&+f_3\{g(X,Y) + (2m-2)\eta(X)\eta(Y)\} + f_3[B(Z,X)\theta(Y) \\
&+B(Z,Y)\theta(X) + (2m-2)\{B(Z,\xi)\theta(X)\eta(Y) + B(Z,\xi)\theta(Y)\eta(X) \\
&+g(X,\nabla_Z\xi)\eta(Y) + g(Y,\nabla_Z\xi)\eta(X) + B(Z,\xi)\theta(Y)\eta(X) \\
&+g(Y,\nabla_Z\xi)\eta(X)\}] + \textstyle\sum_{i=1}^{2m-1}\epsilon_i\{(\nabla_Z B)(W_i,Y)C(X,W_i) \\
&+B(\nabla_Z W_i,Y)C(X,W_i) + B(W_i,Y)C(X,\nabla_Z W_i) \\
&+B(W_i,Y)(\nabla_Z C)(X,W_i)\} - Z(\alpha)B(X,Y) - \alpha(\nabla_Z B)(X,Y).
\end{aligned} \tag{3.10}
$$

Since, M is totally geodesic lightlike hypersurface of a Cosympletic indefinite generalized Sasakian space form, therefore $B(X,Y) = 0$ and $\nabla_X\xi = 0$, $\forall\ X,Y \in \Gamma(TM)$. Hence, from (3.10), we find

$$
\begin{aligned}
(\nabla_Z Ric)(X,Y) &= -(2m-1)(Zf_1)g(X,Y) - Z(f_2)\{g(tX,\phi Y) \\
&+2g(\phi X,tY) + g(E,\phi Y)g(tX,N) - 2g(X,\phi E)g(tY,N)\} \\
&-f_2\{g((\nabla_Z t)X,\phi Y) + g(tX,(\nabla_Z\phi)Y) + g(\nabla_Z E,\phi Y)g(tX,N) \\
&+g(E,(\nabla_Z\phi)Y)g(tX,N) + g(E,\phi Y)g((\nabla_Z t)X,N) \\
&+g(E,\phi Y)g(tX,\nabla_Z N) - 2g(X,\phi E)g((\nabla_Z t)Y,N) \\
&-2g(X,\phi E)g(tY,\nabla_Z N) - 2g(X,\nabla_Z\phi E)g(tY,N)\} \\
&+f_3\{g(X,Y) + (2m-2)\eta(X)\eta(Y)\}.
\end{aligned} \tag{3.11}
$$

From (3.11), it is obvious that $(\nabla_Z Ric)(X,Y) = 0$ if $f_2 = 0$ and f_1, f_3 are constants, which proves the Theorem.

Theorem 3.3. *Let M be a Ricci semi-symmetric lightlike hypersurface of an $(2m+1)$-dimensional indefinite generalized Sasakian space form $\overline{M}(f_1,f_2,f_3)$. If $f_2 = 0$, then, either M is totally geodesic or $Ric(E, A_N E) = 0$ for $E \in \Gamma(TM^\perp)$, where Ric is the Ricci tensor of M and A denotes the shape operator of M.*

Proof: *Suppose M is Ricci semi-symmetric, then from (2.8), we have:*

$$
0 = -Ric(R(X,Y)X_1, X_2) - Ric(X_1, R(X,Y)X_2). \tag{3.12}
$$

Using (3.5) in (3.12), we find

$$
\begin{aligned}
0 = & -B(Y,X_1)Ric(A_NX,X_2) + B(X,X_1)Ric(A_NY,X_2) \\
& -f_1\{g(Y,X_1)Ric(X,X_2) - g(X,X_1)Ric(Y,X_2)\} \\
& -f_2\{g(X,\phi X_1)Ric(tY,X_2) - g(Y,\phi X_1)Ric(tX,X_2) \\
& +2g(X,\phi Y)Ric(tX_1,X_2)\} - f_3\{\eta(X)\eta(X_1)Ric(Y,X_2) \\
& -\eta(Y)\eta(X_1)Ric(X,X_2) + g(X,X_1)\eta(Y)Ric(\xi,X_2) \\
& -g(Y,X_1)\eta(X)Ric(\xi,X_2)\} - B(Y,X_2)Ric(X_1,A_NX) \\
& +B(X,X_2)Ric(X_1,A_NY) - f_1\{g(Y,X_2)Ric(X_1,X) \\
& -g(X,X_2)Ric(X_1,Y)\} - f_2\{g(X,\phi X_2)Ric(X_1,tY) \\
& -g(Y,\phi X_2)Ric(X_1,tX) + 2g(X,\phi Y)Ric(X_1,tX_2)\} \\
& -f_3\{\eta(X)\eta(X_2)Ric(X_1,Y) - \eta(Y)\eta(X_2)Ric(X_1,X) \\
& +g(X,X_2)\eta(Y)Ric(X_1,\xi) - g(Y,X_2)\eta(X)Ric(X_1,\xi)\}.
\end{aligned} \tag{3.13}
$$

Putting $X_1 = E$ in (3.13) and using (2.23), we obtain;

$$
\begin{aligned}
0 = & -f_2\{g(X,\phi E)Ric(tY,X_2) - g(Y,\phi E)Ric(tX,X_2) \\
& +2g(X,\phi Y)Ric(tE,X_2)\} - B(Y,X_2)Ric(E,A_NX) \\
& +B(X,X_2)Ric(E,A_NY) - f_1\{g(Y,X_2)Ric(E,X) \\
& -g(X,X_2)Ric(E,Y)\} - f_2\{g(X,\phi X_2)Ric(E,tY) \\
& -g(Y,\phi X_2)Ric(E,tX) + 2g(X,\phi Y)Ric(E,tX_2)\} \\
& -f_3\{\eta(X)\eta(X_2)Ric(E,Y) - \eta(Y)\eta(X_2)Ric(E,X) \\
& +g(X,X_2)\eta(Y)Ric(E,\xi) - g(Y,X_2)\eta(X)Ric(E,\xi)\}.
\end{aligned} \tag{3.14}
$$

Putting $Y = E$ in (3.14), we get;

$$
\begin{aligned}
0 = & -f_2\{3g(X,\phi E)Ric(tE,X_2) + g(X,\phi X_2)Ric(E,tE) \\
& -g(E,\phi X_2)Ric(E,tX) + 2g(X,\phi E)Ric(E,tX_2)\} \\
& +B(X,X_2)Ric(E,A_NE).
\end{aligned} \tag{3.15}
$$

If $f_2 = 0$ then from (3.15), we have

$$B(X,X_2)Ric(E,A_NE) = 0.$$

So, if $B(X,X_2) = 0$ for any $X, X_2 \in \Gamma(TM)$, then M is totally geodesic. If M is not totally geodesic, it follows that $Ric(E,A_NE) = 0$.

Theorem 3.4. *Let M be a totally geodesic lightlike hypersurface of $(2m+1)$-dimensional indefinite generalized Sasakian space form $\overline{M}(f_1,f_2,f_3)$. Then, M is semi-symmetric if $f_2 = 0$.*

Proof: *Let M be a lightlike hypersurface of indefinite generalized Sasakian space form. Then, we have*

$$
\begin{aligned}
g((R(X,Y).R)(U,V)W,Z) = & \, g(R(X,Y).R(U,V)W,Z) \\
& -g(R(U,V)R(X,Y)W,Z) - g(R(R(U,V)X,Y)W,Z) \\
& -g(R(X,R(U,V)Y)W,Z)
\end{aligned} \tag{3.16}
$$

$\forall \ X,Y,Z,U,V,W \in \Gamma(TM)$.

Using (3.4) and Definition 3.1 in (3.16), we obtain

$g((R(X,Y).R)(U,V)W,PZ)$

$$
\begin{aligned}
&= B(Y,R(U,V)W)g(A_NX,PZ) - B(X,R(U,V)W)g(A_NY,PZ) - \\
&B(V,R(X,Y)W)g(A_NU,PZ) + B(U,R(X,Y)W)g(A_NV,PZ) - \\
&B(Y,W)g(A_NR(U,V)X,PZ) + B(R(U,V)X,W)g(A_NY,PZ) - \\
&B(R(U,V)Y,W)g(A_NX,PZ) + B(X,W)g(A_NR(U,V)Y,PZ) + \\
&f_1\{g(Y,R(U,V)W)g(X,PZ) - g(X,R(U,V)W)g(Y,PZ) - \\
&g(V,R(X,Y)W)g(U,PZ) + g(U,R(X,Y)W)g(V,PZ) - \\
&g(Y,W)g(R(U,V)X,PZ) + g(R(U,V)X,W)g(Y,PZ) \\
&- g(R(U,V)Y,W)g(X,PZ) + g(X,W)g(R(U,V)Y,PZ)\} + \\
&f_2\{g(X,\phi R(U,V)W)g(tY,PZ) - g(Y,\phi R(U,V)W)g(tX,PZ) + \\
&2g(X,\phi Y)g(tR(U,V)W,PZ) - g(U,\phi R(X,Y)W)g(tV,PZ) + \\
&g(V,\phi R(X,Y)W)g(tU,PZ) - 2g(U,\phi V)g(tR(X,Y)W,PZ) - \\
&g(R(U,V)X,\phi W)g(tY,PZ) + g(tR(U,V)X,PZ)g(Y,\phi W) - \\
&2g(R(U,V)X,\phi Y)g(tW,PZ) - g(X,\phi W)g(tR(U,V)W,PZ) + \\
&g(tX,PZ)g(R(U,V)Y,\phi W) - 2g(X,\phi R(U,V)Y)g(tW,PZ)\} + \\
&f_3\{\eta(X)\eta(R(U,V)W)g(Y,PZ) - \eta(Y)\eta(R(U,V)W)g(X,PZ) + \\
&g(X,R(U,V)W)\eta(Y)\eta(PZ) - g(Y,R(U,V)W)\eta(X)\eta(PZ) - \\
&\eta(U)\eta(R(X,Y)W)g(V,PZ) + \eta(V)\eta(R(X,Y)W)g(U,PZ) - \\
&g(U,R(X,Y)W)\eta(V)\eta(PZ) + g(V,R(X,Y)W)\eta(U)\eta(PZ) - \\
&\eta(R(U,V)X)\eta(W)g(Y,Z) + \eta(Y)\eta(W)g(R(U,V)X,PZ) - \\
&g(R(U,V)X,W)\eta(Y)\eta(PZ) + g(Y,W)\eta(R(U,V)X)\eta(PZ) - \\
&g(R(U,V)Y,PZ)\eta(X)\eta(W) + g(X,PZ)\eta(R(U,V)Y)\eta(W) - \\
&g(X,W)\eta(R(U,V)Y)\eta(PZ) + g(R(U,V)Y,W)\eta(X)\eta(PZ)\}.
\end{aligned}
$$

Or,

$$
\begin{aligned}
&g((R(X,Y).R)(U,V)W,PZ) \\
&= g(A_NX,PZ)[B(Y,A_NU)B(V,W) - B(Y,A_NV)B(U,W) + f_1\{g(V,W)B(Y,U) - \\
&g(U,W)B(Y,V)\} + f_2\{\eta(U)\eta(W)B(Y,V) - g(V,\phi W)B(Y,tU) + \\
&2g(U,\phi V)B(Y,tW)\} + f_3\{\eta(U)\eta(W)B(Y,V) - \eta(V)\eta(W)B(U,Y) + \\
&g(U,W)\eta(V)B(Y,\xi) - g(V,W)\eta(U)B(Y,\xi)\}] - g(A_NY,PZ)[B(X,A_NU)B(V,W) - \\
&B(X,A_NV)B(U,W) + f_1\{g(V,W)B(X,U) - g(U,W)B(X,V)\} + \\
&f_2\{\eta(U)\eta(W)B(X,V) - g(V,\phi W)B(X,tU) + 2g(U,\phi V)B(X,tW)\} + \\
&f_3\{\eta(U)\eta(W)B(X,V) - \eta(V)\eta(W)B(U,X) + g(U,W)\eta(V)B(X,\xi) - \\
&g(V,W)\eta(U)B(X,\xi)\}] - g(A_NU,PZ)[B(V,A_NX)B(Y,W) - \\
&B(V,A_NY)B(X,W) + f_1\{g(Y,W)B(V,X) - g(X,W)B(V,Y)\} + \\
&f_2\{g(X,\phi W)B(V,tY) - g(Y,\phi W)B(V,tX) + 2g(X,\phi Y)B(V,tZ)\} + \\
&f_3\{\eta(X)\eta(W)B(V,Y) - \eta(Y)\eta(W)B(V,X) + g(X,W)\eta(Y)B(V,\xi) - \\
&g(Y,W)\eta(X)B(V,\xi)\}] + g(A_NV,PZ)[B(U,A_NX)B(Y,W) - B(U,A_NY)B(X,W) + \\
&f_1\{g(Y,W)B(U,X) - g(X,W)B(U,Y)\} + f_2\{g(X,\phi W)B(U,tY) - \\
&g(Y,\phi W)B(U,tX) + 2g(X,\phi Y)B(U,tZ)\} + f_3\{\eta(X)\eta(W)B(U,Y) - \\
&\eta(Y)\eta(W)B(U,X) + g(X,W)\eta(Y)B(U,\xi) - g(Y,W)\eta(X)B(U,\xi)\}] - \\
&B(Y,W)[g(A_NB(V,X)A_NU,PZ) - g(A_NB(U,X)A_NV,PZ) + \\
&g(A_Nf_1\{g(V,X)U - g(U,X)V\},PZ) + g(A_Nf_2\{g(U,\phi X)tV - g(V,\phi X)tU + \\
&2g(U,\phi V)tX\},PZ) + g(A_Nf_3\{\eta(U)\eta(X)V - \eta(V)\eta(X)U + g(U,X)\eta(V)\xi - \\
&g(V,X)\eta(U)\xi\},PZ)] + g(A_NY,PZ)[B(V,X)B(A_NU,W) - B(U,X)B(A_NV,W) + \\
&f_1\{g(V,X)B(U,W) - g(U,X)B(U,W)\} + f_2\{g(U,\phi X)B(tV,W) - \\
&g(V,\phi X)B(tU,W) + 2g(U,\phi V)B(tX,W)\} + f_3\{\eta(X)\eta(U)B(V,W) - \\
&\eta(X)\eta(V)B(U,W) + g(X,U)\eta(V)B(\xi,W) - g(X,V)\eta(U)B(\xi,W)\}] - \\
&g(A_NX,PZ)[B(U,Y)B(A_NU,W) - B(U,Y)B(A_NV,W) + f_1\{g(V,Y)B(U,W) -
\end{aligned}
$$

$$
\begin{aligned}
&g(U,Y)B(V,W)\} + f_2\{g(U,\phi Y)B(tV,W) - g(V,\phi Y)B(tU,W) + \\
&2g(U,\phi V)B(tY,W)\} + f_3\{\eta(U)\eta(V)B(V,W) - \eta(V)\eta(Y)B(U,W) + \\
&g(U,Y)\eta(V)B(\xi,W) - g(V,Y)\eta(U)B(\xi,W)\}] + B(X,W)[g(A_N B(V,Y)A_N U, PZ) - \\
&g(A_N B(U,Y)A_N V, PZ) + g(A_N f_1\{g(V,Y)U - g(U,Y)V\}, PZ) + \\
&g(A_N f_2\{g(U,\phi Y)tV - g(V,\phi Y)tU + 2g(U,\phi V)tY\}, PZ) + g(A_N f_3\{\eta(U)\eta(Y)V - \\
&\eta(V)\eta(Y)U + g(U,Y)\eta(V)\xi - g(V,Y)\eta(U)\xi\}, PZ) + f_1\{g(Y,R(U,V)W)g(X,PZ) - \\
&g(X,R(U,V)W)g(Y,PZ) - g(V,R(X,Y)W)g(U,PZ) + \\
&g(U,R(X,Y)W)g(V,PZ) + g(R(U,V)X,W)g(Y,PZ) - g(R(U,V)X,PZ)g(Y,W) - \\
&g(PZ,R(U,V)Y)g(X,W) - g(W,R(U,V)Y)g(X,PZ)\} + \\
&f_2\{g(X,\phi R(U,V)W)g(tY,PZ) - g(Y,\phi R(U,V)W)g(tX,PZ) + \\
&2g(X,\phi Y)g(PZ,tR(U,V)W) + g(V,\phi R(X,Y)W)g(tU,PZ) - \\
&g(U,\phi R(X,Y)W)g(tV,PZ) - 2g(U,\phi V)g(PZ,tR(X,Y)W) - \\
&g(R(U,V)X,\phi W)g(tY,PZ) + g(Y,\phi W)g(tR(U,V)X,PZ) - \\
&2g(R(U,V)X,\phi Y)g(tW,PZ) - g(X,\phi W)g(tR(U,V)W,PZ) - \\
&g(R(U,V)Y,\phi W)g(tX,PZ) - 2g(X,\phi R(U,V)Y)g(tW,PZ)\} + \\
&f_3\{\eta(X)\eta(R(U,V)W)g(Y,PZ) - \eta(Y)\eta(R(U,V)W)g(X,PZ) + \\
&\eta(Y)\eta(PZ)g(R(U,V)W,X) - \eta(X)\eta(PZ)g(R(U,V)W,Y) - \\
&g(V,PZ)\eta(U)\eta(R(X,Y)W) + g(U,PZ)\eta(V)\eta(R(X,Y)W) - \\
&g(U,R(X,Y)W)\eta(V)\eta(PZ) + g(V,R(X,Y)W)\eta(U)\eta(PZ) - \\
&\eta(R(U,V)X)\eta(W)g(Y,PZ) + \eta(Y)\eta(W)g((R(U,V)X,PZ) - \\
&g(R(U,V)X,W)\eta(Y)\eta(PZ) + g(Y,W)\eta(R(U,V)X)\eta(PZ) - \\
&g(R(U,V)Y,PZ)\eta(X)\eta(W) + g(X,PZ)\eta(R(U,V)Y)\eta(W) - \\
&g(X,W)\eta(R(U,V)Y)\eta(PZ) + g(R(U,V)Y,W)\eta(X)\eta(PZ)\}.
\end{aligned}
$$

Putting $Y = U = E \in \Gamma(TM^{\perp})$ in above equation and a straight forward calculations, we have

$g((R(X,E).R)(E,V)W,PZ) =$

$$
\begin{aligned}
&-g(A_N E,PZ)[B(V,W)B(X,A_N E) - f_2\{g(V,\phi W)B(X,tE) - \\
&2g(E,\phi V)B(X,tW)\} - B(V,A_N E)B(X,W) + f_2\{g(X,\phi W)B(V,tE) - \\
&g(E,\phi W)B(V,tX) + 2g(X,\phi E)B(V,tZ)\} - B(V,X)B(A_N E,W) - \\
&f_2\{g(E,\phi X)B(tV,W) - g(V,\phi X)B(tE,W)2g(E,\phi V)B(tX,W)\}] + \\
&g(A_N X,PZ)[f_2\{g(V,\phi E)B(tE,W) - 2g(E,\phi V)B(tE,W)\}] + \\
&B(X,W)[g(A_N f_2\{-g(V,\phi E)B(tE,W) + 2g(E,\phi V)tE\},PZ)] + \\
&f_1[g(E,R(E,V)W)g(X,PZ) + g(E,R(X,E)W)g(V,PZ) + \\
&g(X,W)g(R(E,V)E,PZ) - g(R(E,V)E,W)g(X,PZ)\} + \\
&f_2\{g(X,\phi R(E,V)W)g(tE,PZ) - g(E,\phi R(E,V)W)g(tX,PZ) + \\
&2g(X,\phi E)g(PZ,tR(E,V)W) + g(V,\phi R(X,E)W)g(tE,PZ) - \\
&g(E,\phi R(X,E)W)g(tV,PZ) - 2g(E,\phi V)g(PZ,tR(X,E)W) - \\
&g(R(E,V)X,\phi W)g(tE,PZ) + g(E,\phi W)g(tR(E,V)X,PZ) - \\
&2g(R(E,V)X,\phi E)g(tW,PZ) - g(X,\phi W)g(tR(E,V)W,PZ) - \\
&g(R(E,V)E,\phi W)g(tX,PZ) - 2g(X,\phi R(E,V)E)g(tW,PZ)\} + \\
&f_3\{\eta(X)\eta(R(E,V)W)g(E,PZ) - \eta(X)\eta(PZ)g(R(E,V)W,E) - \\
&g(E,R(X,E)W)\eta(V)\eta(PZ) - g(R(E,V)E,PZ)\eta(X)\eta(W) + \\
&g(X,PZ)\eta(R(E,V)E)\eta(W) - g(X,W)\eta(R(E,V)E)\eta(PZ) + \\
&g(R(E,V)E,W)\eta(X)\eta(PZ)\}.
\end{aligned}
$$

Taking $f_2 = 0$ and using the fact that M is totally geodesic in (3.17), we find

$$g((R(X,Y).R)(U,V)W,PZ) = 0,$$

which proves the theorem.

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