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IMPROVED CHEN-RICCI INEQUALITY FOR LAGRANGIAN SUBMANIFOLDS IN QUATERNION SPACE FORMS

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Abstract. In this article, we obtain an improved Chen-Ricci inequality and completely classify Lagrangian submanifolds in quaternion space forms satisfying the equality. Our result is an affirmative answer to Problem 4.6 in [12].

1. Introduction

Let M be a Riemannian n-manifold and X be a unit vector. We choose an orthonormal frame $\{e_1, \cdots, e_n\}$ in T_xM such that $e_1 = X$. We denote the Ricci curvature at X by

$$Ric(X) = K_{12} + \cdots + K_{1n},$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i, e_j.

In [1] B.-Y. Chen proved the following Chen-Ricci inequality on Ricci curvature for any n-dimensional submanifold in Riemannian manifold of constant sectional curvature c:

$$Ric(X) \leq \frac{n-1}{4}c + \frac{n^2}{4}||H||^2.$$

This inequality is not optimal for Lagrangian submanifolds in complex space forms. Using an optimization technique, Oprea in [10] (also see [11]) proved

$$Ric(X) \leq \frac{n-1}{4}(c + n||H||^2),$$

which improves the Chen-Ricci inequality for Lagrangian submanifolds in complex space forms of constant holomorphic sectional curvature c.

In [5] we provided an algebraic proof for the improved Chen-Ricci inequality and completely characterized Lagrangian submanifolds in complex space forms satisfying the equality.

In this article, we extend the improved Chen-Ricci inequality to Lagrangian submanifolds in quaternion space forms. We also provide a detailed affirmative answer to Problem 4.6 in [12], completing the remark 3.2 in [5].

Theorem 3.1 and Corollary 3.2 improve a number of results in [1],[5],[7] and [8] for Lagrangian submanifolds in quaternion space forms.

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2. Preliminaries

Let $\tilde{M}^n$ be a $4n$-dimensional Riemannian manifold with metric g. $\tilde{M}^n$ is called quaternion Kaehler manifold if there exists a 3-dimensional vector bundle V of tensors of type (1,1) over $\tilde{M}^n$ with local basis of almost Hermitian structures I, J and K such that

(a) $IJ = -JI = K, \quad JK = -KJ = I, \quad KI = -IK = J, \quad I^2 = J^2 = K^2 = -Id$,

(b) for any local cross-section η of V, $\tilde{\nabla}_X \eta$ is also a cross-section of V, where X is an arbitrary vector field on $\tilde{M}^n$ and $\tilde{\nabla}$ the Riemannian connection on $\tilde{M}^n$.

In fact, condition (b) is equivalent to the following condition:

(b') there exist local 1-forms p, q and r such that

$$\begin{aligned}
\tilde{\nabla}_X I &= \qquad\quad r(X)J - q(X)K \\
\tilde{\nabla}_X J &= -r(X)I \qquad\quad + p(X)K \\
\tilde{\nabla}_X K &= q(X)I - p(X)J.
\end{aligned}$$

Now let X be a unit vector on $\tilde{M}^n$. Then X, IX, JX and KX form an orthonormal frame on $\tilde{M}^n$. We denote by $Q(X)$ the 4-plane spanned by them. For any two orthonormal vectorsX, Y on $\tilde{M}^n$, if $Q(X)$ and $Q(Y)$ are orthogonal, the plane $\pi(X, Y)$ spanned by X, Y is called a totally real plane. Any 2-plane in a $Q(X)$ is called a quaternionic plane. The sectional curvature of a quaternionic plane π is called the quaternionic sectional curvature of π. A quaternionic Kaehler manifold is a quaternion space form if its quaternionic sectional curvature are equal to a constant, say c. We denote such a 4n-dimensional quaternion space form by $\tilde{M}^n(c)$.

It is known that a quaternionic Kaehler manifold $\tilde{M}^n$ is a quaternion space form if and only if its curvature tensor $\tilde{R}$ is of the following form [6]:

$$\begin{aligned}
\tilde{R}(X,Y)Z = &\frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + \\
&+ g(IY,Z)IX - g(IX,Z)IY + 2g(X,IY)IZ \\
&+ g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ \\
&+ g(KY,Z)KX - g(KX,Z)KY + 2g(X,KY)KZ\}
\end{aligned}$$

Let $f : M \to \tilde{M}^n$ be an isometric immersion of a Riemannian n-manifold M into a 4n-dimensional quaternion space form $\tilde{M}^n(c)$. Then M is called a Lagrangian (or totally real) submanifold if each 2-plane of M is mapped into a toally real plane in $\tilde{M}^n(c)$.

From now on we assume that M is a Lagrangian submanifold of a 4n-dimensional quaternion space form $\tilde{M}^n(c)$. The formulas of Gauss and Weingarten are given respectively by

$$\begin{aligned}
\tilde{\nabla}_X Y &= \nabla_X Y + h(X,Y), \\
\tilde{\nabla}_X \xi &= -A_\xi X + D_X \xi,
\end{aligned} \tag{2.1}$$

for tangent vector fields X and Y and normal vector fields ξ, where D is the normal connection. The second fundamental form h is related to A_ξ by

$$\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle .$$

The mean curvature vector H of M is defined by

$$H = \frac{1}{n} \operatorname{trace} h.$$

We choose a local orthonormal frame field in $\tilde{M}^n(c)$:

$$\begin{aligned}
&e_1, e_2, \ldots\ldots, e_n; && e_{I(1)} = Ie_1, \ldots, e_{I(n)} = Ie_n; \\
&e_{J(1)} = Je_1, \ldots, e_{J(n)} = Je_n; && e_{K(1)} = Ke_1, \ldots, e_{K(n)} = Ke_n,
\end{aligned} \tag{2.2}$$

in such a way that, restricting to M, $e_1, \ldots, e_n$ are tangent to M.

We will use the following convention on the range of indices:

$$
\begin{aligned}
&A, B, C, D = 1, \ldots, n, I(1), \ldots, I(n), J(1), \ldots, J(n), K(1), \ldots, K(n),\\
&i, j, k, l = 1, \ldots, n,\\
&\alpha, \beta = I(1), \ldots, I(n), J(1), \ldots, J(n), K(1), \ldots, K(n),\\
&\phi_1 = I,\ \phi_2 = J,\ \phi_3 = K,\\
&\phi_1(k) = I(k),\ \phi_2(k) = J(k),\ \phi_3(k) = K(k),
\end{aligned}
$$

We set $h_{ij}^{\alpha} = g(h(e_i, e_j), e_\alpha)$. Then for any given r we have (see (2.9) in [4])

$$
h_{ij}^{\phi_r(k)} = h_{ki}^{\phi_r(j)} = h_{jk}^{\phi_r(i)},\ r = 1, 2, 3. \tag{2.3}
$$

Chen introduced the concept of *Lagrangian H-umbilical submanifolds* in [2] to study the "simplest" Lagrangian submanifolds next to the totally geodesic ones. We can extend the notion of *Lagrangian H-umbilical submanifolds* to Lagrangian submanifolds of a quaternion manifold ([9]). By a Lagrangian H-umbilical submanifold of a quaternion manifold $\tilde{M}^n$ we mean a Lagrangian submanifold whose second fundamental form takes the following simple form:

$$
\begin{aligned}
h(e_1, e_1) &= \lambda_1 I(e_1) + \lambda_2 J(e_1) + \lambda_3 K(e_1)\\
h(e_2, e_2) &= \mu_1 I(e_1) + \mu_2 J(e_1) + \mu_3 K(e_1),\\
h(e_1, e_j) &= \mu_1 I(e_j) + \mu_2 J(e_j) + \mu_3 K(e_j),\\
h(e_j, e_k) &= 0, \quad j \neq k, \quad j, k = 2, \ldots, n
\end{aligned}
\tag{2.4}
$$

for some suitable functions λ_r and $\mu_r, r = 1, 2, 3$ with respect to some suitable local orthonormal frame field.

3. Improved Chen-Ricci inequality

We first take a look at the mean curvature vector H. We set $\phi_1 = I,\ \phi_2 = J,\ \phi_3 = K$ as in the previous section. With the orthonormal frames from (2.2), we have

$$
\begin{aligned}
h(e_1, e_1) = \sum_{\alpha=I(1)}^{K(n)} h_{11}^{\alpha} e_\alpha &= h_{11}^{I(1)} e_{I(1)} + \cdots + h_{11}^{I(n)} e_{I(n)} +\\
&\quad + h_{11}^{J(1)} e_{J(1)} + \cdots + h_{11}^{J(n)} e_{J(n)} +\\
&\quad + h_{11}^{K(1)} e_{K(1)} + \cdots + h_{11}^{K(n)} e_{K(n)}\\
&= \sum_{r=1}^{3} \sum_{k=1}^{n} h_{11}^{\phi_r(k)} e_{\phi_r(k)}.
\end{aligned}
$$

Similarly,

$$
h(e_i, e_i) = \sum_{r=1}^{3} \sum_{k=1}^{n} h_{ii}^{\phi_r(k)} e_{\phi_r(k)}.
$$

We set

$$
H_r^j = \frac{1}{n} \sum_{k=1}^{n} h_{kk}^{\phi_r(j)}, r = 1, 2, 3. \tag{3.1}
$$

Then

$$
\begin{aligned}
H &= \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i) \\
&= \frac{1}{n}\sum_{i=1}^{n} h_{ii}^{I(1)} e_{I(1)} + \cdots + \frac{1}{n}\sum_{i=1}^{n} h_{ii}^{I(n)} e_{I(n)} + \\
&\quad + \frac{1}{n}\sum_{i=1}^{n} h_{ii}^{J(1)} e_{J(1)} + \cdots + \frac{1}{n}\sum_{i=1}^{n} h_{ii}^{J(n)} e_{J(n)} + \\
&\quad + \frac{1}{n}\sum_{i=1}^{n} h_{ii}^{K(1)} e_{K(1)} + \cdots + \frac{1}{n}\sum_{i=1}^{n} h_{ii}^{K(n)} e_{K(n)} \\
&= H_1^1 e_{I(1)} + \cdots + H_1^n e_{I(n)} + \\
&\quad + H_2^1 e_{J(1)} + \cdots + H_2^n e_{J(n)} + \\
&\quad + H_3^1 e_{K(1)} + \cdots + H_3^n e_{K(n)} \\
&= \sum_{r=1}^{3}\sum_{k=1}^{n} H_r^k e_{\phi_r(k)}.
\end{aligned}
\tag{3.2}
$$

With $\{e_{I(1)}, \cdots, e_{K(n)}\}$ being orthonormal, we have

$$
||H||^2 = \sum_{r=1}^{3}\sum_{k=1}^{n} (H_r^k)^2. \tag{3.3}
$$

Theorem 3.1. *Let M be a Lagrangian submanifold of real dimension n $(n \geq 2)$ in a $4n$-dimensional quaternion space form $\tilde{M}^n(c)$, x a point in M and X a unit tangent vector in T_xM. Then we have*

$$
Ric(X) \leq \frac{n-1}{4}(c + n||H||^2), \tag{3.4}
$$

where H is the mean curvature vector of M in $\tilde{M}^n(c)$ and $Ric(X)$ is the Ricci curvature of M at X.

The equality sign holds for any unit tangent vector at x if and only if either

(i) *x is a totally geodesic point or*

(ii) *$n = 2$ and x is an H-umbilical point with $\lambda_r = 3\mu_r, r = 1, 2, 3$.*

Proof. We fix the point x in M. Let X be any unit tangent vector at x. We choose an orthonormal frame $e_1, \cdots, e_n, I(e_1), \cdots, K(e_n)$ such that $e_1, \cdots, e_n$ are tangent to M at x with $e_1 = X$. From Gauss equation we have

$$
\tilde{R}(e_1, e_j, e_1, e_j) = R(e_1, e_j, e_1, e_j) - g(h(e_1, e_1), h(e_j, e_j)) + g(h(e_1, e_j), h(e_1, e_j))
$$

or

$$
\tilde{R}(e_1, e_j, e_1, e_j) = R(e_1, e_j, e_1, e_j) - \sum_{r=1}^{3}\sum_{k=1}^{n}(h_{11}^{\phi_r(k)} h_{jj}^{\phi_r(k)} - (h_{1j}^{\phi_r(k)})^2), \forall j \in \overline{2, n}.
$$

Hence we have

$$
(n-1)\frac{c}{4} = Ric(X) - \sum_{r=1}^{3}\sum_{k=1}^{n}\sum_{j=2}^{n}(h_{11}^{\phi_r(k)} h_{jj}^{\phi_r(k)} - (h_{1j}^{\phi_r(k)})^2).
$$

Therefore

$$Ric(X) - (n-1)\frac{c}{4} = \sum_{r=1}^{3}\sum_{k=1}^{n}\sum_{j=2}^{n}\left(h_{11}^{\phi_r(k)}h_{jj}^{\phi_r(k)} - (h_{1j}^{\phi_r(k)})^2\right)$$
$$\leq \sum_{r=1}^{3}\sum_{k=1}^{n}\sum_{j=2}^{n}h_{11}^{\phi_r(k)}h_{jj}^{\phi_r(k)} - \sum_{r=1}^{3}\sum_{j=2}^{n}(h_{1j}^{\phi_r(1)})^2 - \sum_{r=1}^{3}\sum_{j=2}^{n}(h_{1j}^{\phi_r(j)})^2. \quad (3.5)$$

Using (2.3), we have

$$Ric(X) - (n-1)\frac{c}{4} \leq (\sum_{r=1}^{3}\sum_{k=1}^{n}\sum_{j=2}^{n}h_{11}^{\phi_r(k)}h_{jj}^{\phi_r(k)}) - \sum_{r=1}^{3}\sum_{j=2}^{n}(h_{11}^{\phi_r(j)})^2 - \sum_{r=1}^{3}\sum_{j=2}^{n}(h_{jj}^{\phi_r(1)})^2,$$

or

$$Ric(X) - (n-1)\frac{c}{4} \leq \sum_{r=1}^{3}\{\sum_{k=2}^{n}(h_{11}^{\phi_r(k)}\sum_{j=2}^{n}h_{jj}^{\phi_r(k)} - (h_{11}^{\phi_r(j)})^2) +$$
$$+ h_{11}^{\phi_r(1)}\sum_{j=2}^{n}h_{jj}^{\phi_r(k)} - \sum_{j=2}^{n}(h_{jj}^{\phi_r(1)})^2\}. \quad (3.6)$$

From Cauchy-Schwarz's inequality and (3.2), we have

$$(h_{11}^{\phi_r(1)} - \frac{n}{2}H_r^1)^2 + \sum_{j=2}^{n}(h_{jj}^{\phi_r(1)})^2 \geq$$
$$\geq \frac{1}{n}(\frac{1}{2}h_{11}^{\phi_r(1)} + \frac{1}{2}h_{22}^{\phi_r(1)} + \cdots + \frac{1}{2}h_{nn}^{\phi_r(1)})^2 = \frac{n}{4}(H_r^1)^2, \quad r = 1,2,3,$$

or equivalently

$$\sum_{j=2}^{n}(h_{jj}^{\phi_r(1)})^2 - h_{11}^{\phi_r(1)}\sum_{j=2}^{n}h_{jj}^{\phi_r(1)} \geq \frac{n(1-n)}{4}(H_r^1)^2, \quad r = 1,2,3. \quad (3.7)$$

Similarly, by Cauchy-Schwarz's inequality, we have

$$(h_{11}^{\phi_r(k)})^2 + (\frac{n}{2}H_r^k - h_{11}^{\phi_r(k)})^2 \geq \frac{n^2}{8}(H_r^k)^2, \quad r = 1,2,3,$$

which is equivalent to

$$(h_{11}^{\phi_r(k)})^2 - h_{11}^{\phi_r(k)}\sum_{j=2}^{n}h_{jj}^{\phi_r(k)} \geq -\frac{n^2}{8}(H_r^k)^2, \quad r = 1,2,3. \quad (3.8)$$

From (3.6),(3.7) and (3.8), we have

$$Ric(X) - \frac{n-1}{4}c \leq \sum_{r=1}^{3}\{\frac{n^2}{8}\sum_{k=2}^{n}(H_r^k)^2 + \frac{n(n-1)}{4}(H_r^1)^2\}$$
$$\leq \frac{n(n-1)}{4}\sum_{r=1}^{3}\{\sum_{k=2}^{n}(H_r^k)^2 + (H_r^1)^2\}$$
$$= \frac{n(n-1)}{4}||H||^2, \quad (3.9)$$

which implies (3.4).

Now assume the equality sign of (3.4) holds for any unit tangent vector X at x. Inequalities in (3.5), (3.7) and (3.8) become equalities. Thus, we have

$$h_{jk}^{\phi_r(1)} = 0, \quad \forall j,k \geq 2,\ j \neq k, \quad r = 1,2,3, \quad (3.10)$$

$$2h_{11}^{\phi_r(1)} - nH_r^1 = 2h_{22}^{\phi_r(1)} = \cdots = 2h_{nn}^{\phi_r(1)}, \quad r = 1,2,3, \tag{3.11}$$

$$4h_{11}^{\phi_r(k)} = nH_r^k, \;\; k = 2,\cdots,n, \;\; r = 1,2,3. \tag{3.12}$$

Also, by (3.9), we have either

(1) $n \geq 3$ and $H_r^2 = H_r^3 = \cdots = H_r^n = 0, \;\; r = 1,2,3$ or

(2) $n = 2$.

Case(1) $n \geq 3$. We have $H_r^2 = H_r^3 = \cdots = H_r^n = 0, \;\; r = 1,2,3$. From (3.12) we have

$$h_{1j}^{\phi_r(1)} = h_{11}^{\phi_r(j)} = \frac{nH_r^j}{4} = 0, \; \forall j \geq 2, \; r = 1,2,3.$$

From this and (3.10) and (3.11), $(h_{jk}^{\phi_r(1)})$ must be diagonal with $h_{11}^{\phi_r(1)} = (n+1)h_{22}^{\phi_r(1)}$ and $h_{jj}^{\phi_r(1)} = \frac{1}{2}H_r^1, \forall j \geq 2, \;\; r = 1,2,3$.

Now if we compute $Ric(e_2)$ as we do for $Ric(X) = Ric(e_1)$ in (3.5), from the equality we get $h_{2j}^{\phi_r(k)} = h_{jk}^{\phi_r(2)} = 0, \forall k \neq 2, j \neq 2, k \neq j, r = 1,2,3$. From the equality and (3.11), we get

$$\frac{h_{11}^{\phi_r(2)}}{n+1} = h_{22}^{\phi_r(2)} = \cdots = h_{nn}^{\phi_r(2)} = \frac{H_r^2}{2} = 0, \;\; r = 1,2,3.$$

Since the equality holds for all unit tangent vector, the argument is also true for matrices $(h_{jk}^{\phi_r(l)})$. Now finally $h_{2j}^{\phi_r(2)} = h_{22}^{\phi_r(j)} = \frac{H_r^j}{2} = 0, \forall j \geq 3, r = 1,2,3$. Therefore matrix $(h_{jk}^{\phi_r(2)})$ has only two possible nonzero entries (i.e. $h_{12}^{\phi_r(2)} = h_{21}^{\phi_r(2)} = h_{22}^{\phi_r(1)} = \frac{H_r^1}{2}, r = 1,2,3$). Similarly matrix $(h_{jk}^{\phi_r(l)})$ has only two possible nonzero entries

$$h_{1l}^{\phi_r(l)} = h_{l1}^{\phi_r(l)} = h_{ll}^{\phi_r(1)} = \frac{H_r^1}{2}, \;\; \forall l \geq 3, \; r = 1,2,3.$$

We now compute $Ric(e_2)$ as follows:

$$\tilde{R}(e_2,e_j,e_2,e_j) = R(e_2,e_j,e_2,e_j) - g(h(e_2,e_2),h(e_j,e_j)) + (h(e_2,e_j),h(e_2,e_j)),$$

so we have

$$\tilde{R}(e_2,e_j,e_2,e_j) = R(e_2,e_j,e_2,e_j) - \sum_{r=1}^{3} (\frac{H_r^1}{2})^2, \forall j \geq 3. \tag{3.13}$$

From

$$\tilde{R}(e_2,e_1,e_2,e_1) = R(e_2,e_1,e_2,e_1) - g(h(e_2,e_2),h(e_1,e_1)) + g(h(e_2,e_1),h(e_2,e_1)),$$

we get

$$\tilde{R}(e_2,e_1,e_2,e_1) = R(e_2,e_1,e_2,e_1) - (n+1)\sum_{r=1}^{3}\left(\frac{H_r^1}{2}\right)^2 + \sum_{r=1}^{3}\left(\frac{H_r^1}{2}\right)^2. \tag{3.14}$$

By combining (3.13) and (3.14), we get

$$Ric(e_2) - \frac{(n-1)c}{4} = (n+1)\sum_{r=1}^{3}\left(\frac{H_r^1}{2}\right)^2 - \sum_{r=1}^{3}\left(\frac{H_r^1}{2}\right)^2 + (n-2)\sum_{r=1}^{3}\left(\frac{H_r^1}{2}\right)^2 = 2(n-1)\sum_{r=1}^{3}\left(\frac{H_r^1}{2}\right)^2.$$

On the other hand from the equality assumption, we have

$$Ric(e_2) - \frac{(n-1)c}{4} = \frac{n(n-1)}{4}||H||^2 = n(n-1)\sum_{r=1}^{3}\left(\frac{H_r^1}{2}\right)^2.$$

Therefore, we have

$$n(n-1)\sum_{r=1}^{3}\left(\frac{H_r^1}{2}\right)^2 = 2(n-1)\sum_{r=1}^{3}\left(\frac{H_r^1}{2}\right)^2$$

Since $n \geq 3$, we have $H_1^1 = H_2^1 = H_3^1 = 0$. Therefore, $(h_{jk}^{\phi_r(l)})$ are all zero $(r = 1, 2, 3)$ and x is a totally geodesic point.

Case(2) $n = 2$. From (3.12) we have

$$h_{11}^{\phi_r(1)} = 3h_{22}^{\phi_r(1)}, \;\; h_{22}^{\phi_r(2)} = 3h_{11}^{\phi_r(2)}, \quad r = 1, 2, 3.$$

Since X can be any unit vector, we may assume that the mean curvature vector is in $Q(X)$. Then the second fundamental form takes the following form:

$$\begin{aligned} h(e_1, e_1) &= 3\mu_1 I(e_1) + 3\mu_2 J(e_1) + 3\mu_3 K(e_1), \\ h(e_2, e_2) &= \mu_1 I(e_1) + \mu_2 J(e_1) + \mu_3 K(e_1), \\ h(e_1, e_2) &= \mu_1 I(e_2) + \mu_2 J(e_2) + \mu_3 K(e_2), \end{aligned}$$

for some functions μ_1, μ_2 and μ_3 with respect to some local orthonormal frame field. It follows from (2.4) that x is an H-umbilical point with $\lambda_r = 3\mu_r, r = 1, 2, 3$.

The converse can be proved by simple computation.

□

Remark 3.1. Theorem 3.1 is an improvement of a result in [1, page 38] for Lagrangian submanifolds. Theorem 3.1 is also an extention of a theorem in [5] for Lagrangian submanifolds in quaternion space forms.

Remark 3.2. In quaternion space forms, Theorem 3.1 is an improvement of Corollary 2.1 in [8] for Lagrangian submanifolds .

From Theorem 3.1, we have the following

Corollary 3.2. *Let M be a Lagrangian submanifold of real dimension n $(n \geq 2)$ in a $4n$-dimensional quaternion space form $\tilde{M}^n(c)$. If*

$$Ric(X) = \frac{n-1}{4}(c + n||H||^2)$$

for any unit tangent vector X of M, then either M is a totally geodesic submanifold in $\tilde{M}^n(c)$ or $n = 2$ and M is a Lagrangian H-umbilical submanifold of $\tilde{M}^n(c)$ with $\lambda_r = 3\mu_r, r = 1, 2, 3$.

Remark 3.3. Corollary 3.2 is an improvement of Theorem 3.1 in [7] for Lagrangian submanifolds in quaternion space forms.

Remark 3.4. Theorem 3.1 and Corollary 3.2 give a complete solution to Problem 4.6 in [12].

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