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GRAY TENSORS ON LIGHTLIKE HYPERSURFACES

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Abstract. This paper introduces Gray-tensors on lightlike hypersurfaces M^{n+1} of signature $(0, n)$, $(n \geq 1)$ and investigates on their basic properties in connection with their null geometry. In particular, we show that there is an interplay between existence of Gray-tensors of certain type and lightlike warped product structures. As a physical relevance, we show that there exists such a tensor on both globally Killing horizons (GKH) and totally geodesic lightlike triple warped product hypersurfaces.

1. Introduction

Natural linear conditions generalizing Einstein metric equation are discussed in [4] and illustrated by interesting examples. Among such generalizations are $\mathcal{A}$-manifolds (introduced by A. Gray [9]), that is, Riemannian manifolds (M, g) whose Ricci tensor r satisfies $\nabla r(X, X, X) = 0$ for all $X \in \Gamma(TM)$, where ∇ is the Levi-Civita connection of the metric g. Examples of compact manifolds of this type, other than Einstein or locally products, are compact quotients of naturally reductive homogeneous Riemannian manifolds and nilmanifolds covered by the generalized Heisenberg group of A. Kaplan (see [4] and references therein). Also, W. Jelonek in [10] gives explicit examples of compact non-homogeneous proper complete $\mathcal{A}$-manifolds, and an example of locally non-homogeneous proper complete one.

A natural generalization of $\mathcal{A}$-manifolds condition is, in considering on the Riemannian manifold (M, g), a symmetric $(0, 2)$ tensor ϕ (or equivalently, since g is non-degenerate, a symmetric tensor $S \in End(TM)$) satisfying the additional condition $\nabla\phi(X, X, X) = 0$. Such tensors are considered and studied in [10, 11, 12] and called $\mathcal{A}$-tensors (or Killing tensor for ϕ). In particular, a description of compact Einstein-Weyl manifolds is given in [12] in terms of these tensors.

Since any semi-Riemannian manifold has lightlike spaces and taking into account the interesting applications in non-degenerate case, namely the intensive interplay of

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Gray tensors in studying Einstein Weyl structures, warped products, D'Atri spaces Killing tensors ... ([10] - [12]), we reasonably expect a role of Gray tensors in considering geometry of lightlike manifolds. The present paper focus on lightlike hypersurfaces and connect their null geometry to existence and the geometric properties of these tensors. In forthcoming papers we expect establish applications and interplay with Relativistic fluids in space -times with appropriate metric symmetries.

As it is well known, contrary to timelike and spacelike hypersurfaces, the geometry of a lightlike hypersurface M is different and rather difficult since the normal bundle and the tangent bundle have non-zero intersection. At each point $x \in M$, a straight line orthogonal to M lies in T_xM and the family of these straight lines does not determine a normalization of M and consequently an affine connection on M. To overcome this difficulty, a theory on the differential geometry of lightlike hypersurfaces developed by Duggal and Bejancu [3] introduces a non-degenerate screen distribution and construct the corresponding lightlike transversal vector bundle. This enables to define an induced linear connection (depending on the screen distribution, and hence is not unique in general). On the other hand, it is important to notice that the second fundamental form is independent from the choice of the screen distribution.

We outline in section 2 basic informations on normalizations[3] and pseudo-inversion of degenerate metrics [1]. Our approach in studying Gray-tensors comes from an adaptation of techniques in [10, 11] to the case of lightlike hypersurfaces. A known important result on lightlike hypersurfaces (Theorem 2.1 below) states that the induced connection is independent from the screen distribution if and only if the lightlike hypersurface is totally geodesic. Equivalently, the induced connection is torsion-free and metric. In this respect, we introduce in section 3, Gray-tensor (Definition 3.1) on totally geodesic lightlike hypersurfaces endowed with a specific given screen distribution $S(TM)$ where ∇ is then the induced connection on (M, g) in $(\overline{M}, \overline{g})$. Thereafter, we show a technical result on its characterization (Proposition 3.1). Sections 4 and 5 are concerned with a simple and elementary example followed by some explicit constructions of such tensors. In section 6, we study some geometric properties of these tensors and in section 7 we establish for a totally geodesic screen distribution, necessary and sufficient condition for eigenspace distributions of Gray-tensors with exactly three eigenspaces to be integrable (Theorem 7.1). Section 8 is devoted to the special case of totally umbilical screen foliation. In section 9 we establish a sufficient condition for Gray-tensors to be isotropic. Finally, we show in section 10 that there is an interplay between existence of Gray-tensors of certain type and lightlike warped product structure.

2. Preliminaries

Let M be a hypersurface of an $(n+2)$-dimensional pseudo-Riemannian manifold $(\overline{M}, \overline{g})$ of index $0 < \nu < n+2$. In the classical theory of nondegenerate hypersurfaces, the normal bundle has trivial intersection $\{0\}$ with the tangent bundle and plays an important role in the introduction of main geometric objects. In case of lightlike (degenerate, null) hypersurfaces, the situation is totally different. The normal bundle $TM^{\perp}$ is a rank-one distribution on M: $TM^{\perp} \subset TM$ and it coincides with the so called *radical distribution* $RadTM = TM \cap TM^{\perp}$. Hence, the induced metric tensor field g is degenerate and has rank n. The following characterisation is proved in [3].

Proposition 2.1. *Let (M, g) be a hypersurface of an $(n+2)$-dimensional pseudo-Riemannian manifold $(\overline{M}, \overline{g})$. Then the following assertions are equivalent.*

(i) *M is a lightlike hypersurface of $\overline{M}$.*
(ii) *g has constant rank n on M.*
(iii) *$TM^{\perp} = \cup_{x\in M} T_x M^{\perp}$ is a distribution on M.*

A complementary bundle of $TM^{\perp}$ in TM is a rank n nondegenerate distribution on M. It is called a *screen distribution* on M and is often denoted by $S(TM)$. A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M, g, S(TM))$. As $TM^{\perp}$ lies in the tangent bundle, the following result has an important role in studying the geometry of a lightlike hypersurface.

Proposition 2.2. *([3]) Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\overline{M}, \overline{g})$ with a given screen distribution $S(TM)$. Then there exists a unique rank 1 vector subbundle $tr(TM)$ of $T\overline{M}|_M$, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighbourhood $\mathcal{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathcal{U}$ satisfying*

$$\overline{g}(N, \xi) = 1 \tag{2.1}$$

and

$$\overline{g}(N, N) = \overline{g}(N, W) = 0, \qquad \forall\ W \in \Gamma(S(TM)|_{\mathcal{U}}). \tag{2.2}$$

Here and in the sequel we denote by $\Gamma(E)$ the $\mathcal{F}(M)$-module of smooth sections of a vector bundle E over M, $\mathcal{F}(M)$ being the algebra of smooth functions on M. Also, by $\perp$ and $\oplus$ we denote the orthogonal and non-orthogonal direct sum of two vector bundles. By proposition 2.2 we may write down the following decompositions.

$$TM = S(TM) \perp TM^{\perp}, \tag{2.3}$$

$$T\overline{M}|_M = TM \oplus tr(TM) \tag{2.4}$$

and

$$T\overline{M}|_M = S(TM) \perp (TM^{\perp} \oplus tr(TM)). \tag{2.5}$$

As it is well known, we have the following:

Definition 2.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\overline{M}, \overline{g})$ with a given screen distribution $S(TM)$. The induced connection, say ∇, on M is defined by*

$$\nabla_X Y = Q(\overline{\nabla}_X Y), \tag{2.6}$$

where $\overline{\nabla}$ denotes the Levi-civita connection on $(\overline{M}, \overline{g})$ and Q is the projection onto TM with respect to the decomposition (2.4).

Remark 2.1. *Notice that the induced connection ∇ on M depends on both g and the specific given screen distribution $S(TM)$ on M.*

The projections Q and $I - Q$ give rise to the Gauss an Weingarten formulae in the form

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \qquad \forall X, Y \ \in \Gamma(TM), \tag{2.7}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V \qquad \forall X \in \Gamma(TM), \quad \forall\, V \in \Gamma(tr(TM)). \tag{2.8}$$

Here, $\nabla_X Y$ and $A_V X$ belong to $\Gamma(TM)$. Hence
- h is a $\Gamma(tr(TM))$-valued symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$,
- A_V is an $\mathcal{F}(M)$-linear operator on $\Gamma(TM)$, and
- ∇^t is a linear connection on the lightlike transversal vector bundle $tr(TM)$.

Let P denote the projection morphism of $\Gamma(TM)$ onto $\Gamma(S(TM))$ with respect to the decomposition (2.3). We have

$$\nabla_X PY = \overset{\star}{\nabla}_X PY + h^*(X, PY) \qquad \forall X, Y \in \Gamma(TM), \tag{2.9}$$

$$\nabla_X U = -\overset{\star}{A}_U X + \nabla^{*t}_X U \qquad \forall X \in \Gamma(TM), \quad \forall\, U \in \Gamma(TM^{\perp}). \tag{2.10}$$

Here $\overset{\star}{\nabla}_X PY$ and $\overset{\star}{A}_U X$ belong to $\Gamma(S(TM))$, $\overset{\star}{\nabla}$ and ∇^{*t} are linear connections on $S(TM)$ and $TM^{\perp}$, respectively. Hence
- h^* is a $\Gamma(TM^{\perp})$-valued $\mathcal{F}(M)$-bilinear form on $\Gamma(TM) \times \Gamma(S(TM))$, and
- $\overset{\star}{A}_U$ is a $\Gamma(S(TM))$-valued $\mathcal{F}(M)$-linear operator on $\Gamma(TM)$.

They are the second fundamental form and the shape operator of the screen distribution, respectively.

Equivalently, consider a normalizing pair $\{\xi, N\}$ as in proposition 2.2. Then, (2.7) and (2.8) take the form

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N \qquad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}), \tag{2.11}$$

and

$$\overline{\nabla}_X N = -A_N X + \tau(X)N \qquad \forall X \in \Gamma(TM|_{\mathcal{U}}), \tag{2.12}$$

where we put locally on $\mathcal{U}$,

$$B(X, Y) = \overline{g}(h(X, Y), \xi), \tag{2.13}$$

$$\tau(X) = \overline{g}(\nabla^t_X N, \xi). \tag{2.14}$$

It is important to stress the fact that the local second fundamental form B in (2.13) does not depend on the choice of the screen distribution.

We also define (locally) on $\mathcal{U}$ the following:

$$C(X, PY) = \overline{g}(h^*(X, PY), N), \tag{2.15}$$

$$\varphi(X) = -\overline{g}(\nabla^{\star t}_X \xi, N). \tag{2.16}$$

Thus, one has for $X \in \Gamma(TM)$

$$\nabla_X PY = \overset{\star}{\nabla}_X PY + C(X, PY)\xi, \tag{2.17}$$

$$\nabla_X \xi = -\overset{\star}{A}_\xi X + \varphi(X)\xi. \tag{2.18}$$

It is straighforward to verify that for $X, Y \in \Gamma(TM)$,

$$B(X, \xi) = 0, \tag{2.19}$$

$$B(X,Y) = g(\overset{\star}{A}_\xi X, Y), \tag{2.20}$$

$$\overset{\star}{A}_\xi \, \xi = 0. \tag{2.21}$$

The linear connection $\overset{\star}{\nabla}$ in (2.9) is a metric connection on $S(TM)$ and we have for all tangent vector fields X, Y and Z in TM

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \tag{2.22}$$

where η is a 1-form defined by

$$\eta(\cdot) = \overline{g}(N, \cdot). \tag{2.23}$$

The induced connection ∇ is torsion-free, but not necessarily g-metric. Also, on the geodesibility of M the following is known.

Theorem 2.1. *([3, p.88]) Let $(M, g, S(TM))$ be a lightlike hypersurface of a pseudo-Riemannian manifold $(\overline{M}, \overline{g})$. Then the following assertions are equivalent:*

(i) *M is totally geodesic.*
(ii) *h (or equivalently B) vanishes identically on M.*
(iii) *$\overset{\star}{A}_U$ vanishes identically on M, for any $U \in \Gamma(TM^\perp)$*
(iv) *The connection ∇ induced by $\overline{\nabla}$ on M is torsion-free and metric.*
(v) *$TM^\perp$ is a parallel distribution with respect to ∇.*
(vi) *$TM^\perp$ is a Killing distribution on M.*

It turns out that if (M, g) is not totally geodesic, there is no connection that is, at the same time, torsion-free and g-metric.

Now, recall that a large class of differential operators in differential geometry is intrinsically defined by means of the dual metric g^* on the dual bundle $\Gamma(T^*M)$ of 1-forms on M. If the metric g is nondegenerate, the tensor field g^* is nothing else than the inverse of g. We outline here (equivalent) construction in case the metric g is degenerate and refer the reader to [1] for more details.

Let $(M, g, S(TM))$ be a lightlike hypersurface and $\{\xi, N\}$ be a pair of (null-) vectors chosen as in Proposition 2.2. Consider the one-form η as in (2.23). For all $X \in \Gamma(TM)$,

$$X = PX + \eta(X)\xi$$

and $\eta(X) = 0$ if and only if $X \in \Gamma(S(TM))$. Now, we define $\flat$ by

$$\begin{aligned} \flat : \Gamma(TM) &\longrightarrow \Gamma(T^*M) \\ X &\longmapsto X^\flat \end{aligned}$$

such that

$$X^\flat = g(\,X\,,\,\cdot\,) + \eta(X)\eta(\,\cdot\,). \tag{2.24}$$

Clearly, such $\flat$ is an isomorphism of $\Gamma(TM)$ onto $\Gamma(T^*M)$, and can be used to generalize the usual nondegenerate theory. In the nondegenerate case, $\Gamma(S(TM))$ coincides with $\Gamma(TM)$, and as a consequence the 1-form η vanishes identically and the projection morphism P becomes the identity map on $\Gamma(TM)$. We let $\sharp$ denote the inverse of the isomorphism $\flat$ given by (2.24). For $X \in \Gamma(TM)$ (resp. $\omega \in T^*M$), $X^\flat$ (resp. $\omega^\sharp$) is called the dual 1-form of X (resp. the dual vector field of ω) with respect to the degenerate metric g. It follows from (2.24) that if ω is a 1-form on M, we have for $X \in \Gamma(TM)$

$$\omega(X) \ = \ g(\omega^\sharp, X) \ + \ \omega(\xi)\eta(X). \tag{2.25}$$

Now we introduce the so-called associated nondegenerate metric $\tilde{g}$ to the degenerate metric g as follows. For $X, Y \in \Gamma(TM)$, define $\tilde{g}$ by

$$\tilde{g}(X, Y) \ = \ X^\flat(Y). \tag{2.26}$$

Clearly, $\tilde{g}$ is a non degenerate metric on M and plays an important role in defining the usual differential operators *gradient*, *divergence*, *Laplacian* with respect to degenerate metric g on lightlike hypersurfaces. Also, observe that $\tilde{g}$ coincides with g if the latter is nondegenerate. The $(0, 2)$ tensor field $g^{[\ \cdot\ ,\ \cdot\ ]}$, inverse of $\tilde{g}$ is called *the pseudo-inverse of g*.

From now on, unless otherwise stated, the ambient manifold $(\overline{M}, \overline{g})$ has a Lorentzian signature so that all lighlike hypersurfaces considered are of signature $(0, n)$. In particular, it follows that any screen distribution is Riemannian.

As it is well known (theorem 2.1), only totally geodesic lightlike hypersurfaces have their induced connection which is metric and torsion-free. In the next section and the remainder of the text, only such lightlike hypersurfaces will be in consideration. We also assume that the null vector field ξ is globally defined on M. Respective metrics will be denoted $\langle \cdot, \cdot \rangle$ if no ambiguity occurs.

3. Gray-tensors

Definition 3.1. *Let $(M, g, S(TM))$ be a totally geodesic lightlike hypersurface of $(\overline{M}, \overline{g})$, ∇ the induced (Levi-Civita) connection on M. By Gray-tensor on $(M, g, S(TM))$, we mean a screen preserving element $S \in End(TM)$ for which*

(a) $\langle SX, Y \rangle = \langle X, SY \rangle$ *for all* X, Y *in* $\Gamma(TM)$,

(b) $X^\flat(\nabla S(X, X)) = 0$ *for all* X *in* $\Gamma(TM)$,

hold, where $\flat$ denotes the duality isomorphism between TM and $TM^\star$ with respect to the degenerate metric tensor g and the screen distribution $S(TM)$.

It should be noticed that screen preserving means P and S commute. One also write $S \in G(\nabla)$ if S is a Gray-tensor. A Gray-tensor is called isotropic if it is $Rad(TM)$-valued, otherwise, it is called a proper Gray-tensor.

Killing tensors on M are symmetric $(0, 2)$-tensors, say ϕ such that

$$\phi(X, Y) = \langle SX, Y \rangle, \quad \forall\ X, Y \in \Gamma(TM)$$

for some Gray-tensor S. Observe that ϕ is a degenerate $(0, 2)$-tensor since at each $u \in M$ its nullity space $\eta_{\phi|_u} \supset RadTM|_u$, i.e

$$\phi(X, \xi) = \phi(\xi, X) = 0, \quad \forall\ X \in TM,\ \forall \xi \in \Gamma(RadTM).$$

It also satisfies

$$\nabla\phi(X, X, X) = \langle \nabla S(X, X), X \rangle, \quad \forall\ X \in \Gamma(TM).$$

Since ∇ is a metric connection, we have

$$\nabla_X \xi = \varphi(X)\xi, \quad \forall\ X \in \Gamma(TM),$$

for some global 1-form φ on M. The following proposition is equivalent to the Riemannian case [10].

Proposition 3.1. *Let $(M, g, S(TM))$ be a totally geodesic lightlike hypersurface, S a symmetric $(1,1)$tensor and $\phi(X,Y) = \langle SX, Y\rangle$ for all X in TM. The following assertions are equivalent.*

(a) $S \in G(\nabla)$.
(b) *For every geodesic γ on (M, g), the real valued function $t \longmapsto \phi(\gamma'(t), \gamma'(t))$ is constant on $dom\gamma$ and, if γ is a null geodesic, $S(\gamma'(t))$ is parallel along γ.*
(c) $\Sigma_{cyclic}\nabla_X\phi(Y,Z) = -\Sigma_{cyclic}\eta(X)\eta(\nabla S(Y,Z))$.

Proof. The equivalence (a) and (c) is immediate using the definition of $\flat$ and the bipolarization of relation (b) in definition 3.1. Let us show the equivalence (a) and (b). Assume (a) and consider γ a geodesic on M. We have

$$\frac{d}{dt}\phi(\gamma'(t), \gamma'(t)) \quad = \quad \nabla_{\gamma'(t)}\phi(\gamma'(t), \gamma'(t)).$$

We distinguish two cases: γ is a null geodesic or not.

If γ is a non null geodesic, from (3) we have

$$\begin{aligned}\frac{d}{dt}\phi(\gamma'(t), \gamma'(t)) &= \nabla_{\gamma'(t)}\phi(\gamma'(t), \gamma'(t)). \\ &= \gamma'(t)^\flat(\nabla S(\gamma'(t), \gamma'(t))) = 0,\end{aligned}$$

i.e ϕ is constant on $dom\gamma$.

If γ is a null geodesic, it follows definition of ϕ that it vanishes identically on $dom\gamma$. In addition, $\gamma'(t)$ is proportional to ξ for all t in $dom\gamma$. Thus, there exists a nowhere vanishing function $t \to \lambda_0(t)$ on $dom\gamma$ such that

$$\nabla S(\gamma'(t), \gamma'(t)) = (\lambda_0(t))^2 \nabla S(\xi, \xi) \in \Gamma(RadTM|_\gamma). \tag{3.1}$$

Using (b) in definition 3.1, we have $\eta(\nabla S(\gamma'(t), \gamma'(t))) = 0 \quad \forall t \in dom\gamma$. This together with (3.1) lead to $\nabla S(\gamma'(t), \gamma'(t)) = 0 \quad \forall t \in dom\gamma$. Finally, since γ is a geodesic, we have $\nabla_{\gamma'(t)}S(\gamma'(t)) = 0$, and (b) is proved.

Conversely, assume (b) holds and let $X \in T_{x_0}M$, $x_0 \in M$. Consider γ a geodesic satisfying initial conditions $\gamma(0) = x_0$ and $\gamma'(0) = X$. One has

$$X^\flat(\nabla S(X,X)) = \nabla_{\gamma'(t)}\phi(\gamma'(t), \gamma'(t))|_{t=0} + \eta(\gamma'(t))\eta(\nabla S(\gamma'(t), \gamma'(t)))|_{t=0} = 0,$$

i.e (a) is proved and the proof is complete.□

Remark 3.1.

(a) Observe that for X, $Y$and Z in $S(TM)$, relation (c) in proposition 3.1 reduces to

$$\nabla_X\phi(Y,Z) + \nabla_Y\phi(Z,X) + \nabla_Z\phi(X,Y) = 0. \tag{3.2}$$

(b) Since M has signature $(0,n)$, $n = dimM - 1$, the Gray-tensor S induces by restriction on the nondegenerate (Riemannian) screen distribution $S(TM)$, a $\mathcal{A}$-tensor S' with respect to the (unique) Levi-Civita connection $\overset{\star}{\nabla}$ induced by ∇ on $S(TM)$. Indeed, $S' \in End(S(TM))$ by screen preserving of S and it is known [10] that in this case, (3.2) is equivalent to being $\mathcal{A}$-tensor for S'. So, the Gray-tensor S splits as

$$S = S' \circ P + \eta(\cdot)S\ \xi. \tag{3.3}$$

One can show that, if σ is a Riemannian $\mathcal{A}$-tensor on $S(TM)$ and if in addition the screen distribution is totally geodesic in M, then, for $\lambda_0 \in C^\infty(M)$, the $(1,1)$-tensor defined on M by

$$S = \sigma \circ P + \lambda_0 \eta(\cdot)\xi \tag{3.4}$$

is a Gray-tensor on M, provided $\xi \cdot \lambda_0 = 0$.

4. An elementary example

Let us recall that a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is a $\mathcal{A}$- manifold if for every geodesic γ on $(\overline{M}, \overline{g})$, the real valued function $t \longrightarrow \overline{R}ic(\gamma'(t), \gamma'(t))$ is constant on domγ; where $\overline{R}ic$ is the $(0,2)$ Ricci tensor on $(\overline{M}, \overline{g})$ [13], [10]. Now consider a Lorentzian Einstein $\mathcal{A}$- manifold $(\overline{M}, \overline{g})$ whose Riemann curvature satisfies the ambient holonomy condition $\overline{R}(\xi, \cdot) \in T^\star M \otimes T^\star M \otimes S(TM)$ where ξ is a characteristic Killing vector field on a lightlike hypersurface $(M, g, S(TM))$. Observe that with this holonomy condition, the screen distribution $S(TM)$ is not necessarily totally geodesic . It is for example the case when we consider a normalization satisfying $\tau = 0$, $C \neq 0$ with C being a Codazzi tensor on M. In this case the induced Ricci curvature Ric on (M, g) reduces to

$$Ric(X, Y) = \overline{R}ic(X, Y) = \Lambda g(X, Y), \tag{4.1}$$

for all tangent vector fields X and Y [2], where the last equality holds with the Einstein condition on $(\overline{M}, \overline{g})$, Λ being a constant. Let S denote a $S(TM)$- valued symmetric tensor such that $Ric(X, Y) = g(SX, Y)$. Such a S is then given by $SX = \Lambda PX$ where P denotes the projection morphism of TM onto $S(TM)$. Obviously, we have $S\xi = 0$. Now, let γ denote a geodesic in M (and hence in $\overline{M}$ since M is totally geodesic in $\overline{M}$). Taking into account 4.1, we have

$$\frac{d}{dt} Ric(\gamma'(t), \gamma'(t)) = \frac{d}{dt} \overline{R}ic(\gamma'(t), \gamma'(t)) = 0,$$

since $(\overline{M}, \overline{g})$ is a $\mathcal{A}$-manifold. Thus the function $t \longrightarrow Ric(\gamma'(t), \gamma'(t))$ is constant on *dom*γ. Suppose now that γ is a null geodesic. We have necessarily for each $t \in dom\gamma$, $\gamma'(t) = \lambda \xi_{\gamma(t)}$ as M has signature $(0, n)$. Then $S(\gamma'(t)) = 0$ and (trivially) $\nabla_{\gamma'(t)} S(\gamma'(t)) = 0$, that is $S(\gamma'(t))$ is parallel along γ. Finally, using item (b) in Proposition 3.1, we conclude that the Ricci endomorphism given by $SX = \Lambda PX$ is a Gray tensor on M.

5. Some Constructions.

Let (N, g_N) and (F, g_F) be a lightlike and a Riemannian manifold of dimension n and m respectively. Let $\pi : N \times F \longrightarrow N$ and $\varrho : N \times F \longrightarrow F$ denote the projection maps given by $\pi(x, y) = x$ and $\varrho(x, y) = y$ for $(x, y) \in N \times F$, respectively, where the projection π on N is done with respect to a nondegenerate screen distribution $S(TN)$. The product manifold $M = N \times F$, endowed with the degenerate metric defined by

$$g(X, Y) = g_N(\pi_\star X, \pi_\star Y) + f(\pi(x, y)) g_F(\varrho_\star X, \varrho_\star Y), \tag{5.1}$$

for all X, Y tangent to M, where $\star$ is the symbol of the tangent map and $f : N \longrightarrow \mathbb{R}^\star_+$ is some positive smooth function on N is called a lightlike warped product and denoted $M = (N \times_f F, g)$.

Remark 5.1. *In [7], this warped product is called of class A. The class B one is concerned with two lightlike factors.*

(a) Let $M = \mathbb{L} \times M_1 \times_f M_2$ be a totally geodesic lightlike triple warped product hypersurface, with f a smooth positive function on M_1, $\mathbb{L}$ a (one dimensional) global null curve, (M_i, g_i) Riemannian manifolds $(i = 1, 2)$. Since M is totally geodesic, it is possible to use a normalization for which the 1-form τ (or equivalently φ) vanishes identically. Let ∇^i $(i = 1, 2)$ denote the Levi-Civita connection on (M_i, g_i). We have

$$g = g_1 + (f\pi_1(x))^2 g_2,$$

and the induced connection ∇ on M is given for X, Y tangent to $M' = M_1 \times_f M_2$ by

$$\begin{aligned} \nabla_X Y &= \nabla^1_{X_1} Y_1 + \nabla^2_{X_2} Y_2 + [X_1(\psi) Y_2 + Y_1(\psi) X_2 - g(X_2, Y_2) grad\psi] \\ &\quad + C(X, Y)\xi, \end{aligned}$$

where π_1 denotes the projection on the factor M_1 of M, $X = (X_1, 0) + (0, X_2) = (X_1, X_2)$, $Y = (Y_1, 0) + (0, Y_2) = (Y_1, Y_2)$ on $M_1 \times M_2$, $\nabla^i_{X_i} Y_i|_p \in T_p M_i$ with the vector $(\nabla^1_{X_1} Y_1|_p, 0_q) \in T_{(p,q)} M_1 \times M_2$ etc., $\psi = \ln f$ and $grad\psi$ its gradient with respect to g, and C the second fundamental form of the screen distribution $S(TM) = TM_1 \oplus TM_2$. Note that for $X \in \Gamma(TM)$, due to $[\xi, X] = 0$, we have

$$\nabla_\xi X = \nabla_X \xi = -\tau(X)\xi = 0.$$

Now, assume that $S(TM)$ is totally geodesic in M (and hence in the ambient space $\overline{M} \supset M$) and define a $(1, 1)$ tensor on M by

$$\begin{cases} S(\xi) = \mu\xi, & \mu \in \mathbb{R} \\ S(X) = 0, & X \in D_1 = TM_1 \\ S(X) = \lambda X, & \lambda = Cf^2,\ C \in \mathbb{R}^\star. \end{cases}$$

S is a well defined $(1, 1)$ tensor on M that preserves the screen distribution and is obviously symmetric. Let $X = \eta(X)\xi + X_1 + X_2 \in TM$. Our aim is to show that $X^\flat(\nabla S(X, X)) = 0$. We have $SX = \mu\eta(X)\xi + \lambda X_2$ and direct computation gives

$$\nabla_X(SX) = -\lambda g(X_2, Y_2) grad\psi + 3Cf^2 X_1(\psi) X_2 + \lambda \nabla^2_{X_2} X_2.$$

Also,

$$S(\nabla_X X) = \lambda \left[2X_1(\psi) X_2 + \nabla^2_{X_2} X_2\right].$$

Then,

$$\nabla_X(SX) - S(\nabla_X X) = \lambda \left[-g(X_2, X_2) grad\psi + X_1(\psi) X_2\right]. \tag{5.2}$$

Therefore

$$\begin{aligned} X^\flat(\nabla S(X, X)) &= X^\flat(\nabla_X(SX) - S(\nabla_X X)) \\ &= \eta(X)\xi^\flat[\nabla_X(SX) - S(\nabla_X X)] \\ &\quad + (X_1 + X_2)^\flat[\nabla_X(SX) - S(\nabla_X X)] \end{aligned}$$

$$\begin{aligned} &= \eta(X)\eta[\nabla_X(SX) - S(\nabla_X X)] \\ &\quad +(X_1+X_2)^\flat[\nabla_X(SX) - S(\nabla_X X)]. \end{aligned}$$

Since by (5.2), $\nabla_X(SX) - S(\nabla_X X)$ is $\Gamma(S(TM))$-valued, we have

$$\eta[\nabla_X(SX) - S(\nabla_X X) = 0].$$

But the second term is

$$\begin{aligned} (X_1+X_2)^\flat[\nabla_X(SX) - S(\nabla_X X)] &= \lambda(X_1+X_2)^\flat[-g(X_2,X_2)grad\psi \\ &\quad +X_1(\psi)X_2] \\ &= \lambda[-\langle X_1, grad\psi\rangle\langle X_2,X_2\rangle \\ &\quad +X_1(\psi)\langle X_2,X_2\rangle] = 0. \end{aligned}$$

Thus, $X^\flat(\nabla S(X,X)) = 0$ and S defines a Gray-tensor on $(M, g, S(TM))$.

(b) **Killing horizons**. Let (M, g) be a lightlike hypersurface of a pseudo-Riemannian manifold $(\overline{M}, \overline{g})$ and $\overline{G}$ a continuous k-parameters group of isometry acting on $(\overline{M}, \overline{g})$. By local isometry horizon (LIH in short) with respect to $\overline{G}$ it is meant a lightlike hypersurface that is invariant under $\overline{G}$ and for which each null geodesic is a trajectory of the group. In case $\overline{G}$ is 1-parameter, the LIH is said to be a *Killing horizon*. It turns out that a Killing horizon is a lightlike hypersurface whose null tangent vector can be normalized to coincide with a Killing vector field [5]. Taking into account theorem 2.1, Killing horizons are totally geodesic in $(\overline{M}, \overline{g})$. By *global hypersurface* in a Killing horizon M we mean a topological hypersurface which is crossed (orthogonally and) exactly once by any null geodesic trajectory of M. A Killing horizon admitting such a hypersurface will be called *a globally Killing horizon* (GKH). On the latter, it is possible to construct a special screen distribution as follows. Let $(\varphi_t)_{t\in I\subset\mathbb{R}}$ be the 1-parameter group with respect to which M is a Killing horizon, and H a global hypersurface in M. By definition of H it follows that for each $p \in M$, there exists a unique $(t, q) \in I \times H$ such that $p = \varphi_t(q)$. We set $S(T_pM) = \varphi_{t\star q}(T_qH)$. Clearly, such a $S(TM)$ defines an integrable screen distribution on M, we denote $S(TM, \varphi_t, H)$. Recall that throughout the text, the ambient manifold $(\overline{M}, \overline{g})$ has Lorentzian signature so that global hypersurfaces are Riemannian. Also, the normalized null tangent vector on the Killing horizon will be denoted ξ. Consider now a globally Killing horizon for which local geodesic symmetries preserve a global hypersurface, say H, and volume of its regions. The Ricci endomorphism (or the Ricci tensor) of such a H is an $\mathcal{A}$-tensor [9], say σ. Now define on $(M, g, S(TM, \varphi_t, H))$ a $(1,1)$-tensor by

$$SX = \mu\eta(X)\xi + \sigma(PX), \qquad \mu \in \mathbb{R}, \tag{5.3}$$

where P denote the projection morphism of the bundle TM onto the screen distribution $S(TM, \varphi_t, H)$ with respect to the decomposition (2.3). Clearly, such a S is g-symmetric, and preserves $S(TM, \varphi_t, H)$. Also, observe that since local geodesic symmetries preserve H, the screen distribution $S(TM, \varphi_t, H)$ is totally geodesic in M. Finally, using (b) in remark 3.1, it follows that S is a Gray-tensor on M.

6. Some Facts

Fact 6.1. *Any Gray-tensor on $(M, g, S(TM))$ is diagonalizable.*

Proof. First, observe that the global null vector ξ is an eigenvector field of S. Since $\langle S\xi, X\rangle = \langle \xi, SX\rangle = 0$ for all X in $\Gamma(TM)$, it follows $S\xi \in \Gamma(RadTM)$ and there exists a smooth function λ_0 such that $S\xi = \lambda_0\xi$. Since $S(TM)$ is Riemannian, we know that the restriction S' is diagonalizable and the same is for S using (3.3).□

Now, define the integer-valued function

$$x \to E_S(x) = Card\{\text{distinct eigenvalues of } S_x\}$$

and set

$$M_S = \{x \in M : E_S \text{ is constant in a neighbourhood of } x\}$$

The set M_S is open and dense in M. On each component U of M_S, the dimension, say p_α, of the eigenspace $D_\alpha = Ker(S - \lambda_\alpha I)$ associated to the eigenfunction λ_α is constant. From now on, we assume all manifolds connected unless otherwise stated and $M = M_S$. Also, note that

$$TM = \sum_{\alpha}^{k} D_\alpha$$

with $D_0 = RadTM = span\{\xi\}$. We use the following range of indices: $0 \le \alpha \le k$ and $1 \le i \le k$. We have the following technical result.

Fact 6.2. *Let S denote a Gray-tensor on $(M, g, S(TM))$, and $\lambda_0, \lambda_1, \cdots, \lambda_k$ in $C^\infty(M)$ be eigenfunctions of S. Then,*

$$\begin{aligned}\forall\ X \in \Gamma(D_i), \nabla S(X, X) &= -\frac{1}{2}\langle X, X\rangle \nabla^g \lambda_i \\ &+ \left[\frac{1}{2}\langle X, X\rangle \eta(\nabla^g\lambda_i) + (\lambda_i - \lambda_0)C(X, X)\right]\xi,\end{aligned} \tag{6.1}$$

and

$$D_\alpha \subset Kerd\lambda_\alpha \qquad 0 \le \alpha \le k. \tag{6.2}$$

If $i \neq j$, $X \in \Gamma(D_i)$ and $Y \in \Gamma(D_j)$ then

$$\langle \nabla_X X, Y\rangle = \frac{1}{2}\frac{Y \cdot \lambda_i}{\lambda_j - \lambda_i}\langle X, X\rangle. \tag{6.3}$$

If $X \in \Gamma(D_0)$ or $Y \in \Gamma(D_0)$

$$\langle \nabla_X X, Y\rangle = 0. \tag{6.4}$$

Proof. For $X \in \Gamma(D_i)$ and $Y \in \Gamma(TM)$we have

$$\nabla S(Y, X) = (Y \cdot \lambda_i)X + (\lambda_i I - S)\nabla_Y X. \tag{6.5}$$

Then,

$$\begin{aligned}\langle \nabla S(Y, X), X\rangle &= (Y \cdot \lambda_i)\langle X, X\rangle + \langle(\lambda_i I - S)\nabla_Y X, X\rangle \\ &= (Y \cdot \lambda_i)\langle X, X\rangle + \langle \nabla_Y X, \lambda_i X - \lambda_i X\rangle \\ &= (Y \cdot \lambda_i)\langle X, X\rangle\end{aligned}$$

that is

$$\langle \nabla S(Y, X), X\rangle = (Y \cdot \lambda_i)\langle X, X\rangle. \tag{6.6}$$

Therefore, taking $Y = X$ leads to

$$0 = \langle \nabla S(X,X), X\rangle = (X\cdot\lambda_i)\langle X,X\rangle \qquad 1\le i\le k.$$

Since $X \in \Gamma(D_i) \subset \Gamma(S(TM))$ (Riemannian), we have $\langle X, X\rangle \neq 0$ and $X\cdot\lambda_i = 0, 1 \le i \le k$, that is $D_i \subset Kerd\lambda_i, 1\le i\le k$. Also, integrable curves of ξ are null geodesics. Then $\nabla S(\xi,\xi) = 0 = (\xi\cdot\lambda_0)\xi$ and $(\xi\cdot\lambda_0) = 0$. Thus, $D_0 \subset Kerd\lambda_0$ and (6.2) is proved. From (6.5) and (6.2) it follows that

$$\nabla S(X,X) = (\lambda_i I - S)\nabla_X Y X. \tag{6.7}$$

Observe that for X,Y and Z in $\Gamma(S(TM))$, (3.2) is equivalent to

$$\langle\nabla S(X,Y), Z\rangle + \langle\nabla S(Y,Z), X\rangle + \langle\nabla S(Z,X), Y\rangle = 0.$$

Also, $(\langle\nabla S(X,Y), X\rangle = \langle\nabla S(X,X), Y\rangle$. Hence $2\langle\nabla S(X,X), Y\rangle + \langle\nabla S(Y,X), X\rangle = 0$. Taking into account (6.6) yields $2\langle\nabla S(X,X), Y\rangle + (Y\cdot\lambda_i)\langle X,X\rangle = 0$, i.e

$$\langle 2\nabla S(X,X) + \langle X,X\rangle\nabla^g\lambda_i, Y\rangle = 0 \qquad \forall Y\in\Gamma(S(TM)). \tag{6.8}$$

Then, since (6.8) holds trivially for $Y \in \Gamma(RadTM)$,

$$\langle 2\nabla S(X,X) + \langle X,X\rangle\nabla^g\lambda_i, Y\rangle = 0 \qquad \forall Y\in\Gamma(TM). \tag{6.9}$$

Thus,

$$2\nabla S(X,X) + \langle X,X\rangle\nabla^g\lambda_i \in RadTM = Span\{\xi\}.$$

It follows that

$$\nabla S(X,X) = -\frac{1}{2}\langle X,X\rangle\nabla^g\lambda_i + q(X)\xi, \forall Y\in\Gamma(TM), \tag{6.10}$$

where $q(X)$ is a quadratic function in X. From (6.10), we have

$$\eta(\nabla S(X,X)) = -\frac{1}{2}\langle X,X\rangle\eta(\nabla^g\lambda_i) + q(X). \tag{6.11}$$

Now, using (3.3), we derive for $X \in \Gamma(D_i)$,

$$\nabla S(X,X)) = \overset{\star}{\nabla} S'(X,X) + (\lambda_i - \lambda_0)C(X,X)\xi, \tag{6.12}$$

and

$$\eta\left(\nabla S(X,X)\right) = (\lambda_i - \lambda_0)C(X,X). \tag{6.13}$$

Thus, combining (6.11) and (6.13) lead to

$$q(X) = \frac{1}{2}\langle X,X\rangle\eta(\nabla^g\lambda_i) + (\lambda_i-\lambda_0)C(X,X). \tag{6.14}$$

Substitute in (6.10) to get the announced relation in (6.1).

For $X \in \Gamma(D_i)$, $Y\in\Gamma(D_j)$ with $i \neq j$,

$$\begin{aligned}\langle\nabla S(X,X), Y\rangle &= \langle(\lambda_i I - S)\nabla_X X, Y\rangle\\ &= \langle\nabla_X X, (\lambda_i - \lambda_j)Y\rangle.\end{aligned}$$

Thus, by (6.1),

$$-\frac{1}{2}\langle X,X\rangle\langle\nabla^g\lambda_i, Y\rangle = (\lambda_i - \lambda_j)\langle\nabla_X X, Y\rangle.$$

and

$$\langle \nabla_X X, Y \rangle = \frac{1}{2}\langle X, X\rangle \frac{Y \cdot \lambda_i}{\lambda_j - \lambda_i}\langle X, X\rangle.$$

Finally, it is clear that if $X \in \Gamma(D_0)$ or $Y \in \Gamma(D_0)$, one has $\langle \nabla_X X, Y\rangle = 0$, and the proof is complete.□

Corollary 6.1. *The following assertions are equivalent.*

(a) $\forall\ X \in \Gamma(D_i), \nabla_X X \in \Gamma(D_i)$.
(b) $\forall\ X, Y \in \Gamma(D_i), \nabla_X Y + \nabla_Y X \in \Gamma(D_i)$.
(c) $\forall\ X \in \Gamma(D_i), \nabla S(X, X) = 0$.
(d) $\forall\ X, Y \in \Gamma(D_i), \nabla S(X, Y) + \nabla S(Y, X) = 0$.
(e) $\nabla^g \lambda_i$ *is* D_0*-valued vector field and* $\forall\ X \in \Gamma(D_i),\ C(X, X) = 0, \quad 1 \le i \le k$.

Proof. The equivalences $(a) \iff (b)$ and $(c) \iff (d)$ are obvious as polarizations. Let us show $(a) \iff (c)$. We have

$$\nabla_X X \in\ \Gamma(D_i) \overset{(6.7)}{\Longrightarrow} \nabla S(X, X) = 0.$$

Conversely, if for all X in $\Gamma(D_i)$, $\nabla S(X, X) = 0$, then by (6.7), $(\lambda_i I - S)\nabla_X X = 0$, i.e $\nabla_X X \in \Gamma(D_i)$, thus $(a) \iff (c)$. Finally, using (6.1) we obtain

$$\nabla S(X, X) = 0 \iff -\frac{1}{2}\langle X, X\rangle P\nabla^g \lambda_i + (\lambda_i - \lambda_0)C(X, X)\xi = 0,$$

which is equivalent to $P\nabla^g \lambda_i = 0$ and $C(X, X) = 0$, i.e (e).□

Note that D_0 is of rank one, then it is integrable. Also, for X, Y in $\Gamma(D_i)$, we have

$$\nabla S(X, Y) - \nabla S(Y, X) = (\lambda_i I - S)([X, Y])$$

so that D_i is integrable if and only if $\forall\ X,\ Y$ in $\Gamma(D_i),\ \nabla S(X, Y) - \nabla S(Y, X)$. Moreover, we obtain the following.

Fact 6.3. *If* $\nabla^g \lambda_i$ *is* D_0*-valued and for all* $X \in \Gamma(D_i)$, $C(X, X) = 0$, *then the following assertions are equivalent on* M.

(a) D_i *is integrable.*
(b) *For all* $X,\ Y$ *in* $\Gamma(D_i)$, $\nabla S(X, Y) = 0$.
(c) D_i *is autoparallel.*

Proof. For the first equivalence, we shall prove $(a) \Longrightarrow (b)$ and observe that $(b) \Longrightarrow (a)$ is obvious. Assume that (a) holds. From corollary 6.1(valid since (e) holds by hypothesis), $\nabla S(X, Y) + \nabla S(Y, X) = 0$ and integrability implies $\nabla S(X, Y) = \nabla S(Y, X)$. Thus, $\nabla S(X, Y) = 0$ and $(a) \Longrightarrow (b)$. Finally, from $\nabla_X Y + \nabla_Y X \in \Gamma(D_i)$ and $\nabla_X Y - \nabla_Y X = [X, Y] \in \Gamma(D_i)$ we obtain the equivalence $(a) \iff (c)$.□

7. Gray-tensors with exactly three eigenspaces

We consider and investigate some geometric properties of Gray-tensors with exactly three eigenspaces $D_0 = Ker(\lambda_0 I - S)$, $D_\alpha = Ker(\alpha I - S)$ and $D_\beta = Ker(\beta I - S)$ with $S(TM) = D_\alpha \oplus D_\beta$. In Riemannian setting, a classical theorem due to Jelonek [10] states that, for a $\mathcal{A}$-tensor with exactly two eigenvalues λ, μ and a constant trace, the eigenvalues are necessarily constant, and the eigenspace distributions are both integrable if and only if the $\mathcal{A}$-tensor is parallel. The following is a lightlike version of this result with three eigenvalues.

Theorem 7.1. *Let S be a Gray-tensor on $(M, g, S(TM))$ with exactly three eigenfunctions $\lambda_0 = cte$, α, β and a constant trace. Then $\nabla^g\alpha$ and $\nabla^g\beta$ are $D_0 = RadTM$-valued. In addition, If $S(TM)$ is totally geodesic then the distributions D_α and D_β are both integrable if and only if ∇S vanishes on $S(TM) \times S(TM)$.*

Proof. Since S is smooth, $x \to p(x) = \dim D_\alpha(x)$ and $x \to q(x) = \dim D_\beta(x)$ are discrete differentiable functions on $M_S = M$, so they are constant functions we denote by p and q respectively. From

$$\lambda_0 + p\alpha + q\beta = trS = cte$$

we derive

$$p\nabla^g\alpha + q\nabla^g\beta = 0, \qquad (\nabla^g\lambda_0 = 0). \tag{7.1}$$

Observe that $\langle\nabla^g\alpha, \nabla^g\beta\rangle = 0$. Then from (7.1) we obtain $p\langle\nabla^g\alpha, \nabla^g\alpha\rangle = 0$ and $q\langle\nabla^g\beta, \nabla^g\beta\rangle = 0$. Hence $\nabla^g\alpha$ and $\nabla^g\beta$ are RadTM-valued since p and q are non zero.

Assume D_α is integrable and consider $X,\ V \in \Gamma(D_\alpha)$ and $Y \in \Gamma(D_\beta)$. We have

$$\begin{aligned}
\langle\nabla S(V, Y), X\rangle &= \langle\nabla_V(SY) - S(\nabla_V Y), X\rangle \\
&= -\beta\langle Y, \nabla_V X\rangle - \alpha\langle\nabla_V Y, X\rangle \\
&= -\beta\langle Y, \nabla_V X\rangle + \alpha\langle\nabla_V X, Y\rangle \\
&= (\alpha - \beta)\langle\nabla_V X, Y\rangle.
\end{aligned}$$

But the last term vanishes since D_α is autoparallel from (b) in Fact 6.3. Thus, we obtain for all X, V in D_α, Y in D_β,

$$\langle\nabla S(V, Y), X\rangle = 0. \tag{7.2}$$

Now, let U in D_β. Since $C(X, Y) = 0$ we have

$$\nabla S(X, Y) = (\beta I - S)\nabla_X Y \in S(TM). \tag{7.3}$$

Hence

$$\langle\nabla S(X, Y), U\rangle = \langle(\beta I - S)\nabla_X Y, U\rangle = \langle\nabla_X Y, (\beta I - S)U\rangle = 0. \tag{7.4}$$

Similar computation assuming D_β integrable leads to

$$\langle\nabla S(Y, X), U\rangle = 0, \tag{7.5}$$

and

$$\langle\nabla S(Y, X), V\rangle = 0, \tag{7.6}$$

for all Y, U in $\Gamma(D_\beta)$ and X, V in $\Gamma(D_\alpha)$. Thus, By (7.2),(7.4)-(7.6) and (b) in Fact 6.3 ∇S vanishes on $S(TM) \times S(TM)$.

The converse is immediate from Fact 6.3.□

8. Totally umbilical screen foliation

In general a distribution $D \subset TM$ is called umbilical if there exist a vector field $\varsigma \in \chi(M)$ such that

$$\nabla_X X = p(\nabla_X X) + \langle X, X\rangle\varsigma, \tag{8.1}$$

for every local section $X \in \Gamma(D)$, where p denotes the "orthogonal" projection $p : TM \longrightarrow D$. In case D is integrable, then it is called totally umbilical. The vector field ς in the definition is called the mean curvature vector of the distribution D. In particular, the screen distribution $S(TM)$ is totally umbilical if on any coordinate neighbourhood $\mathcal{U} \subset M$ there exists a smooth function ρ such that

$$C(X, PY) = \rho\, g(X, Y). \tag{8.2}$$

Now, we state the following

Proposition 8.1. *Let S be a Gray-tensor on $(M, g, S(TM))$ where the screen distribution is totally umbilical. Then all the eigenspace distributions $D_\alpha = Ker(\alpha I - S)$ are umbilical.*

Proof. Note that $TM = D_0 \oplus \sum_{i=1}^{k} D_i$ with $D_0 = RadTM$. Since $\nabla_\xi \xi \in D_0$, it is obvious that D_0 is umbilical. Also, for $X \in \Gamma(D_i)$,

$$\nabla_X X = \overset{\star}{\nabla}_X X + C(X, X)\xi = \overset{\star}{\nabla}_X X + \rho g(X, X)\xi. \tag{8.3}$$

Let $p_i : TM \longrightarrow D_i$ denote the projection morphism onto D_i, we write

$$\nabla_X X = p_i(\nabla_X X) + h_i(X, X).$$

It follows that for $Y \in S(TM)$,

$$\langle \nabla_X X, Y\rangle = \langle p_i(\nabla_X X), Y\rangle + \langle h_i(X, X), Y\rangle,$$

that is

$$\begin{aligned}
\langle h_i(X, X), Y\rangle &= \sum_{\substack{j=1\\ j\neq i}}^{k} \langle \nabla_X X, P_j Y\rangle \\
&\overset{(6.3)}{=} \frac{1}{2}\langle X, X\rangle \sum_{\substack{j=1\\ j\neq i}}^{k} \frac{\langle \nabla^g \lambda_i, P_j Y\rangle}{\lambda_j - \lambda_i} \\
&= -\frac{1}{2}\langle X, X\rangle \sum_{\substack{j=1\\ j\neq i}}^{k} \langle \frac{\nabla^g \lambda_i}{\lambda_i - \lambda_j}, P_j Y\rangle \\
&\overset{(6.2)}{=} -\frac{1}{2}\langle X, X\rangle \sum_{\substack{j=1\\ j\neq i}}^{k} \langle P_j \nabla^g \ln|\lambda_i - \lambda_j|, Y\rangle.
\end{aligned}$$

Hence, the $S(TM)$ component of $h_i(X, X)$ is

$$\xi_i = -\frac{1}{2}\langle X, X\rangle \sum_{\substack{j=1\\ j\neq i}}^{k} P_j(\nabla^g \ln|\lambda_i - \lambda_j|). \tag{8.4}$$

Then, from (8.3) we have

$$h_i(X, X) = \langle X, X\rangle(\xi_i + \rho\xi). \tag{8.5}$$

Hence D_i is umbilical $(1 \leq i \leq k)$, with $\varsigma_i = \xi_i + \rho\xi$ as mean curvature vector field.□

9. Almost product foliation

By integrable almost product structure it is meant a sequence $(D_0, \cdots, D_k)$ of distributions for which all the distributions $D_{\alpha_1} \oplus D_{\alpha_2} \oplus \cdots \oplus D_{\alpha_p}$ are integrable for any $0 \leq \alpha_1 \leq \cdots \leq \alpha_p \leq k$ and $p \in \{0, 1, \cdots, k\}$. A distribution D_i $(1 \leq i \leq k)$ is called D_0-almost autoparallel (resp. D_0-almost parallel) if for any X, Y in $\Gamma(D_i)$, $\nabla_X Y \in \Gamma(D_0 \oplus D_i)$ (resp. $\forall X \in \Gamma(TM), \forall Y \in \Gamma(D_i)$, $\nabla_X Y \in \Gamma(D_0 \oplus D_i)$).

The following result deals with quasi isotropy of S. More precisely, we have

Theorem 9.1. *Let S be a Gray-tensor on $(M, g, S(TM))$ with eigenfunctions $\lambda_0, \lambda_1, \cdots, \lambda_k$. Assume $\nabla^g \lambda_0$, $\nabla^g \lambda_1$, $\ldots$, $\nabla^g \lambda_k$ are $RadTM = D_0$- valued and the $D_\alpha = Ker(\lambda_\alpha I - S)$ define an integrable almost product structure on M. Then, $\nabla S|_{S(TM)\times TM} \in \Gamma(D_0)$.*

Proof. First, note that for $X \in S(TM)$, $\nabla S(X, \xi) = (X \cdot \lambda_0)\xi \in D_0$. Now, for $X \in \Gamma(D_i)$, we have from (6.1) and $\nabla^g \lambda_i \in \Gamma(_0)$,

$$\nabla S(X, X) = (\lambda_i - \lambda_0)\ C(X, X)\ \xi\ \in \Gamma(D_0). \tag{9.1}$$

Integrability of each D_i leads to $\nabla S(X, Y) = \nabla S(Y, X)$ for X and Y in D_i. Also, the integrability of the almost product structure implies $S(TM)$ is integrable and consequently C is symmetric on $S(TM) \times S(TM)$. So, for $X,\ Y \in \Gamma(D_i)$, we obtain by bipolarization of (9.1),

$$\nabla S(X, Y) = (\lambda_i - \lambda_0)\ C(X, Y)\ \xi\ \in \Gamma(D_0). \tag{9.2}$$

If $X \in \Gamma(D_i)$, $Y \in \Gamma(D_j)$, $i \neq j$, we have from (6.3) and $\nabla^g \lambda_i \in \Gamma(D_0)$,

$$\langle \nabla_X X, Y\rangle \;=\; \frac{1}{2}\frac{\langle \nabla^g \lambda_i, Y\rangle}{\lambda_j - \lambda_i}\langle X, X\rangle \;=\; 0,$$

Thus, $\nabla_X X \in D_0 \oplus D_i$ if $X \in \Gamma(D_i)$. It follows that

$$\nabla_X Y + \nabla_Y X \in D_0 \oplus D_i, \text{for } X \text{ and } Y \ \in \Gamma(D_i).$$

Since D_i is integrable, $\nabla_X Y - \nabla_Y X \in \Gamma(D_i) \subset \Gamma(D_0 \oplus D_i)$. Hence, for X and Y in $\Gamma(D_i)$,

$$\nabla_X Y \in \Gamma(D_0 \oplus D_i), \tag{9.3}$$

and each D_i is D_0-almost autoparallel.

Let $i,\ j,\ l$ be pairwise different numbers and $X \in \Gamma(D_i)$, $Y \in \Gamma(D_j)$ and $Z \in \Gamma(D_l)$. By Koszul formula and integrability of the almost product structure, it follows that

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle \;&=\; X \cdot \langle Y, Z\rangle + Y \cdot \langle X, Z\rangle - Z \cdot \langle X, Y\rangle \\ &\quad + \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle - \langle [Y, Z], X\rangle = 0. \end{aligned}$$

Hence, for $X \in \Gamma(D_i)$, $Y \in \Gamma(D_j)$, $(i \neq j)$,

$$\nabla_X Y \in \Gamma(D_0 \oplus D_i \oplus D_j). \tag{9.4}$$

Also, consider $X,\ Z \in \Gamma(D_i)$, $Y \in \Gamma(D_j)$, $(i \neq j)$, we have

$$0 = \langle Z, Y\rangle \;\;\Rightarrow\;\; 0 = \langle \nabla_X Z, Y\rangle + \langle Z, \nabla_X Y\rangle \overset{(9.3)}{=} \langle Z, \nabla_X Y\rangle.$$

Then, using (9.4) we derive

$$\nabla_X Y \ \in \Gamma(D_0 \oplus D_j), \text{ for } X \in \Gamma(D_i),\ Y \in \Gamma(D_j),\ (i \neq j). \tag{9.5}$$

Consequently, from (9.3) and (9.5), it follows

$$\nabla_X Y \ \in \Gamma(D_0 \oplus D_i), \ \text{for } X \in \Gamma(S(TM)) \text{ and } Y \in \Gamma(D_i). \tag{9.6}$$

Finally, we have from (6.5) and (9.5) that for $X \in \Gamma(S(TM))$, $Y \in \Gamma(D_j)$,

$$\begin{aligned}
\nabla S(X,Y) &= \langle \nabla^g \lambda_j, X\rangle Y + (\lambda_j I - S)\nabla_X Y \\
&= (\lambda_j I - S)(\eta(\nabla_X Y)\xi + p_j(\nabla_X Y)) \\
&= (\lambda_j - \lambda_0)\ \eta(\nabla_X Y)\xi \\
&= (\lambda_j - \lambda_0)\ C(X,Y)\xi \ \in \Gamma(D_0), \qquad (9.7)
\end{aligned}$$

which completes the proof.□

Remark 9.1. *It follows that, under hypothesis of theorem 9.1, we have by (9.7),*

$$\nabla S(X,Y) = (\lambda_j - \lambda_0)\ C(X,Y)\xi, \tag{9.8}$$

for $X \in \Gamma(S(TM))$ and $Y \in \Gamma(D_j)$.

Corollary 9.1. *Let S be a Gray-tensor on $(M, g, S(TM))$ with constant eigenfunctions $(\lambda_0, \lambda_1, \cdots, \lambda_k) \in \mathbb{R}^{k+1}$ and integrable almost product structure given by its eigenspace distributions $D_\alpha = Ker(\lambda_\alpha I - S)$. Then S is an isotropic Gray-tensor.*

Proof. From theorem 9.1, it suffices to show that $\nabla S(\xi, X) \in \Gamma(D_0)$ for X in $\Gamma(TM)$. But since $D_0 \oplus D_i$ is integrable, $\nabla_\xi X \in \Gamma(D_0 \oplus D_i)$ for $X \in \Gamma(D_i)$. Thus,

$$\begin{aligned}
\nabla S(\xi, X) &= (\xi \cdot \lambda_i)X + (\lambda_i - \lambda_0)p_0(\nabla_\xi X) \\
&= (\lambda_i - \lambda_0)p_0(\nabla_\xi X) \in \Gamma(D_0).\square
\end{aligned}$$

Corollary 9.2. *Let S be a Gray-tensor on $(M, g, S(TM))$ with eigenfunctions $(\lambda_0, \lambda_1, \cdots, \lambda_k)$. Assume that $\nabla^g\lambda_0, \nabla^g\lambda_1, \cdots, \nabla^g\lambda_k$ are $RadTM = D_0$-valued and the $D_\alpha = Ker(\lambda_\alpha I - S)$ define an integrable almost product structure on M. If leaves of the screen distribution are totally geodesic in M, then $\nabla S = 0$ on $S(TM) \times S(TM)$.*

Proof. The foliation determined by the screen distribution is totally geodesic if and only if $C = 0$. Then our claim follows (9.8) in remark 9.1.□

10. Gray-tensors and lightlike warped product

Lightlike warped products are introduced in [7], and used in [8] to study the problem of finding globally null manifolds with constant scalar curvature. The following result shows that there is an interplay between existence of Gray-tensors of certain type and lightlike warped product structure. In some sense it represents a more general converse to example (a) in section 5.

Theorem 10.1. *Let (M, g) be a Killing horizon with a complete simply connected Riemannian global hypersurface, say H, S a Gray-tensor on $(M, g, S(TM, \varphi_t, H))$ with $k+1$ eigenfunctions λ_0 , $\lambda_1, \ldots , \lambda_k$ and eigendistributions $D_0 = Ker(S - \lambda_0 I) = RadTM$, $D_i = Ker(S - \lambda_i I)$, $i = 1, \ldots, k$. If*

(a) *$\lambda_1 = \mu = constant$,*
(b) *The almost product structure $(D_0, D_1, \ldots, D_k)$ is integrable,*

(c) $\oplus_{\substack{j>1\\ j\neq i}} D_j \subset Kerd\lambda_i,\ i=1,\dots,k,$

then,

$$M = \mathbb{L}\times M_1 \times_{f_2} M_2 \times \cdots, \times_{f_k} M_k,$$

where $\mathbb{L}$ *is a one-dimensional integral curve of the global null (radical) vector field on* M *and* M_i $(1\le i\le k)$ *are leaves of* D_i *and* $f_i^2 = |\lambda_i - \mu|$, $(2\le i\le k)$ *are smooth positive functions on the factor* M_1.

Proof. First of all, let us recall that any codimension-one foliation which admits an orthogonal Killing field must be totally geodesic (see [6] for instance). As (M, g) is a Killing horizon, the characteristic orthogonal line bundle (or radical distribution) is a Killing distribution (Theorem 2.1). Thus, $S(TM, \varphi_t, H)$ defines a codimension-one foliation in M which admits an orthogonal Killing field, say ξ. Hence, leaves of $S(TM, \varphi_t, H)$ are totally geodesic in M. It follows that M is a product manifold $\mathbb{L}\times M'$ where $\mathbb{L}$ is a one-dimensional integral curve of the global null (radical) vector field ξ on M and M' a leaf of $S(TM, \varphi_t, H)$. Let g' denote the Riemannian metric induced on M' and $\pi : \mathbb{L}\times M' \longrightarrow M'$ the natural projection map onto M'. Then the lightlike hypersurface (M, g) is isometric to $(\mathbb{L}\times M', g = \pi^\star g')$. Now, by item (b) in remark 3.1, S induces by restriction an $\mathcal{A}$-tensor S' on the leal M' with respect to the Levi-Civita connection $\overset{\star}{\nabla}$ it inherits from M and $\lambda_1,\dots,\lambda_k$ are eigenfunctions of S', with eigendistributions D_i, $i = 1,\dots,k$. In particular, as $\lambda_1 = \mu =$ constant and

$$\oplus_{\substack{j>1\\ j\neq i}} D_j \subset Kerd\lambda_i,\ i=1,\dots,k,$$

the final result follows [11]. Indeed, since in addition to the above facts, H (and hence M') is a complete (and simply connected) Riemannian hypersurface of M, we have

$$(M', g') = M_1 \times_{f_2} M_2 \times \cdots \times_{f_k} M_k,$$

where $TM_i = D_i$ and $f_i = \sqrt{|\lambda_i - \mu|}$, $2 \le i \le k$. Then,

$$M = \mathbb{L}\times M_1 \times_{f_2} M_2 \times \cdots \times_{f_k} M_k$$

is a multiply warped product manifold where $f_2,\dots,f_k$ are smooth positive functions on the factor M_1 of the lightlike product manifold $\mathbb{L}\times M_1$.□

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