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ON NULL 2-TYPE SUBMANIFOLDS OF EUCLIDEAN SPACES

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Abstract. Let M^n be an n-dimensional submanifold of the Euclidean space E^{n+2} with non-parallel mean curvature vector and flat normal connection. We prove that if M is of null 2-type and flat with constant mean curvature, then the dimension of the first normal space $N_1(M)$ of M is one. Then we show that M is an open portion of an n-dimensional helical cylinder of E^{n+2} if and only if M is flat and of null 2-type with constant mean curvature.

1. Introduction

In [2], B.Y. Chen gave a classification of null 2-type surfaces in the Euclidean space E^3 and he proved that they are circular cylinders. Later, in [3], he proved that a surface M in the Euclidean space E^4 is of null 2-type with parallel normalized mean curvature vector if and only if M is an open portion of a circular cylinder in a hyperplane of E^4, and the only null 2-type surfaces of the Euclidean space E^4 with constant mean curvature are open portion of helical cylinders which are the product surfaces of a straight line and a helix.

In [8], S.J. Li showed that a surface M in E^m with parallel normalized mean curvature vector is of null 2-type if and only if M is an open portion of a circular cylinder. Also, in [9], S.J. Li proved that for a non-pseudo-umbilical Chen surface M in E^m with constant mean curvature, if M is of null 2-type, then M is flat and lies fully in E^3, E^4, E^5 or E^6 of E^m (for the definition of Chen surfaces please see [1, 7]).

In [6], A. Ferrandez and P. Lucas proved that Euclidean hypersurfaces of null 2-type and having at most two distinct principal curvatures are locally isometric to a generalized cylinder.

In [4, 5], the author studied 3-dimensional null 2-type submanifolds of the Euclidean space E^5. In [4], he proved that a 3-dimensional submanifold M of the Euclidean space E^5 having two distinct principal curvatures in the parallel mean curvature direction and having a second fundamental form of a constant square length is of null 2-type if and only if M is locally isometric to one of $E \times S^2 \subset E^4 \subset E^5$, $E^2 \times S^1 \subset E^4 \subset E^5$ or $E \times S^1(a) \times S^1(a)$. In [5], he showed that for a 3-dimensional

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submanifold M of the Euclidean space E^5 such that M is not of 1-type, if M is of null 2-type and flat with constant mean curvature and non-parallel mean curvature vector, then the normal bundle of M is flat. And also, he proves that M is an open portion of a 3-dimensional helical cylinder of E^5 if and only if M is of null 2-type and flat with constant mean curvature and non-parallel mean curvature vector.

Considering the result on the normally flatness of M^3 in E^5, we want to generalize the main result given on the classification of null 2-type submanifolds ([5]) to an n-dimensional submanifold M^n of the Euclidean space E^{n+2} by assuming the normally flatness as one of the hypotheses. We prove that for an n-dimensional normally flat submanifold M of the Euclidean space E^{n+2} with non-parallel mean curvature vector, if M is of null 2-type and flat with constant mean curvature, then $\dim(N_1(M)) = 1$ and then we show that M is an open portion of an n-dimensional helical cylinder of E^{n+2} if and only if M is flat and of null 2-type with constant mean curvature and non-parallel mean curvature vector. This work generalizes the results given in [3], [5] and [9] about helical cylinders and null 2-type submanifolds with non-parallel mean curvature vector.

2. Preliminaries

Let M be an n-dimensional submanifold of an m-dimensional Euclidean space E^m. Denote by Δ the Laplacian of M associated with the induced metric. A submanifold M of E^m is said to be of finite type if the position vector x of M in E^m can be decomposed in the following form:

$$x = x_0 + x_1 + \cdots + x_k, \tag{2.1}$$

where x_0 is a constant vector and $x_1, \dots, x_k$ are non-constant maps satisfying $\Delta x_i = \lambda_i x_i$, $i = 1, \dots k$. If all eigenvalues $\lambda_1, \dots, \lambda_k$ are mutually different, then the submanifold M is said to be of k-type and if, in particular, one of $\lambda_1, \dots, \lambda_k$ is zero, the submanifold M is said to be of null k-type.

Let M be an n-dimensional submanifold in an m-dimensional Euclidean space E^m. We denote by h, A, H, ∇ and $\nabla^\perp$, the second fundamental form, the Weingarten map, the mean curvature vector, the induced Riemannian connection and the normal connection of the submanifold M in E^m, respectively. We choose an orthonormal local frame $\{e_1, \dots, e_m\}$ on M such that $e_1, \dots, e_n$ are tangent to M and e_{n+1} is the direction of H, i.e., the normalized mean curvature vector. Denote by $\{\omega^1, \dots, \omega^m\}$ the dual frame and $\{\omega_B^A\}$, $A, B = 1, \dots, m$, the connection forms associated to $\{e_1, \dots, e_m\}$. We use the following convention on the range of indices: $1 \le A, B, C, \dots \le m$, $1 \le i, j, k, \dots \le n$, $n+1 \le \beta, \nu, \gamma, \dots \le m$. Denoting by D the Riemannian connection of E^m, we put $D_{e_k} e_i = \sum \omega_i^j(e_k) e_j + \sum h^\beta(e_i, e_k) e_\beta$ and $D_{e_k} e_\nu = \sum \omega_\nu^j(e_k) e_j + \sum \omega_\nu^\beta(e_k) e_\beta$. By Cartan's Lemma, we have

$$\omega_i^\beta = \sum_{j=1}^n h_{ij}^\beta \omega^j, \quad h_{ij}^\beta = h_{ji}^\beta,$$

where h_{ij}^β are the coefficients of the second fundamental form in the direction e_β. The mean curvature vector H is given by

$$H = \frac{1}{n} \sum_{\beta=n+1}^m \mathrm{tr}(h^\beta) e_\beta. \tag{2.2}$$

Using the connection equations $\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_j^k(e_i) e_k$, we obtain the equations of Gauss and Codazzi relative to the basis $\{e_1, \ldots, e_n\}$, respectively, as

$$\begin{aligned} e_\ell(\omega_i^j(e_k)) - e_k(\omega_i^j(e_\ell)) = & \sum_{r=1}^{n} \{ \omega_i^r(e_\ell)\omega_r^j(e_k) - \omega_i^r(e_k)\omega_r^j(e_\ell) \\ & + \omega_i^j(e_r)[\omega_k^r(e_\ell) - \omega_\ell^r(e_k)]\} + \sum_{\nu=n+1}^{m} (h_{ik}^\nu h_{j\ell}^\nu - h_{jk}^\nu h_{i\ell}^\nu), \\ & 1 \le i < j \le n, \ \ 1 \le \ell < k \le n \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} e_j(h_{ik}^\nu) - e_k(h_{ij}^\nu) = & \sum_{r=1}^{n} \{ h_{ir}^\nu[\omega_k^r(e_j) - \omega_j^r(e_k)] + h_{rk}^\nu \omega_i^r(e_j) - h_{rj}^\nu \omega_i^r(e_k)\} \\ & + \sum_{\beta=n+1}^{m} (h_{ij}^\beta \omega_\beta^\nu(e_k) - h_{ik}^\beta \omega_\beta^\nu(e_j)), \\ & \nu = n+1, \ldots, m, \ \ i = 1, \ldots, n, \ 1 \le j < k \le n. \end{aligned} \tag{2.4}$$

Also the Ricci equation is given by

$$\langle R^\perp(e_i, e_j)e_\beta, e_\gamma \rangle = \langle [A_{e_\beta}, A_{e_\gamma}](e_i), e_j \rangle, \tag{2.5}$$

where $R^\perp$ is the curvature tensor of the normal bundle.

The first normal space $N_1(M)$ of M at each point $p \in M$ in E^m is defined as the orthogonal complement of the subspace $\{\xi \in T_p^\perp M | A_\xi = 0\}$ in the normal space $T_p^\perp M$.

The product of a circular helix with nonzero torsion that lies in a 3-dimensional linear subspace E^3 of an m-dimensional Euclidean space E^m and a $(k-1)$-plane of E^m is called a k-dimensional helical cylinder in the Euclidean space E^m.

3. Null 2-type submanifolds

If M is a null 2-type submanifold of E^m, then we have the following decomposition of the position vector x of M in E^m:

$$x = x_1 + x_2, \quad \Delta x_1 = 0, \ \ \Delta x_2 = \lambda x_2, \tag{3.1}$$

for some non-constant vectors x_1 and x_2 on M where λ is a non-zero constant. Since we have $\Delta x = -nH$, then (3.1) implies

$$\Delta H = \lambda H. \tag{3.2}$$

In the theory of finite type immersions we know that a submanifold M of E^m is of 1-type if and only if either M is a minimal submanifold of E^m or M is a minimal submanifold of a hypersphere of E^m. Also, it is known from a lemma in [3] that for an n-dimensional submanifold M of a Euclidean space E^m, if there is a constant $\lambda \ne 0$ such that $\Delta H = \lambda H$, then M is either of 1-type or of null 2-type.

We need the following lemma given in [3].

Lemma 3.1. [3] *Let M be an n-dimensional submanifold of the Euclidean space E^m such that M is not of 1-type. Then M is of null 2-type if and only if we have*

$$\frac{n}{2}\nabla\alpha^2 + 2\mathrm{tr} A_{\nabla^\perp H} = 0, \tag{3.3}$$

$$\Delta\alpha = \lambda\alpha - \alpha\|A_{n+1}\|^2 - \alpha < \nabla^{\perp} e_{n+1}, \nabla^{\perp} e_{n+1} >, \tag{3.4}$$

$$\alpha \operatorname{tr}(A_{n+1}A_\beta) = 2\omega^\beta_{n+1}(\nabla\alpha) + \alpha \operatorname{tr}(\nabla\omega^\beta_{n+1}) - \alpha < \nabla^{\perp} e_{n+1}, \nabla^{\perp} e_\beta >, \tag{3.5}$$

where $\beta = n+2, \ldots, m$, $\alpha^2 =< H, H >$, $\|A_{n+1}\|^2 = \operatorname{tr}(A_{e_{n+1}}A_{e_{n+1}})$ *and* $\nabla\alpha$ *is the gradient of* α

Using Lemma 3.1 we obtain the following.

Lemma 3.2. *Let M be an n-dimensional normally flat submanifold of the Euclidean space E^{n+2} with non-parallel mean curvature vector such that M is not of 1-type. If M is of null 2-type with constant mean curvature α, then the followings hold:*

$$\omega^{n+2}_{n+1}(e_i)h^{n+2}_{ii} = 0, \quad i = 1, \ldots, n, \tag{3.6}$$

$$\|A_{n+1}\|^2 + \sum_{i=1}^{n} (\omega^{n+2}_{n+1}(e_i))^2 = \lambda, \tag{3.7}$$

$$\operatorname{tr}(A_{n+1}A_{n+2}) = \operatorname{tr}(\nabla\omega^{n+2}_{n+1}), \tag{3.8}$$

where $\|A_{n+1}\|^2 = \operatorname{tr}(A_{e_{n+1}}A_{e_{n+1}})$.

Proof. As M is normally flat in E^{n+2}, we can choose an orthonormal tangent basis $\{e_1, \ldots, e_n\}$ such that A_{n+1} and A_{n+2} are diagonal, that is, $A_\beta(e_i) = h^\beta_{ii}e_i$, $\beta = n+1, n+2$. Since M is assumed not to be of 1-type, then $\alpha \neq 0$ and we can locally choose an orthonormal normal basis $\{e_{n+1}, e_{n+2}\}$ on M such that $e_{n+1} = \frac{H}{\alpha}$ which is non-parallel in the normal bundle. Thus $\nabla^{\perp}_{e_i} e_{n+1} = \omega^{n+2}_{n+1}(e_i)e_{n+2} \neq 0$, that is, $\omega^{n+2}_{n+1}(e_i) \neq 0$ at least for one $i \in \{1, \ldots, n\}$. Also, $\nabla^{\perp}_{e_i} e_{n+2} = \omega^{n+1}_{n+2}(e_i)e_{n+1}$. Hence $< \nabla^{\perp}_{e_i} e_{n+1}, \nabla^{\perp}_{e_i} e_{n+2} >= 0$ and $< \nabla^{\perp}_{e_i} e_{n+1}, \nabla^{\perp}_{e_i} e_{n+1} >= \sum_{i=1}^{n}(\omega^{n+2}_{n+1}(e_i))^2$.

Considering α is a constant, the proof of (3.6), (3.7), and (3.8) follow immediately from (3.3), (3.4) and (3.5), respectively. □

Theorem 3.1. *Let M be an n-dimensional normally flat submanifold of the Euclidean space E^{n+2} with non-parallel mean curvature vector such that M is not of 1-type. If M is of null 2-type and flat with constant mean curvature, then* $\dim N_1(M) = 1$.

Proof. Considering the hypotheses of the theorem and Lemma 3.2, we can choose an orthonormal tangent and normal basis on M as in the proof of Lemma 3.2. We then have $n\alpha = h^{n+1}_{11} + \cdots + h^{n+1}_{nn}$, $h^{n+2}_{11} + \cdots + h^{n+2}_{nn} = 0$, $\omega^{n+2}_{n+1} \neq 0$ and the Weingarten maps A_{n+1} and A_{n+2} are diagonal.

Because of the flatness of M, the equations of Gauss (2.3) become

$$h^{n+1}_{ii}h^{n+1}_{jj} + h^{n+2}_{ii}h^{n+2}_{jj} = 0, \quad 1 \leq i < j \leq n \tag{3.9}$$

and the equations of Codazzi (2.4) become

$$e_k(h^{n+1}_{ii}) = -h^{n+2}_{ii}\omega^{n+1}_{n+2}(e_k), \tag{3.10}$$

$$e_k(h^{n+2}_{ii}) = -h^{n+1}_{ii}\omega^{n+2}_{n+1}(e_k), \tag{3.11}$$

where $i \neq k$.

Since the mean curvature direction is non-parallel, then $\nabla^{\perp}_{e_i} e_{n+1} \neq 0$, that is, $\omega^{n+2}_{n+1}(e_i) \neq 0$ at least for one $i \in \{1, \ldots, n\}$. Without lose of generality, suppose that $\omega_{n+1}^{n+2}(e_1) \neq 0$. As we have the hypotheses of Lemma 3.2, then the equation (3.6) for $i=1$ implies that $h_{11}^{n+2}=0$. Hence we get $h_{22}^{n+2}+\cdots+h_{nn}^{n+2}=0$.

We now show that $h_{11}^{n+1} \neq 0$ according to the chosen $\omega_{n+1}^{n+2}(e_1) \neq 0$. For $k=1$, when we take the sum of the equations of Codazzi (3.11) on i from 2 to n we obtain

$$e_1(h_{22}^{n+2}+\cdots+h_{nn}^{n+2}) = -(h_{22}^{n+1}+\cdots+h_{nn}^{n+1})\omega_{n+1}^{n+2}(e_1).$$

As $h_{22}^{n+2}+\cdots+h_{nn}^{n+2}=0$ and $\omega_{n+1}^{n+2}(e_1) \neq 0$ we get

$$h_{22}^{n+1}+\cdots+h_{nn}^{n+1}=0 \tag{3.12}$$

and thus $h_{11}^{n+1}=n\alpha$ which is different from zero as M is not of 1-type. Using $h_{11}^{n+2}=0$, from the equations of Gauss (3.9) for $i=1$ we get $h_{11}^{n+1}h_{jj}^{n+1}=0$ which gives $h_{jj}^{n+1}=0$ for $j=2,\ldots,n$. Hence $A_{n+1}=\operatorname{diag}(n\alpha,0,\ldots,0)$.

Since all h_{ii}^{n+1}'s are constant, then the equations of Codazzi (3.10) for $k=1$ and $i=2,\ldots,n$ give us $h_{ii}^{n+2}\omega_{n+2}^{n+1}(e_1)=0$, i.e., $h_{ii}^{n+2}=0$ for $i=2,\ldots,n$, as $\omega_{n+2}^{n+1}(e_1) \neq 0$. Therefore $A_{n+2}=0$.

As a result, we have $A_{n+1} \neq 0$ and $A_{n+2}=0$ which means that $\dim N_1(M)=1$.
□

Theorem 3.2. *Let M be an n-dimensional normally flat submanifold of the Euclidean space E^{n+2} with non-parallel mean curvature vector such that M is not of 1-type. Then, M is an open portion of an n-helical cylinder of E^{n+2} if and only if M is of null 2-type and flat with constant mean curvature.*

Proof. Let M be a n-dimensional helical cylinder in E^{n+2}. Then, by a suitable choice of the Euclidean coordinates M takes the following form

$$x(u_1,\ldots,u_n) = (a\cos u_1, a\sin u_1, bu_1, u_2,\ldots,u_n)$$

for some constant $a>0$ and $b\neq 0$. By a straight forward calculation it is seen that the Laplacian Δ of M is given by

$$\Delta = -\frac{1}{c^2}\frac{\partial^2}{\partial u_1^2} - \sum_{i=2}^{n}\frac{\partial^2}{\partial u_i^2}, \quad c=\sqrt{a^2+b^2}. \tag{3.13}$$

Let us put

$$x_1 = (0,0,bu_1,u_2,\ldots,u_n) \quad \text{and} \quad x_2 = (a\cos u_1, a\sin u_1, 0,\ldots,0).$$

Then it is easily seen that

$$\Delta x_1 = 0, \quad \Delta x_2 = \frac{1}{c^2}x_2. \tag{3.14}$$

This shows that M is of null 2-type.

Let us put

$$e_1 = \frac{1}{c}(-a\sin u_1, a\cos u_1, b, 0, 0), \quad e_i = (0,\ldots,0,1,0,\ldots,0),\ i=2,\ldots,n,$$

$$e_{n+1} = (-\cos u_1, -\sin u_1, 0,0,\ldots,0), \quad e_{n+2} = \frac{1}{c}(-b\sin u_1, b\cos u_1, -a, 0,\ldots,0),$$

where "1" in the vector e_i is in the $(i+2)$th place. Then, by a straight forward calculation we obtain

$$\begin{aligned}
&\omega^1 = c\,du_1,\ \omega^i = du_i,\ i = 2\dots,n,\\
&\omega^i_j = 0,\ i,j = 1,\dots,n,\\
&\omega^{n+1}_1 = \frac{a}{c^2}\omega^1,\ \omega^{n+2}_1 = 0,\ \omega^{n+1}_j = \omega^{n+2}_j = 0,\ j = 2,\dots,n,\\
&\omega^{n+2}_{n+1} = -\tfrac{b}{c^2}\omega^1.
\end{aligned} \tag{3.15}$$

All these show that M is flat, the mean curvature $\alpha = |H| = \frac{a}{3c^2}$ is constant and $e_{n+1} = \frac{H}{\alpha}$ is non-parallel. It is also observed that $A_{n+2} = 0$ and hence M is normally flat because of the equation of Ricci (2.5).

Conversely, let M be a flat and null 2-type submanifold of E^{n+2} with constant mean curvature and non-parallel mean curvature vector. By Theorem 3.1 we have $\dim(N_1(M)) = 1$. As it is shown in the proof of Theorem 3.1 we can have $A_{n+1} = \mathrm{diag}(n\alpha, 0,\dots,0)$, $A_{n+2} = 0$ and $\mu_0 = \omega^{n+2}_{n+1}(e_1) \neq 0$. Thus, by the equations of Codazzi (3.11) for $i = 1$, we obtain $\omega^{n+2}_{n+1}(e_k) = 0$, $k = 2,\dots,n$. Since we have the hypotheses of Lemma 3.2, then the equation(3.7) implies that $\mu_0 = \omega^{n+2}_{n+1}(e_1)$ is a constant. Therefore we have

$$\omega^{n+1}_1 = n\alpha\omega^1,\ \omega^{n+2}_1 = 0,\ \ \omega^{n+1}_j = \omega^{n+2}_j = 0,\ j = 2,\dots,n,\ \ \omega^{n+2}_{n+1} = \mu_0\omega^1. \tag{3.16}$$

By considering the flatness of M and (3.16), the connection forms ω^A_B of M coincide with the connection forms of the helical cylinder which are given in (3.15). Thus, as a result of the fundamental theorem of submanifolds, M is locally isometric to an n-dimensional helical cylinder of E^{n+2}. □

This work generalizes the results given in [3], [5] and [9] about helical cylinders and null 2-type submanifolds with non-parallel mean curvature vector.

Theorem 3.3. *Let M be an n-dimensional submanifold of the Euclidean space E^{n+2} such that M is not of 1-type. Then, M is an open portion of an n-helical cylinder of E^{n+2} if and only if M is of null 2-type and flat with* $\dim N_1(M) = 1$ *and non-parallel mean curvature vector.*

Proof. Suppose that M is a null 2-type and flat submanifold of E^{n+2} with $\dim(N_1(M)) = 1$ and non-parallel mean curvature vector. Then we have $\omega^i_j = 0$ and $A_{n+2} = 0$. By a direct computation and with the help of $A_{n+2} = 0$, the equation (3.3) becomes

$$A_{n+1}(\nabla\alpha) = -\frac{n}{2}\alpha\nabla\alpha, \tag{3.17}$$

that is, $\nabla\alpha$ is an eigenvector of A_{n+1} with the eigenvalue $-\frac{n}{2}\alpha$ on $U = \{p \in M : \nabla\alpha \neq 0 \text{ at } p\}$. If we choose e_1 parallel to $\nabla\alpha$, then $h^{n+1}_{11} = -\dfrac{n\alpha}{2}$, $h^{n+1}_{22} + \dots + h^{n+1}_{nn} = \dfrac{3n\alpha}{2}$, and $e_i(\alpha) = 0$, $i = 2,\dots,n$.

By the equations of Codazzi (2.4) for $j = 1$, $i = k$ and $\nu = n+1$, we have $e_1(h^{n+1}_{ii}) = 0$, $i = 2,\dots,n$. Hence we obtain $\frac{3n}{2}e_1(\alpha) = e_1(h^{n+1}_{22} + \dots + h^{n+1}_{nn}) = 0$. Therefore, $\nabla\alpha = 0$ on U, i.e., $U = \emptyset$, and α is a constant. Also, as $\dim N_1(M) = 1$ it follows from the equation of Ricci (2.5) that normal space is flat. As a result, we have all the assumptions of Theorem 3.2 from which M is an open portion of an n-helical cylinder. The converse is given in the proof of Theorem 3.2. □

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