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EMBEDDING FINITE PROJECTIVE GEOMETRIES INTO FINITE PROJECTIVE PLANES

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Abstract. We show that *every* finite projective geometry can be embedded in a projective plane of suitable order. Specifically: in $\mathrm{PG}(2,q^n)$, the set of points $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfying the equation $x^{q+1}+xy^q+\tau yz^q+sz^{q+1}=0$, with $\tau \neq 0$ and $-s$ not a $(q+1)^{th}$ power, contains q^n+1 points. If the points $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -s/\tau \\ 1 \end{pmatrix}$ are removed, the remaining subset is a disjoint union of $q-1$ equicardinal subsets, each of which is isomorphic to $\mathrm{PG}(n-1,q)$.

We will represent the points of a projective plane by column vectors, but in the interest of economy of space we will write $\begin{pmatrix} x & y & z \end{pmatrix}^T$ instead of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. However, when no coordinates are necessary - which happens in a few places in the proof of the Theorem - we will use lower case boldface letters to denote points.
The following symbols will be used:

Q: the subset of the finite field $\mathrm{GF}(q^n)$ comprising the nonvanishing $(q+1)^{th}$ powers;
ZQ: the subset of $\mathrm{GF}(q^n)$ comprising the $(q-1)^{th}$ powers ($0 \in ZQ$);
$\Xi(x) = x^{q^{n-1}} + x^{q^{n-2}} + \cdots + x^q + x$, over $\mathrm{GF}(q^n)$.

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We shall denote by A, the set of points $\begin{pmatrix} x & y & z \end{pmatrix}^T$ satisfying the equation

$$x^{q+1} + xy^q + \tau yz^q + sz^{q+1} = 0\,, \tag{1}$$

with $\tau \neq 0$ and $-s \notin Q \cup \{0\}$.
We will also say that a line in the projective plane $\mathrm{PG}(2, q^n)$ is a *short secant* or a *full secant* if it intersects the set A at two points or at more than two points, respectively.

Lemma 1. *A full secant has $q+1$ points in common with the set A.*

Proof. Let $\begin{pmatrix} a & b & c \end{pmatrix}^T$, $\begin{pmatrix} d & e & f \end{pmatrix}^T$, $\begin{pmatrix} a+\ell d & b+\ell e & c+\ell f \end{pmatrix}^T \in A$ for some $\ell \neq 0$. Then, by virtue of (1), we have

$$a^{q+1} + ab^q + \tau bc^q + sc^{q+1} = 0\,, \tag{2}$$

$$d^{q+1} + de^q + \tau ef^q + sf^{q+1} = 0\,, \tag{3}$$

$$(a+\ell d)^{q+1} + (a+\ell d)(b+\ell e)^q + \tau(b+\ell e)(c+\ell f)^q + s(c+\ell f)^{q+1} = 0\,. \tag{4}$$

Note that our assumption that $-s \notin Q$ implies $be \neq 0$ and also $\ell \neq -b/e$.
Upon expanding the left side of equation (4), one obtains an expression which reduces, because of equations (2), (3), to a binomial $\alpha\ell^q + \beta\ell$, where α, β, depend upon the values of a,b,c,d,e,f,τ,s. We cannot have $\alpha = \beta = 0$, because that would entail that ℓ can be any element of our field, including $-b/e$, which has been ruled out in the preceding paragraph.
If $\alpha = 0$ and $\beta \neq 0$ or if $\alpha \neq 0$ and $\beta = 0$, we get $\ell = 0$, i.e. the line $[\begin{pmatrix} a & b & c \end{pmatrix}^T, \begin{pmatrix} d & e & f \end{pmatrix}^T]$ is a short secant. The same thing takes place if $\alpha\beta \neq 0$ and $-\beta/\alpha \notin ZQ$.
If $-\beta/\alpha \in ZQ$, the equation $\alpha\ell^q + \beta\ell = 0$ yields $q-1$ nonvanishing solutions for ℓ, and the ratio of any two solutions is a member of the $\mathrm{GF}(q)$ subfield. In this case the line in question has $q+1$ points in common with A. □

The restriction $-s \notin Q$ precludes the possibility $y = 0$ in equation (1). Then $z = 0 \Rightarrow$ either $x = 0$ or $x = -y$, whereas $x = 0 \Rightarrow$ either $z = 0$ or $y = -sz/\tau$. Therefore there are exactly three points $\begin{pmatrix} x & y & z \end{pmatrix}^T \in A$ with $xyz = 0$.

Lemma 2. *The lines joining the points $\begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^T$, $\begin{pmatrix} 0 & -s/\tau & 1 \end{pmatrix}^T$, to any other point in A are short secants.*

Proof. The lines $[\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T, \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^T]$ and $[\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & -s/\tau & 1 \end{pmatrix}^T]$ are short secants, clearly.
The line $[\begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & -s/\tau & 1 \end{pmatrix}^T]$ has equation $\tau x + \tau y + sz = 0$. If a point $\begin{pmatrix} a & b & 1 \end{pmatrix}^T$ lies on this line, we have $s = -\tau(a+b)$. If $\begin{pmatrix} a & b & 1 \end{pmatrix}^T \in A$, we also have $a^{q+1} + ab^q + \tau b + s = 0$. Substitute here the expression for s that we have just obtained to arrive at $\tau = (a+b)^q$. As a consequence, we obtain $s = -(a+b)^q(a+b) = -(a+b)^{q+1}$, in conflict with our assumption that $-s \notin Q$. We have thus established that the line $[\begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & -s/\tau & 1 \end{pmatrix}^T]$ is a short secant.

Next consider the lines $[\begin{pmatrix}-1 & 1 & 0\end{pmatrix}^T, \begin{pmatrix}a & b & 1\end{pmatrix}^T]$, $\begin{pmatrix}a & b & 1\end{pmatrix}^T \in A$. If $\ell \neq 0$ and the point $\begin{pmatrix}a-\ell & b+\ell & 1\end{pmatrix}^T \in A$, then $(a-\ell)^{q+1}+(a-\ell)(b+\ell)^q+\tau(b+\ell)+s=0$. Upon expanding the left side of this equation and using the fact that $\begin{pmatrix}a & b & 1\end{pmatrix}^T \in A$, i.e. that

$$a^{q+1}+ab^q+\tau b+s=0, \tag{5}$$

we are left with $\tau = (a+b)^q$. Substitute this expression for τ into (5) to arrive again at the contradiction $s=-(a+b)^{q+1}$.
Finally, we look at the lines $[\begin{pmatrix}0 & -s/\tau & 1\end{pmatrix}^T, \begin{pmatrix}a & b & 1\end{pmatrix}^T]$, $\begin{pmatrix}a & b & 1\end{pmatrix}^T \in A$. If the point $\begin{pmatrix}a & b-\ell s/\tau & 1+\ell\end{pmatrix}^T \in A$, we get $a^{q+1}+a(b-\ell s/\tau)^q+\tau(b-\ell s/\tau)(1+\ell^q)+s(1+\ell)^{q+1}=0$.
By virtue of (5) again, this reduces to $\tau b+s=as^q/\tau^q$. Upon substituting this expression for $\tau b+s$ into (5), we obtain $\tau a+\tau b+s=0$, whence $\tau a+as^q/\tau^q=0$, i.e. $-s^q \in Q$, which is equivalent to $-s \in Q$, the same contradiction again. □

Lemma 3. *The line joining two points* $\begin{pmatrix}a & b & 1\end{pmatrix}^T, \begin{pmatrix}c & d & 1\end{pmatrix}^T \in A$, *with* $a \neq c$, *contains* $q-1$ *more points of* A *if* $c/a \in ZQ$, *and does not contain any other point of* A *otherwise.*

Proof. By assumption, equation (5) holds, and also

$$c^{q+1}+cd^q+\tau d+s=0. \tag{6}$$

Assume that the point $\begin{pmatrix}a+\ell c & b+\ell d & 1+\ell\end{pmatrix}^T$, $\ell \neq 0$, is also in A:

$$(a+\ell c)^{q+1}+(a+\ell c)(b+\ell d)^q+\tau(b+\ell d)(1+\ell^q)+s(1+\ell)^{q+1}=0. \tag{7}$$

By virtue of (5), (6), this equation reduces to $(ac^q+ad^q+\tau b+s)\ell^q+(a^qc+b^qc+\tau d+s)\ell=0$.
Now substitute into this equation the expressions for $\tau b+s$ and $\tau d+s$ obtained from (5), (6), to arrive at the equation

$$a(c+d-a-b)^q\ell^q = c(c+d-a-b)^q\ell. \tag{8}$$

We will now show that $c+d-a-b \neq 0$. Assume, contrariwise, that $a+b=c+d$, and subtract equation (6) from (5): $a(a+b)^q-c(c+d)^q+\tau(b-d)=0$.
If $a+b=c+d$, this equation becomes $(a-c)(a+b)^q+\tau(b-d)=0$, whence $(a-c)(a+b)^q=\tau(d-b)=\tau(a-c)$. As $a \neq c$ by assumption, we end up with $\tau=(a+b)^q$. Equation (5), rewritten as $a(a+b)^q+\tau b+s=0$, now becomes $a(a+b)^q+b(a+b)^q+s=0$, whence $s=-(a+b)^{q+1}$, contradicting our assumption that $-s \notin Q$.
Having established that $a+b \neq c+d$, equation (8) reduces to

$$\ell^{q-1}=c/a. \tag{9}$$

This shows that $c/a \in ZQ$ and necessity has been demonstrated.
The proof for sufficiency is a fairly simple matter and we omit it. □

Theorem 1. *The subset of* A *comprising the points* $\begin{pmatrix} x & y & z \end{pmatrix}^T$ *with* $xyz \neq 0$ *and the point* $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$ *can be partitioned into* $q-1$ *subsets, each of which is a projective geometry* $PG(n-1,q)$, *with collinearity inherited from the projective plane* $PG(2,q^n)$.
Every line of the plane which joins two points of A *not in the same* $PG(n-1,q)$ *subset, is a short secant.*

Proof. If $xyz \neq 0$, we let $z=1$ and write equation (1) as

$$y^q = -\frac{\tau}{x}y - \frac{x^{q+1}+s}{x}. \tag{10}$$

Denote $-\tau/x = \lambda$, $-(x^{q+1}+s)/x = \theta$. Then equation (10) becomes

$$y^q = \lambda y + \theta. \tag{11}$$

It was shown in [1, Theorem 19] that equation (11) possesses a unique solution for y if $\lambda \notin ZQ$, whereas for $\lambda \in ZQ$ it has q solutions if $\Xi(\theta/\omega^q)=0$ and no solution otherwise, where ω is any one of the $q-1$ elements of $\mathrm{GF}(q^n)$ satisfying $\lambda = \omega^{q-1}$.
We treat first the case $\lambda = -\tau/x \notin ZQ$.
Our field comprises $(q^n-1)(q-2)/(q-1)$ elements that are not members of ZQ. Hence, as τ is fixed, there are this many elements x for which $\lambda \notin ZQ$. Each of these x's leads to a unique value of y from equation (10), thereby producing $(q^n-1)(q-2)/(q-1)$ points $\begin{pmatrix} x & y & 1 \end{pmatrix}^T$ satisfying equation (1). Moreover, these points fall into $q-2$ mutually disjoint subsets A_1, A_2, $\ldots$, A_{q-2}, where $|A_i| = (q^n-1)/(q-1)$ for all i and such that two points $\begin{pmatrix} a & b & 1 \end{pmatrix}^T$, $\begin{pmatrix} c & d & 1 \end{pmatrix}^T \in A$ are in the same subset A_i if and only if $c/a \in ZQ$. It follows then from Lemma 3 that all the secants within each A_i are full - and they contain $q+1$ points of A_i, in virtue of Lemma 1 - while the lines joining two points from different A_i's do not intersect the set $\cup_{i=1}^{q-2} A_i$ again.
To prove that the A_i's are projective geometries, consider three noncollinear points $\begin{pmatrix} a & b & 1 \end{pmatrix}^T$, $\begin{pmatrix} c & d & 1 \end{pmatrix}^T$, $\begin{pmatrix} e & f & 1 \end{pmatrix}^T \in A_i$ for a fixed i. We will show that a line joining a point $\begin{pmatrix} a+\ell c & b+\ell d & 1+\ell \end{pmatrix}^T \in [\begin{pmatrix} a & b & 1 \end{pmatrix}^T, \begin{pmatrix} c & d & 1 \end{pmatrix}^T]$, $\ell \neq 0$, to a point $\begin{pmatrix} a+me & b+mf & 1+m \end{pmatrix}^T \in [\begin{pmatrix} a & b & 1 \end{pmatrix}^T, \begin{pmatrix} e & f & 1 \end{pmatrix}^T]$, $m \neq 0$, intersects the line through the points $\begin{pmatrix} c & d & 1 \end{pmatrix}^T$, $\begin{pmatrix} e & f & 1 \end{pmatrix}^T$ within the same A_i subset. By assumption, equations (5), (6), (7), (9), hold, and also

$$e^{q+1} + ef^q + \tau f + s = 0, \tag{12}$$

$$(a+me)^{q+1} + (a+me)(b+mf)^q + \tau(b+mf)(1+m)^q + s(1+m)^{q+1} = 0. \tag{13}$$

Equations (5) and (12) reduce equation (13) to

$$m^{q-1} = e/a. \tag{14}$$

It is easy to see that the line joining the points $\begin{pmatrix} a+\ell c & b+\ell d & 1+\ell \end{pmatrix}^T$ and $\begin{pmatrix} a+me & b+mf & 1+m \end{pmatrix}^T$ and the line $[\begin{pmatrix} c & d & 1 \end{pmatrix}^T, \begin{pmatrix} e & f & 1 \end{pmatrix}^T]$ intersect at the point $\begin{pmatrix}\ell c - me & \ell d - mf & \ell - m\end{pmatrix}^T$. We have to show that this point is in A_i, i.e. that it satisfies equation (1):

$$(\ell c - me)^{q+1} + (\ell c - me)(\ell d - mf)^q + \tau(\ell d - mf)(\ell - m)^q + s(\ell - m)^{q+1} = 0.$$

Expand the left side of this equation, then use equations (6), (12), to reduce it to

$$\ell^q m(c^q e + d^q e + \tau f + s) + \ell m^q(ce^q + cf^q + \tau d + s) = 0.$$

Substitute here $\tau f + s = -e^{q+1} - ef^q$ and $\tau d + s = -c^{q+1} - cd^q$, and divide by ℓm:

$$\ell^{q-1} e(c + d - e - f)^q + m^{q-1} c(e + f - c - d)^q = 0.$$

Use now the expressions for ℓ^{q-1} and m^{q-1} as given by (9) and (14) to arrive at the following obvious identity:

$$\frac{c}{a} e(c + d - e - f)^q + \frac{e}{a} c(e + f - c - d)^q = 0.$$

We have thus established that the A_i's are projective geometries indeed.

We pass now to the case in which $\lambda = -\tau/x \in ZQ$.

For a fixed τ, there are $(q^n - 1)/(q - 1)$ values of x for which $-\tau/x \in ZQ$. For these elements x, equation (10) may or may not possess solutions for y, as mentioned earlier. If it does have solutions, it has q of them, leading to q points with the same x-coordinate. These points, together with $\begin{pmatrix}0 & 1 & 0\end{pmatrix}^T$, make up a full secant.

As explained above, it follows from [1, Theorem 19] that in order to decide whether equation (10) has solutions, one needs to consider the expression $\Xi(\theta/\omega^q)$, where $\omega^{q-1} = -\tau/x$, $\theta = -(x^{q+1} + s)/x$.

We have

$$\frac{\theta}{\omega^q} = -\frac{x^{q+1} + s}{x\omega^q} = -\left(\frac{x}{\omega}\right)^q + \frac{s}{\tau\omega} = \left(\frac{\tau}{\omega^q}\right)^q + \frac{s}{\tau\omega} = \left(\frac{\tau^{q+1}}{\tau^q\omega^q}\right)^q + \frac{s}{\tau\omega}.$$

Hence, as Ξ is an additive function and also $\Xi(x^q) = \Xi(x)$ (easy consequences of the definition of Ξ), we have the following implications:

$$\Xi\left(\frac{\theta}{\omega^q}\right) = 0 \Rightarrow \Xi\left(\frac{s}{\tau\omega}\right) = -\Xi\left(\frac{\tau^{q+1}}{\tau^q\omega^q}\right)^q = -\Xi\left(\frac{\tau^{q+1}}{\tau^q\omega^q}\right) = -\Xi\left(\frac{\tau^{1+1/q}}{\tau\omega}\right) \Rightarrow$$
$$\Xi\left(\frac{s + \tau^{1+1/q}}{\tau\omega}\right) = 0.$$

It has been shown in [1, Theorem 15] that for any $a \in$GF(q), the equation $\Xi(x) = a$ possesses q^{n-1} distinct roots. If $a = 0$, one root is clearly 0, but we cannot accept it, because $s + \tau^{1+1/q} = 0 \Rightarrow -s \in Q$. Therefore we have $q^{n-1} - 1$ acceptable solutions for the expression $\left(s + \tau^{1+1/q}\right)/\tau\omega$. As s and τ are fixed, this yields $q^{n-1} - 1$ values for ω.

If ω is one solution, then so is $a\omega$ for all $a \in$GF(q), because the definition of Ξ implies $\Xi(ac) = a \cdot \Xi(c)$ for every $a \in$GF(q) and any c. Since $a \in$GF$(q) \Rightarrow (a\omega)^{q-1} = \omega^{q-1}$, it follows that all the $q - 1$ elements $a\omega$, as a ranges through GF$(q) \setminus \{0\}$, yield the same value for $x = -\tau/\omega^{q-1}$. We have thus arrived at $(q^{n-1} - 1)/(q - 1)$ distinct values of x for which equation (10) gives q values of y, for a total of $(q^n - q)/(q - 1)$ points. These, together with $\begin{pmatrix}0 & 1 & 0\end{pmatrix}^T$, make up a set B comprising $(q^n - 1)/(q - 1)$ points.

As each x gives rise to q values of y, we see that in the set B, all the secants

through $\begin{pmatrix}0 & 1 & 0\end{pmatrix}^T$ are full.
In order to establish that B is a geometry, we have to demonstrate that if a line meets two sides of a triangle whose vertices are in B, it intersects the third side at a point within B as well. If none of the three vertices is $\begin{pmatrix}0 & 1 & 0\end{pmatrix}^T$, the proof is the same as for the A_i's. But in the case in which one vertex is $\begin{pmatrix}0 & 1 & 0\end{pmatrix}^T$, it is necessary to show that if $\begin{pmatrix}c & d & 1\end{pmatrix}^T, \begin{pmatrix}e & f & 1\end{pmatrix}^T \in B$, then the line joining a point $\begin{pmatrix}c & d+\ell & 1\end{pmatrix}^T \in [\begin{pmatrix}0 & 1 & 0\end{pmatrix}^T, \begin{pmatrix}c & d & 1\end{pmatrix}^T]$, $\ell \neq 0$, to a point $\begin{pmatrix}e & f+m & 1\end{pmatrix}^T \in [\begin{pmatrix}0 & 1 & 0\end{pmatrix}^T, \begin{pmatrix}e & f & 1\end{pmatrix}^T]$, $m \neq 0$, intersects the line $[\begin{pmatrix}c & d & 1\end{pmatrix}^T, \begin{pmatrix}e & f & 1\end{pmatrix}^T]$ within B.
The lines $[\begin{pmatrix}c & d+\ell & 1\end{pmatrix}^T, \begin{pmatrix}e & f+m & 1\end{pmatrix}^T]$ and $[\begin{pmatrix}c & d & 1\end{pmatrix}^T, \begin{pmatrix}e & f & 1\end{pmatrix}^T]$ meet at the point $\begin{pmatrix}e\ell - cm & f\ell - dm & \ell - m\end{pmatrix}^T$, and we have to demonstrate that

$$(e\ell-cm)^{q+1}+(e\ell-cm)(f\ell-dm)^q+\tau(f\ell-dm)(\ell-m)^q+s(\ell-m)^{q+1}=0. \quad (15)$$

As $\begin{pmatrix}c & d+\ell & 1\end{pmatrix}^T$, $\begin{pmatrix}e & f+m & 1\end{pmatrix}^T \in B \subset A$, we have

$$c^{q+1}+c(d+\ell)^q+\tau(d+\ell)+s=0 \qquad \text{and} \qquad e^{q+1}+e(f+m)^q+\tau(f+m)+s=0.$$

Since $\begin{pmatrix}c & d & 1\end{pmatrix}^T, \begin{pmatrix}e & f & 1\end{pmatrix}^T \in B$ as well, these equations yield $\ell^{q-1} = -\tau/c$ and $m^{q-1} = -\tau/e$.
Upon multiplying out the left side of equation (15) and using the fact that the points $\begin{pmatrix}c & d & 1\end{pmatrix}^T$ and $\begin{pmatrix}e & f & 1\end{pmatrix}^T$ are in A, it reduces to $\ell^q m(ce^q + cf^q + \tau d + s) + \ell m^q(c^q e + d^q e + \tau f + s) = 0$.
Since $\tau d + s = -c^{q+1} - cd^q$ and $\tau f + s = -e^{q+1} - ef^q$, we arrive, after dividing by ℓm, at $\ell^{q-1}c(e+f-c-d)^q + m^{q-1}e(c+d-e-f)^q = 0$. But $\ell^{q-1} = -\tau/c$ and $m^{q-1} = -\tau/e$, so that the last equation is an obvious identity.
We pass now to the last paragraph of the theorem: note that for $q = 2$ it is vacuous, because in this case there is only one $\mathrm{PG}(n-1, 2)$, namely B.
It has been shown earlier in the proof that lines joining two points from different A_i's are short secants. It has also been shown (Lemma 2) that the lines joining the points $\begin{pmatrix}-1 & 1 & 0\end{pmatrix}^T, \begin{pmatrix}0 & -s/\tau & 1\end{pmatrix}^T$ to any other point in A are short secants.
What is left to do is demonstrate that for any $i \in \{1, 2, \ldots, q-2\}$, a line $[\mathbf{a}, \mathbf{b}]$ with $\mathbf{a} \in A_i$ and $\mathbf{b} \in B$, has no other point in common with A. Assume, to the contrary, that there is another point $\mathbf{c} \in A_i$ so that coll($\mathbf{a}, \mathbf{b}, \mathbf{c}$. Then since A_i must comprise $q-1$ more points collinear with $\mathbf{a}$ and $\mathbf{c}$, we obtain a full secant with more than $q+1$ points. This violates Lemma 1.
The same contradiction is arrived at if coll($\mathbf{a}, \mathbf{b}, \mathbf{c}$) with $\mathbf{c} \in B$.
If, on the other hand, $\mathbf{c} \in A_j, j \neq i$, and coll($\mathbf{a}, \mathbf{b}, \mathbf{c}$), then the line $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ would not have any other point within A: we already know that the line $[\mathbf{a}, \mathbf{c}]$ has no other point in common with $\cup_{i=1}^{q-2} A_i$, and it cannot in tersect the set B at a point other than $\mathbf{b}$, either, as that would lead again to a full secant with more than $q+1$ points. Therefore the line $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ would be a three-point full secant. But $3 < q+1$ for $q > 2$, hence the conclusion of Lemma 1 would be violated again. □

Corollary 1. $|A| = q^n + 1$.

Proof. The set A consists of two points $\begin{pmatrix}-1 & 1 & 0\end{pmatrix}^T, \begin{pmatrix}0 & -s/\tau & 1\end{pmatrix}^T$, plus the mutually disjoint subsets $A_1, A_2, \ldots, A_{q-2}, B$, each of which has cardinality $(q^n -$

$1)/(q-1)$.
The conclusion follows readily. $\square$

Example 1. Let w be a primitive root of the finite field $\mathrm{GF}(2^4)$, where $w^4 = w + 1$ over $\mathrm{GF}(2)$. In the projective plane $\mathrm{PG}(2, 2^4)$, consider the 17 points satisfying the equation $x^3 + xy^2 + yz^2 + wz^3 = 0$:

		1	2	3	4	5	6	7
$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\w\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}w\\w\\1\end{pmatrix}$	$\begin{pmatrix}w\\w^7\\1\end{pmatrix}$	$\begin{pmatrix}w^4\\w^{10}\\1\end{pmatrix}$	$\begin{pmatrix}w^4\\w^{14}\\1\end{pmatrix}$	$\begin{pmatrix}w^6\\w^4\\1\end{pmatrix}$	$\begin{pmatrix}w^6\\w^{14}\\1\end{pmatrix}$

8	9	10	11	12	13	14	15
$\begin{pmatrix}w^{11}\\w^5\\1\end{pmatrix}$	$\begin{pmatrix}w^{11}\\w^8\\1\end{pmatrix}$	$\begin{pmatrix}w^{12}\\w^{14}\\1\end{pmatrix}$	$\begin{pmatrix}w^{12}\\1\\1\end{pmatrix}$	$\begin{pmatrix}w^{13}\\w^9\\1\end{pmatrix}$	$\begin{pmatrix}w^{13}\\w^{11}\\1\end{pmatrix}$	$\begin{pmatrix}1\\w^6\\1\end{pmatrix}$	$\begin{pmatrix}1\\w^{13}\\1\end{pmatrix}$

The 15 numbered points make up a projective geometry $\mathrm{PG}(3, 2)$. Its 35 lines are:
(1 2 3), (1 4 5), (1 6 7), (1 8 9), (1 10 11), (1 12 13), (1 14 15), (2 4 15), (2 5 14), (2 6 9), (2 7 8), (2 10 13), (2 11 12), (3 4 14), (3 5 15), (3 6 8), (3 7 9), (3 10 12), (3 11 13), (4 6 10), (4 7 11), (4 8 12), (4 9 13), (5 6 11), (5 7 10), (5 8 13), (5 9 12), (6 12 14), (6 13 15), (7 12 15), (7 13 14), (8 10 14), (8 11 15), (9 10 15), (9 11 14).

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