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PROJECTIVE EQUIVALENCE OF QUADRICS IN KLINGENBERG PROJECTIVE SPACES OVER A SPECIAL LOCAL RING

MAREK JUKL AND VÁCLAV SNÁŠEL

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Abstract. This article is devoted to the projective equivalence of quadrics in Klingenberg projective spaces over a special type of local ring. Signature of quadrics is introduced by a suitable way (as a generalization of this one for quadrics in projective spaces over fields) and projective equivalence criterion is found.

1. Introduction

W. Klingenberg [2] has introduced certain incidence structure which may be considered as a generalization of projective spaces (over fields). This structure was later denominated by his name. F. Machala [1] studied Klingenberg projective spaces over a local ring.

One special type of local ring is used in this article, namely *plural algebra* of finite order which may be taken as a natural generalization of the s.c. dual numbers. It is used for example in mathematical statistics or dynamics of solid body.

The free finite dimensional modules over special local rings were studied by author and the results may be used for description of Klingenberg projective spaces over plural algebras.

Definition 1.1. A *plural algebra over a field T of order m* is any linear algebra $\mathbf{A}$ on T having as a vector space over T a basis

$$\{1, \eta, \eta^2, \dots, \eta^{m-1}\}, \text{ with } \eta^m = 0.$$

In the case $T = \mathbb{R}$ we use the notion *real plural algebra* or *plural numbers*.

Throughout this paper we will denote by $\mathbf{A}$ denote the real plural algebra of order m.

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Remark 1.1. Any element β of $\mathbf{A}$ has the unique expression in the form

$$\beta = \sum_{i=0}^{m-1} b_i \eta^i .$$

We get a system of homomorphisms $p_0, \dots, p_{m-1}$ of $\mathbf{A}$ onto $\mathbb{R}$ which are naturally given for all $k = 0, \dots, m-1$ by

$$p_k(\beta) = b_k .$$

Remark 1.2. Every plural algebra $\mathbf{A}$ is a local ring with the maximal ideal $\eta\mathbf{A}$. The ideals $\eta^j\mathbf{A}$, $1 \le j \le m$, are just all ideals of $\mathbf{A}$.

Remark 1.3. Let $\mathbf{M}$ be a free finite dimensional module over $\mathbf{A}$. It is well known that since $\mathbf{A}$ is a local ring all bases of $\mathbf{M}$ have the same number of elements and from every system of generators of $\mathbf{M}$ we may select a basis of $\mathbf{M}$.

Moreover in our case the module $\mathbf{M}$ has the following qualities (proved by the author in [3]):

(1) Any linearly independent system can be completed to a basis of $\mathbf{M}$.
(2) A submodule of $\mathbf{M}$ is a free module if and only if it is a direct summand of $\mathbf{M}$.

Free finite dimensional modules over a local ring $\mathbf{R}$ are called $\mathbf{R}$-*spaces* (see e.g. [5]) and their direct summands $\mathbf{R}$-*subspaces*.

Now, we may formulated the remark 2. by the following way:

(3) $\mathbf{A}$-subspaces of $\mathbf{A}$-space $\mathbf{M}$ are just all free submodules of $\mathbf{M}$.

For $\mathbf{A}$-subspaces of $\mathbf{M}$ it holds (proved in [3]):

(4) Let K, L be $\mathbf{A}$-subspaces of the $\mathbf{A}$-space $\mathbf{M}$. Then $K + L$ is an $\mathbf{A}$-subspace if and only if $K \cap L$ is an $\mathbf{A}$-subspace. In this case the dimensions of $\mathbf{A}$-subspaces fulfil the following relation:

$$\dim (K + L) + \dim (K \cap L) = \dim K + \dim L.$$

It follows from this remarks that the following lemma holds.

Lemma 1.1. *Let* $\mathbf{M}$ *be an* $\mathbf{A}$*-space and let* $\bar{\mathbf{M}}$ *be a vector space* $\mathbf{M}/\eta\mathbf{M}$*. Then elements* $\mathbf{u}_1, \dots, \mathbf{u}_k$ *form a linearly independent system in* $\mathbf{M}$ *if and only if cosets* $\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_k$ *form a linearly independent system in* $\bar{\mathbf{M}}$*.*

According to [1] we define:

Definition 1.2. Let $\mathbf{R}$ be a local ring with the maximal ideal r. Let us denote $\mathbf{M} = \mathbf{R}^{n+1}$, $\bar{\mathbf{M}} = \mathbf{M}/r\mathbf{M}$, $\bar{\mathbf{R}} = \mathbf{R}/r$, and let μ be a natural homomorphism $\mathbf{M} \to \bar{\mathbf{M}}$.

Then the incidence structure P_R such that

(1) the points are just all submodules $[\mathbf{x}]$ of $\mathbf{M}$ such that $\mu(\mathbf{x})$ is a non-zero element of $\bar{\mathbf{M}}$,
(2) the lines are just all submodules $[\mathbf{x},\mathbf{y}]$ of $\mathbf{M}$ such that $[\mu(\mathbf{x}), \mu(\mathbf{y})]$ is a 2-dimensional subspace of $\bar{\mathbf{M}}$,
(3) the incidence relation is the inclusion,

is called *n-dimensional coordinate projective Klingenberg space over the ring* $\mathbf{R}$.

If $X = [\mathbf{x}]$ is a point of P_R, then $\mathbf{x}$ will be called an *arithmetic representative of* X.

We obtain from Definition 1.2, Lemma 1.1 and Remark 1.3 the following description of Klingenberg projective spaces in our case. We may see that for the case m=0 (where m is the order of plural algebra $\mathbf{A}$) Klingenberg projective space becomes "usual" projective space (over a field).

Corollary 1.1.

(1) *The points of the Klingenberg space P_A are just all 1-dimensional $\mathbf{A}$-subspaces of $\mathbf{M}$. The k-dimensional subspaces of the Klingenberg space P_A are just all $(k+1)$-dimensional $\mathbf{A}$-subspaces of $\mathbf{M}$, $1 \le k \le n-1$,*

(2) *$X = [\mathbf{x}]$ is a point of P_A if and only if $\mathbf{x} \in \mathbf{M} - \eta\mathbf{M}$.*

2. Projective equivalence of quadrics in Klingenberg projective spaces

Definition 2.1. Let an automorphism f of the $\mathbf{A}$-space $\mathbf{M}$ be given[1]. Then the mapping $C_f\colon P_A \to P_A$ defined by

$$\forall\, X = [\mathbf{x}] \in P_A\colon C_f(X) = [f(\mathbf{x})]$$

is called a *collineation in Klingenberg projective space P_A (induced by the automorphism f).*

Remark 2.1. Without troubles we may prove that two automorphisms f, g of the $\mathbf{A}$-space $\mathbf{M}$ induce the same collineation in P_A (i.e. $C_f = C_g$) iff there exists a unit $\alpha \in \mathbf{A} - \eta\mathbf{A}$ with $g = \alpha f$.

Definition 2.2. Let a quadratic form φ_2 on $\mathbf{M}$ be given. Then the set Q_{φ_2} defined by

$$Q_{\varphi_2} = \{X = [\mathbf{x}] \in P_A;\ \mathbf{x} \notin \eta\mathbf{M} \wedge \varphi_2(\mathbf{x}) = 0\}$$

is called a *quadric in P_A (determined by the quadratic form φ_2).*

Now let us introduce the notion of *equivalence of quadrics* by a very natural way:

Definition 2.3. Quadrics Q and Q' on P_A will be called *projectively equivalent* if there exists a collineation C in P_A such that

$$Q' = C(Q).$$

Evidently, this relation is equivalence on the set of quadrics on P_A.

Definition 2.4. Let a quadric Q_{φ_2} on P_A be given. An arithmetic basis $\mathcal{A} = \{\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_n\}$ will be called a *normal polar basis of P_A with respect to the Q_{φ_2}* if

(1) $\mathcal{A}$ is a polar basis of $\mathbf{M}$ with respect to the quadratic form φ_2,

(2) for every i, $1 \le i \le n$, there exists k, $0 \le k \le m$, such that

$$\varphi_2(\mathbf{u}_i) = \pm\eta^k.$$

It is proved in [4] that for every quadratic form on a free module over $\mathbf{A}$ there exists a normal polar basis. Therefore we get clearly:

Proposition 2.1. *Let a quadric Q on P_A be given. Then there exists at least one normal polar basis of P_A with respect to the Q.*

The following notion is a natural generalization of the "usual" signature of a quadric in the projective space over real numbers.

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[1]the notion *automorphism* is considered in the obvious sense

Definition 2.5. Let a quadric Q_{φ_2} on P_A and a normal polar basis $\mathcal{A}$ of P_A with respect to Q_{φ_2} be given. Then the set

$$\mathfrak{S}(Q_{\varphi_2}) = \{(p_0, p_1, \ldots, p_{m-1}), (q_0, q_1, \ldots, q_{m-1})\}$$

where the numbers $p_0, p_1 \ldots, p_{m-1}, q_0, q_1, \ldots, q_{m-1}$ are defined for all k, $0 \le k \le m-1$ as follows

$$p_k = \mathrm{card}\{\mathbf{u} \in \mathcal{A};\ \varphi_2(\mathbf{u}) = \eta^k\}, \quad n_k = \mathrm{card}\{\mathbf{u} \in \mathcal{A};\ \varphi_2(\mathbf{u}) = -\eta^k\},$$

will be called *plural signature of the quadric* Q_{φ_2}.

The aim of this article is to find a projective equivalence criterion for quadrics.

Let Q_{φ_2} be a quadric and C_f a collineation in P_A. Evidently, the image $C(Q_{\varphi_2})$ is a quadric determined by the quadratic form which is equal to the composition $f^{-1}\varphi_2$.

First, let a quadratic form φ_2 on $\mathbf{M}$ be given. If we introduce, for a certain normal polar basis of the $\mathbf{A}$-space $\mathbf{M}$ with respect to the φ_2, the system of numbers $p_0, \ldots, p_{m-1},\ q_0, \ldots, q_{m-1}$ by the same way as in definition 2.5 we may derive (see [4]) that the ordered $2m$-tuple

$$\mathfrak{S}(\varphi_2) = (p_0, p_1, \ldots, p_{m-1}, q_0, q_1, \ldots, q_{m-1})$$

is independent of the choose of the normal polar basis.

Further, let us consider two quadratic forms φ_2 and ψ_2 on $\mathbf{M}$. Now, using the fact above we may prove that there exists an automorphism f on $\mathbf{M}$ with $f\psi_2 = \varphi_2$ iff $\mathfrak{S}(\varphi_2) = \mathfrak{S}(\psi_2)$. (If $\mathcal{A}_\varphi$, resp. $\mathcal{A}_\psi$, are normal polar bases[2] with respect to φ_2, resp. ψ_2, then the automorphism maps $\mathcal{A}_\varphi$ onto $\mathcal{A}_{\psi_2}$).

Second, let Q be a quadric on P_A and let quadratic form φ_2 as well as ψ_2 determines this quadric. It may be shown that it is possible iff there exists a unit $\alpha \in \mathbf{A} - \eta\mathbf{A}$ with $\varphi_2 = \alpha\psi_2$ or an automorphism f on $\mathbf{M}$ such that $f\psi_2 = \varphi_2$.

Let $\varphi_2 = \alpha\psi_2$. It is clear that in the case $p_0(\alpha) > 0$ we have $\mathfrak{S}(\varphi_2) = \mathfrak{S}(\psi_2)$. Denoting $\mathfrak{S}(\varphi_2) = (p_0, \ldots, p_{m-1}, q_0, \ldots, q_{m-1})$ we obtain for $p_0(\alpha) < 0$ that $\mathfrak{S}(\psi_2) = (q_0, \ldots, q_{m-1}, p_0, \ldots, p_{m-1})$. With respect to this fact the signature of quadric was introduced by the way above (definition 2.5) correctly.

Thirdly, let two quadrics Q, Q' on P_A be given.

Let this quadrics have the same plural signature. It follows from this and from the previous paragraph that there exist quadratic forms φ_2, ψ_2 such that

$$Q = Q_{\varphi_2}, Q' = Q_{\psi_2} \text{ and } \mathfrak{S}(\varphi_2) = \mathfrak{S}(\psi_2).$$

As mentioned above we have guaranteed the existence of an automorphism f on $\mathbf{M}$ with $f\psi_2 = \varphi_2$, which gives $C_f(Q) = Q'$. We obtain the projective equivalence of these given quadrics.

Now, let us suppose that Q, Q' are projectively equivalent quadrics. Thus there exists a collineation C_f in P_A with $Q' = C_f(Q)$. If φ_2 is an arbitrary quadratic form determining Q (i.e. $Q = Q_{\varphi_2}$) then $\psi_2 = f^{-1}\varphi_2$ is a quadratic form which determines quadric $C_f(Q)$ (i.e. $Q' = Q_{\psi_2}$). It yields that $\mathfrak{S}(Q_{\varphi_2}) = \mathfrak{S}(Q_{\psi_2})$, immediately.

We have proved the following theorem, which brings projective equivalence criterion.

[2]this notion is for quadratic forms introduced analogously as for the case of quadrics

Theorem 2.1. *Let quadrics Q, Q' on P_A be given. Then these quadrics are projectively equivalent if and only if*

$$\mathfrak{S}(Q_{\varphi_2}) = \mathfrak{S}(Q_{\psi_2}).$$

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Department of Algebra and Geometry, Faculty of Science, Palacký University Olomouc, Tr. 17 listopadu 12, 779 00 Olomouc, Czech Republic
E-mail address: `marek.jukl@upol.cz`

Department of Computer Science, Faculty of Electrical Engineering and Computer Science, Technical University of Ostrava, 17. listopadu 15, 08 33 Ostrava - Poruba, Czech Republic
E-mail address: `vaclav.snasel@vsb.cz`