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AN IMPROVED CHEN-RICCI INEQUALITY

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Abstract. Oprea proves that $Ric(X) \leq \frac{n-1}{4}(c + n||H||^2)$ improving the Chen-Ricci inequality for Lagrangian submanifolds in complex space forms by using an optimization technique. In this article, we give an algebraic proof of the inequality and completely classify Lagrangian submanifolds in complex space forms satisfying the equality, which is not discussed in Oprea's paper.

1. Introduction

Let M^n be a Riemannian n-manifold and X be a unit vector. We choose an orthonormal frame $\{e_1, \cdots, e_n\}$ in T_xM^n such that $e_1 = X$. We denote the Ricci curvature at X by

$$Ric(X) = K_{12} + \cdots + K_{1n},$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i, e_j.

In [6] B.-Y. Chen proves the following inequality on Ricci curvature for any n-dimensional submanifold in Riemannian manifold of constant sectional curvature c:

$$Ric(X) \leq \frac{n-1}{4}c + \frac{n^2}{4}||H||^2.$$

This inequality on Ricci curvature is now named as the Chen-Ricci inequality (see [12]). This inequality is not optimal for Lagrangian submanifolds in complex space forms. Using an optimization technique, Oprea in [11] proves

$$Ric(X) \leq \frac{n-1}{4}(c + n||H||^2),$$

which improves the Chen-Ricci inequality for Lagrangian submanifolds in complex space forms of constant holomorphic sectional curvature c.

In this article, we provide an algebraic proof for the improved Chen-Ricci inequality. Equality conditions are obtained. We also completely characterize Lagrangian submanifolds satisfying the equality, which is not discussed in Oprea's paper.

Theorem 3.1 and Corollary 3.2 improve a couple of results in [6] and [10].

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2. Preliminaries

Let $f : M \to \tilde{M}^n$ be an isometric immersion of a Riemannian n-manifold M into a Kaehler n-manifold $\tilde{M}^n$. Then M is called a Lagrangian (or totally real) submanifold if the almost complex structure J of $\tilde{M}^n$ carries each tangent space of M into its corresponding normal space.

By a complex-space-form $\tilde{M}^n(c)$ we mean a Kaehler manifold with constant holomorphic sectional curvature c.

An n-dimensional submanifold M of a Riemannian manifold (N, g) is called totally umbilical (respectively, totally geodesic) if its second fundamental form h in N satisfies

$$h(X, Y) = g(X, Y)H$$

(respectively, $h \equiv 0$), where

$$H = \frac{1}{n}\,\text{trace}\, h$$

is the mean curvature vector of M in N. For a totally umbilical submanifold M the shape operator A_H at H has exactly one eigenvalue; moreover, $A_\xi = 0$ for each normal vector ξ perpendicular to H.

Totally umbilical submanifolds, if they exist, are the simplest submanifolds next to totally geodesic submanifolds in a Riemannian manifold. However, it is well known that there exist no totally umbilical Lagrangian submanifolds in a complex-space-form $\tilde{M}^n(c)$ with $n \geq 2$ except the totally geodesic ones (see [8]).

Consequently, it is natural to look for and to investigate the "simplest" Lagrangian submanifolds next to the totally geodesic ones in complex-space-forms $\tilde{M}^n(c)$. In order to do so B.-Y. Chen introduced the concept of *Lagrangian H-umbilical submanifolds* (cf. [5, 7]).

By a Lagrangian H-umbilical submanifold of a Kaehler manifold $\tilde{M}^n$ we mean a Lagrangian submanifold whose second fundamental form takes the following simple form:

$$\begin{aligned} &h(e_1, e_1) = \lambda J e_1, \quad h(e_2, e_2) = \cdots = h(e_n, e_n) = \mu J e_1, \\ &h(e_1, e_j) = \mu J e_j, \quad h(e_j, e_k) = 0, \;\; j \neq k, \quad j, k = 2, \ldots, n \end{aligned} \tag{2.1}$$

for some suitable functions λ and μ with respect to some suitable orthonormal local frame field.

We need the following lemmas in section 3.

Lemma 2.1. *Let $(x_1, x_2, \ldots, x_n)$ be a point in R^n. If $x_1 + x_2 + \cdots + x_n = na$, we have*

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq na^2.$$

The equality sign holds if and only if $x_1 = x_2 = \cdots = x_n = a$.

Proof. $x_1 + x_2 + \cdots + x_n = na$ is a plane tangent to the sphere $x_1^2 + x_2^2 + \cdots + x_n^2 = na^2$ at the point $(a, a, \ldots, a)$. Lemma follows from the fact that the distance between any point in the plane and the origin is bigger than or equal to the radius of the sphere and the minimum occurs at the point $(a, a, \ldots, a)$. □

Lemma 2.2. *Let $f_1(x_1, x_2, \ldots, x_n)$ be a function in R^n defined by*

$$f_1(x_1, x_2, \ldots, x_n) = x_1 \sum_{j=2}^{n} x_j - \sum_{j=2}^{n} x_j^2$$

If $x_1 + x_2 + \cdots + x_n = 2na$, we have

$$f_1(x_1, x_2, \ldots, x_n) \leq \frac{n-1}{4n}(x_1 + x_2 + \cdots + x_n)^2$$

The equality sign holds if and only if $\frac{1}{n+1} x_1 = x_2 = \cdots = x_n = a$.

Proof. From $x_1 + x_2 + \cdots + x_n = 2na$, we have

$$(x_1 - na) + x_2 + \cdots + x_n = na.$$

By Lemma 2.1, we have

$$(x_1 - na)^2 + x_2^2 + \cdots + x_n^2 \geq na^2.$$

With the equality sign holds if and only if $\frac{1}{n+1}x_1 = x_2 = \cdots = x_n = a$. Therefore we have

$$\left(x_1^2 - 2nax_1 + n^2a^2\right) + x_2^2 + \cdots + x_n^2 \geq na^2$$

and

$$\left(n^2 - n\right) a^2 \geq x_1 \left(2na - x_1\right) - x_2^2 - \cdots - x_n^2,$$

i.e.

$$x_1 \sum_{j=2}^{n} x_j - \sum_{j=2}^{n} x_j^2 \leq \left(n^2 - n\right) a^2 = \frac{n-1}{4n} \left(x_1 + x_2 + \cdots + x_n\right)^2.$$

□

Lemma 2.3. *Let $f_2(x_1, x_2, \ldots, x_n)$ be a function in R^n defined by*

$$f_2(x_1, x_2, \ldots, x_n) = x_1 \sum_{j=2}^{n} x_j - x_1^2$$

If $x_1 + x_2 + \cdots + x_n = 4a$, we have

$$f_2(x_1, x_2, \ldots, x_n) \leq \frac{1}{8} \left(x_1 + x_2 + \cdots + x_n\right)^2.$$

The equality sign holds if and only if $x_1 = a, x_2 + \cdots + x_n = 3a$.

Proof. Let $u = x_1, v = x_2 + \cdots + x_n - 2a$, we have

$$u + v = x_1 + x_2 + \cdots + x_n - 2a = 2a.$$

By Lemma 2.1, we have

$$u^2 + v^2 = x_1^2 + (x_2 + \cdots + x_n - 2a)^2 \geq 2a^2.$$

With the equality sign holds if and only if $x_1 = a, x_2 + \cdots + x_n = 3a$. Therefore we have

$$(x_2 + \cdots + x_n - 2a)^2 \geq 2a^2 - x_1^2$$

and

$$(x_2 + \cdots + x_n)^2 - 4a\left(x_2 + \cdots + x_n\right) + 4a^2 \geq 2a^2 - x_1^2$$

i.e.

$$\left(x_2 + \cdots + x_n\right)\left[\left(x_2 + \cdots + x_n\right) - 4a\right] + 2a^2 \geq -x_1^2.$$

Since $x_1 = 4a - (x_2 + \cdots + x_n)$, we now have

$$2a^2 \geq x_1 \left(x_2 + \cdots + x_n\right) - x_1^2.$$

Therefore

$$f_2(x_1, x_2, \ldots, x_n) = x_1 \left(x_2 + \cdots + x_n\right) - x_1^2 \leq 2a^2 = \frac{1}{8} \left(x_1 + x_2 + \cdots + x_n\right)^2.$$

□

3. An Improved inequality for Ricci curvature

Let $\tilde{M}^n(c)$ be a complex-space-form with constant holomorphic sectional curvature c. If M^n is a Lagrangian submanifold of real dimension n in $\tilde{M}^n(c)$. It is well known that

$$A_{JY}X = -Jh(X,Y) = A_{JX}Y \tag{3.1}$$

for any tangent vector fields in M^n. Let $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ be an orthonormal frame field. Then

$$h^i_{jk} = h^j_{ik}, \forall i,j,k \in \overline{1,n}, \tag{3.2}$$

where h^i_{jk} is the Je_i component of the vector $h(e_j, e_k)$.

Theorem 3.1. *Let M^n be a Lagrangian submanifold of real dimension n ($n \geq 2$) in a complex-space-form $\tilde{M}^n(c)$, x a point in M^n and X a unit tangent vector in T_xM^n. Then we have*

$$Ric(X) \leq \frac{n-1}{4}(c + n||H||^2), \tag{3.3}$$

where H is the mean curvature vector of M^n in $\tilde{M}^n(c)$ and $Ric(X)$ is the Ricci curvature of M^n at X.

The equality sign holds for any unit tangent vector at x if and only if either

(i) *x is a totally geodesic point or*

(ii) *$n = 2$ and x is an H-umbilical point with $\lambda = 3\mu$.*

Proof. We fix the point x in M^n. Let X be any unit tangent vector at x. We choose an orthonormal frame $\{e_1, \cdots, e_n\}$ in T_xM^n such that $e_1 = X$ and $\{Je_1, \cdots, Je_n\}$ an orthonormal frame in $T^\perp_x M^n$. From Gauss equation we have

$$\tilde{R}(e_1, e_j, e_1, e_j) = R(e_1, e_j, e_1, e_j) - \tilde{g}(h(e_1, e_1), h(e_j, e_j)) + \tilde{g}(h(e_1, e_j), h(e_1, e_j))$$

or

$$\tilde{R}(e_1, e_j, e_1, e_j) = R(e_1, e_j, e_1, e_j) - \sum_{r=1}^{n}(h^r_{11}h^r_{jj} - (h^r_{1j})^2), \forall j \in \overline{2,n}.$$

Hence we have

$$(n-1)\frac{c}{4} = Ric(X) - \sum_{r=1}^{n}\sum_{j=2}^{n}(h^r_{11}h^r_{jj} - (h^r_{1j})^2).$$

Therefore

$$\begin{aligned} Ric(X) - (n-1)\frac{c}{4} &= \sum_{r=1}^{n}\sum_{j=2}^{n}\left(h^r_{11}h^r_{jj} - (h^r_{1j})^2\right) \\ &\leq \sum_{r=1}^{n}\sum_{j=2}^{n} h^r_{11}h^r_{jj} - \sum_{j=2}^{n}(h^1_{1j})^2 - \sum_{j=2}^{n}(h^j_{1j})^2. \end{aligned} \tag{3.4}$$

Using (3.2), we have

$$Ric(X) - (n-1)\frac{c}{4} \leq (\sum_{r=1}^{n}\sum_{j=2}^{n} h^r_{11}h^r_{jj}) - \sum_{j=2}^{n}(h^j_{11})^2 - \sum_{j=2}^{n}(h^1_{jj})^2. \tag{3.5}$$

Now we assume

$$f_1\left(h^1_{11}, h^1_{22}, \ldots, h^1_{nn}\right) = h^1_{11}\sum_{j=2}^{n} h^1_{jj} - \sum_{j=2}^{n}\left(h^1_{jj}\right)^2,$$

$$f_r\left(h^r_{11}, h^r_{22}, \ldots, h^r_{nn}\right) = h^r_{11}\sum_{j=2}^{n} h^r_{jj} - \left(h^r_{11}\right)^2, \forall r \in \overline{2,n}.$$

Since $nH^1 = h^1_{11} + h^1_{22} + \ldots + h^1_{nn}$, by Lemma 2.2 we have

$$f_1\left(h^1_{11}, h^1_{22}, \ldots, h^1_{nn}\right) \le \frac{n-1}{4n}(nH^1)^2 = \frac{n(n-1)}{4}(H^1)^2. \tag{3.6}$$

Similarly, by Lemma 2.3, we have for $2 \le r \le n$ that

$$f_r\left(h^r_{11}, h^r_{22}, \ldots, h^r_{nn}\right) \le \frac{1}{8}(nH^r)^2 = \frac{n^2}{8}(H^r)^2 \le \frac{n(n-1)}{4}(H^r)^2. \tag{3.7}$$

From (3.5),(3.6) and (3.7), we have

$$Ric(X) - \frac{n-1}{4}c \le \frac{n(n-1)}{4}\sum_{r=1}^{n}(H^r)^2 = \frac{n(n-1)}{4}||H||^2,$$

which implies (3.3).

Now assume $n \ge 3$ and the equality sign of (3.3) holds for any unit tangent vector X at x. By (3.7), we have $H^r = 0$ for $r \ge 2$ (or simply choose Je_1 parallel to H). Combining this and Lemma 2.3 we have

$$h^1_{1j} = h^j_{11} = \frac{nH^j}{4} = 0, \forall j \ge 2.$$

From (3.4), we have $h^1_{jk} = 0, \forall j,k \ge 2, j \ne k$. From Lemma 2.2, (h^1_{jk}) must be diagonal with $h^1_{11} = (n+1)a$ and $h^1_{jj} = a, \forall j \ge 2$, where $a = \frac{H^1}{2}$.

Now if we compute $Ric(e_2)$ as we do for $Ric(X) = Ric(e_1)$ in (3.4), from the equality we get $h^r_{2j} = h^2_{jr} = 0, \forall r \ne 2, j \ne 2, r \ne j$. From the equality and Lemma 2.2, we get

$$\frac{h^2_{11}}{n+1} = h^2_{22} = \cdots = h^2_{nn} = \frac{H^2}{2} = 0.$$

Since the equality holds for all unit tangent vector, the argument is also true for matrices (h^r_{jk}). Now finally $h^2_{2j} = h^j_{22} = \frac{H^j}{2} = 0, \forall j \ge 3$. Therefore matrix (h^2_{jk}) has only two possible nonzero entries (i.e. $h^2_{12} = h^2_{21} = h^1_{22} = \frac{H^1}{2}$). Similarly matrix (h^r_{jk}) has only two possible nonzero entries

$$h^r_{1r} = h^r_{r1} = h^1_{rr} = \frac{H^1}{2}, \forall r \ge 3.$$

We now compute $Ric(e_2)$ as follows:

$$\tilde{R}(e_2, e_j, e_2, e_j) = R(e_2, e_j, e_2, e_j) - \tilde{g}(h(e_2, e_2), h(e_j, e_j)) + \tilde{g}(h(e_2, e_j), h(e_2, e_j)),$$

so we have

$$\tilde{R}(e_2, e_j, e_2, e_j) = R(e_2, e_j, e_2, e_j) - (\frac{H^1}{2})^2, \forall j \ge 3. \tag{3.8}$$

From

$$\tilde{R}(e_2, e_1, e_2, e_1) = R(e_2, e_1, e_2, e_1) - \tilde{g}(h(e_2, e_2), h(e_1, e_1)) + \tilde{g}(h(e_2, e_1), h(e_2, e_1)),$$

we get

$$\tilde{R}(e_2, e_1, e_2, e_1) = R(e_2, e_1, e_2, e_1) - (n+1)\left(\frac{H^1}{2}\right)^2 + \left(\frac{H^1}{2}\right)^2. \tag{3.9}$$

By combining (3.8) and (3.9), we get

$$Ric(e_2) - \frac{(n-1)c}{4} = (n+1)\left(\frac{H^1}{2}\right)^2 - \left(\frac{H^1}{2}\right)^2 + (n-2)\left(\frac{H^1}{2}\right)^2 = 2(n-1)\left(\frac{H^1}{2}\right)^2.$$

On the other hand from the equality assumption, we have

$$Ric(e_2) - \frac{(n-1)c}{4} = \frac{n(n-1)}{4}||H||^2 = n(n-1)\left(\frac{H^1}{2}\right)^2.$$

Therefore, we have

$$n(n-1)\left(\frac{H^1}{2}\right)^2 = 2(n-1)\left(\frac{H^1}{2}\right)^2$$

Since $n \neq 1$, we have either $H^1 = 0$ or $n = 2$.

If $H^1 = 0$, (h^r_{jk}) are all zero and x is a totally geodesic point.

If $n = 2$, then we have

$$h(e_1, e_1) = \lambda J e_1, h(e_2, e_2) = \mu J e_1, h(e_1, e_2) = \mu J e_2$$

with $\lambda = 3\mu = \frac{3H^1}{2}$.

The converse can be proved by simple computation.

□

Remark 3.1. Oprea proved inequality (3.3) in [11] using a maximization technique, but did not discuss the equality case. Here, we prove the inequality using algebraic inequalities. The benefit of our proof is that we can determine the equality condition in better form. In this way we can completely characterize Lagrangian submanifolds satisfying the equality case of the inequality. Theorem 3.1 improves a result in [6, page 38] for Lagrangian submanifolds.

Example 3.1. *It is easy to see that the Whitney 2-sphere in* $\mathbf{C}^2$ *satisfies the equality of (3.3).*

Remark 3.2. By using the same approach, we may extend Theorem 3.1 to Lagrangian submanifolds in Quaternion projective spaces. This way we improve Theorem 3.1 on page 300 in [10].

From Theorem 3.1, we have the following

Corollary 3.2. *Let* M^n *be a Lagrangian submanifold of real dimension* n $(n \geq 2)$ *in a complex-space-form* $\tilde{M}^n(c)$. *If*

$$Ric(X) = \frac{n-1}{4}(c + n||H||^2)$$

for any unit tangent vector X *of* M^n, *then either* M^n *is a totally geodesic submanifold in* $\tilde{M}^n(c)$ *or* $n = 2$ *and* M^n *is a Lagrangian H-umbilical submanifold of* $\tilde{M}^n(c)$ *with* $\lambda = 3\mu$.

Remark 3.3. Lagrangian H-umbilical submanifolds in complex space forms satisfying the condition $\lambda = 3\mu$ have been completely classified (see [1, 2, 3, 4, 9] and [7, pp. 331-332] for details).

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