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BIMINIMAL CURVES IN EUCLIDEAN SPACES

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Abstract. We study with biminimal curves, that is curves which are critical points of the bienergy for normal variations. We give a description of the Euler-Lagrange equation associated to biminimal curves on a Riemannian manifold. We describe curves of AW(k) type of Euclidean n-space $\mathbb{E}^n$. We show that every biminimal curves of $\mathbb{E}^n$ are of AW(1)-type. We also show that λ-biminimal curves of $\mathbb{E}^n$ are of AW(3)-type.

1. Introduction

Biharmonic functions play important roles in elasticity and hydrodynamics. In 1964, Eells and Sampson introduced the notion of biharmonic maps between Riemannian manifolds, which is a generalization of harmonic maps [6].

Let (M, g) and $(N, \widetilde{g})$ be m and n-dimensional differentiable manifolds respectively. A smooth map $x : (M, g) \to (N, \widetilde{g})$ is called harmonic if it is a critical point of the energy functional

$$E : C^\infty(M, N) \to \mathbb{R} \ \ ; \ E(x) = \frac{1}{2}\int_M |dx|^2 \, v_g \tag{1.1}$$

and the corresponding Euler-Lagrange equation is given by the vanishing of the torsion field

$$\tau(x) = trace \nabla dx. \tag{1.2}$$

Further, the map x is called biharmonic if it is a critical point for all variations of the bienergy functional (see [4]),

$$E_2(x) = \frac{1}{2}\int_m |\tau(x)|^2 \, v_g. \tag{1.3}$$

An isometric immersion $x : (M^m, g) \to (N^n, \widetilde{g})$ $(m \leq n)$ between Riemannian manifolds or its image is called biminimal if it is the critical point of the bienergy functional E_2 variations normal to the image $x(M) \subset N$, that is $\frac{dE_2}{dt}(x_t)\,|_{t=0} = 0$,

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for any smooth variations of the map $x_t : (-\varepsilon, \varepsilon) \times M \to N$, $x_0 = x$, such that $v = \frac{dx_t}{dt}|_{t=0}$ to $x(M)$ [4].

The Euler-Lagrange operator attached to bienergy, called the bitension field and computed by Jiang in [8], is;

$$J(\tau(x)) = \tau_2(x) = -\left(\Delta\tau(x) - traceR^N(dx, \tau(x)dx)\right) \tag{1.4}$$

where J is the Jacobi operator of x and R^N is the Riemannian curvature operator of N. Further, $\tau_2(x)$ vanishes if and only if the map x is biharmonic.

In a different settings, in [5], B. Y. Chen defined biharmonic submanifolds $M \subset \mathbb{E}^n$ of the Euclidean space as those with harmonic mean curvature vector field, that is, $\Delta H = 0$, where Δ is the Laplacian operator. If we apply the definition of the biharmonic maps to Riemannian immersions into the Euclidean space we recover Chen' s notation of biharmonic submanifold.

In the instance of an isometric immersion $x : M \to N$ requiring that the normal part of $\tau_2(x)$ is zero characterized the biminimal maps that is the mean curvature vector field H of x satisfies

$$\tau_2^{\perp}(x) = (\Delta H - traceR^N(dx, H)dx)^{\perp} = 0, \quad \tau(x) = H. \tag{1.5}$$

The isometric immersion x is called minimal if it is a critical point of the volume function

$$V = \frac{1}{2}\int_m dv_g \tag{1.6}$$

and the corresponding Euler-Lagrange equation is $H = 0$.

It is obvious that biharmonic immersions are biminimal. A generalization of biminimal immersions are λ-biminimal immersions, they are defined as the critical points with respect to normal variations of the fixed energy, of the constrained bienergy functional

$$E_{2,\lambda}(x) = \frac{1}{2}\int_m |\tau(x)|^2 \, dv + \frac{1}{2}\int_m |dx|^2 \, dv. \tag{1.7}$$

The Euler-Lagrange equation for λ-biminimal immersion is,

$$[\tau_2, \lambda]^{\perp} = [\tau_2]^{\perp} - 2\lambda \, [\tau]^{\perp} = 0, \tag{1.8}$$

where $[,]^{\perp}$ denotes the normal part of $[.]$ (see [13]).

It is obvious that biharmonic immersions are biminimal.

In [1] K. Arslan and A. West studied with the isometric immersions of AW(k) type. Further K. Arslan and C. Ozgür considered the immersed curves in $\mathbb{E}^n$ of type AW(k) [2]. We show that every biminimal curves of $\mathbb{E}^n$ are of AW(1)-type. We also show that λ-biminimal curves of $\mathbb{E}^n$ are of AW(3)-type.

2. Biharmonic Curves

Let M be a m-dimensional Riemannian manifold and $\gamma : I \subset \mathbb{R} \to M$ parametrized by arc length, that is $|\gamma^{\cdot}| = 1$. Then γ is called a Frenet curve of osculating order $r, 1 \leq r \leq m$, if there exists orthonormal vector fields $V_1, V_2, ..., V_r$ along γ such that

$$\begin{aligned} V_1 &= \dot{\gamma} = T \\ \nabla_T V_1 &= \kappa_1 V_2 \\ \nabla_T V_2 &= -\kappa_1 V_1 + \kappa_2 V_2 \\ &.... \\ \nabla_T V_r &= -\kappa_{r-1} V_{r-1} \end{aligned}$$

where $\kappa_1, \kappa_2, ..., \kappa_{r-1}$ are positive functions on I [12].

Remark 2.1. A geodesic is a Frenet curve of osculating order 1; a circle is a Frenet curve of osculating order 2 with κ_1 =constant ; a helix of order r; $r \geq 3$ is a Frenet curve of osculating order r with $\kappa_1, \kappa_2, ..., \kappa_{r-1}$ constants; a helix of order 3 is called simply helix.

We now restrict our attention to isometric immersion $\gamma : I \subset \mathbb{R} \to (M, g)$ from an interval I to a Riemannian manifold. The image $C = \gamma(I)$ is the trace of a curve in M and γ is a parametrization of C by arc-length. In this case the tension field becomes $\tau(\gamma) = \nabla_T T$ and the biharmonic equation reduces to

$$\tau_2(\gamma) = -\nabla_T^2 \tau(T) - R(T, \tau(\gamma))T = 0 \tag{2.1}$$

Note that γ is a part of a geodesic of M if and only if γ is harmonic [4].

The tension field $\tau(\gamma)$ of γ becomes;

$$\tau(\gamma) = k_1 V_2 = H \tag{2.2}$$

where H is the mean curvature vector of γ.

The bitension field of γ becomes;

$$-\tau_2(\gamma) = \Delta H + trace R(T, H)T. \tag{2.3}$$

By the use of Frenet equations (2.3) we get

$$-\tau_2(\gamma) = (k_1^{''} - k_1^3 - k_1 k_2^2)V_2 - 3k_1 k_1^{'} T + (k_1^{'} k_2 + k_1 k_2^{'})V_3 + k_2 k_3 V_4 + k_1 R(T, V_2)T. \tag{2.4}$$

From the biharmonic equation if f is harmonic then it is biharmonic, thus geodesics are a subclass of biharmonic curves.

The converse in not true and this makes the class of biharmonic curves richer than that of geodesics.

It is then natural to ask what geometric properties characterize biharmonic curves.

Moreover, $\lambda-$biminimal curves in a Riemannian manifold M are characterized by Monteldo in [4].

Proposition 2.1. [4] *Let $\gamma : I \subset \mathbb{R} \to (M^m, g)$ $(m \geq 2)$ be an isometric curve from an open interval of $\mathbb{R}$ into a Riemannian manifold (M^m, g). Then γ is a $\lambda-$biminimal curve if and only if there exists a real number λ such that*

$$\begin{aligned} (k_1^{''} - k_1^3 - k_1 k_2^2) + k_1 g(V_1, V_2)T, V_2) - \lambda k_1 &= 0 \\ (k_1^2 k_2)^{'} + k_1^2 g(R(T, V_2)T, V_3) &= 0 \\ k_1 k_3 + k_1 g(R(T, V_2)T, V_4) &= 0 \\ k_1 g(R(T, V_2)T, V_j) &= 0 \end{aligned} \tag{2.5}$$

where, $j = 5, ..., m$ and R is the curvature vector of (M^m, g).

Remark 2.2. Asking for $\lambda-$biminimal curve γ to be biharmonic requires the supplementary condition $\tau_2(x)^{\top}=0$, equivalent to $k_1k_1^{'}=0$, that is; either k_1 is constant or γ is a geodesic ($k_1=0$) of M, where $(\ )^{\top}$ denotes the tangential part.

Corollary 2.1. [4] *An isometric curve γ on a surface M of Gaussian curvature G is biminimal if and only if its signed curvature k_1 satisfies the ordinary differential equation:*

$$k_1^{''}-k_1^3+k_1G-\lambda k_1=0 \tag{2.6}$$

for some $\lambda\in\mathbb{R}$.

Corollary 2.2. [4] *An isometric curve γ on a Riemannian three manifold of constant sectional curvature c is $\lambda-$biminimal if and only if its curvatures k_1 and k_2 fulfill the system:*

$$\begin{aligned} k_1^{''}-k_1^3-k_1k_2^2+k_1c-\lambda k_1=0,\ \lambda\in\mathbb{R}, \\ k_1^2k_2=constant \end{aligned} \tag{2.7}$$

where $c=g(R(T,V_2)T,V_2)$ and $g(R(T,V_2)T,V_3)=0$.

From now on we consider the isometric immersion $\gamma:I\subset\mathbb{R}\to\mathbb{E}^n$ given with arclength parameter. The tension field $\tau(\gamma)$ and bitension field $\tau_2(\gamma)$ of γ become

$$\begin{aligned} \tau(\gamma) &= k_1V_2=H, \\ -\tau_2(\gamma) &= \Delta H \\ &= \Delta^DH-\widetilde{A}H+(\Delta H)^{\top} \\ &= (\Delta H)^{\perp}+(\Delta H)^{\top} \end{aligned} \tag{2.8}$$

respectively, where $\widetilde{A}$ is the Simon's operator defined by

$$\widetilde{A}H=<\widetilde{A}_HT,T>H=<H,H>H=k_1^3V_2,$$

and

$$\begin{aligned} \Delta^DH=(k_1^{''}-k_1k_2^2)V_2+(k_1^{'}k_2+k_1k_2^{'})V_3+k_1k_3V_4 \\ (\Delta H)^{\top}=-3k_1k_1^{'}T. \end{aligned} \tag{2.9}$$

We consider the following cases;

a): Curves satisfying $\boldsymbol{\Delta^DH=0}$

In [15] Ü. Lümiste classified the curves in $\mathbb{E}^n$ satisfying the condition $\overline{\nabla}_{\gamma'}\overline{\nabla}_{\gamma'}h=0$,where h is the second fundamental form of the immersed curve. Further in [3] M. Barros and O. J. Garay proved the same result.

Theorem 2.1. *([15],[3]) A curve $\gamma(I)$ with $\overline{\nabla}^2_{\gamma'}h=0$ (or $\Delta^DH=0$) in $\mathbb{E}^n$ is either*

i) $\mathbb{E}^1$, $S^1(r)$ in $\mathbb{E}^2$,

ii) cornu spiral (clothoid) $C^1(a)$ (i.e. a plane curve whose curvature k_1 is proportional to the arclength: $k_1=as$) in $\mathbb{E}^2$, or

iii) spherical cornu spiral $C^1_s(b,c)$ in $S^2(\sqrt{\frac{b}{c}})$ with the Frenet curvatures

$$k_1(s)=\sqrt{b(s-a)^2+\frac{c^2}{b}},\quad k_2(s)=\frac{bc}{b^2(s-a)^2+c^2}$$

such that

$$\frac{1}{(k_1)^2}+\frac{(\dot{k_1})^2}{k_1^2k_2^4}=\frac{b}{c}.$$

b): Curves satisfying $\Delta^D H+\lambda H=0$

In [10] B. Kılıç classified the immersed curves in $\mathbb{E}^n$ satisfying the condition $\Delta^D H+\lambda H=0, \lambda\in R$. isometric immersions satisfying this condition is called *harmonic 1-type.*

Theorem 2.2. [10] *Let $\gamma: I\subset\mathbb{R}\rightarrow\mathbb{E}^n$ be an immersed curve given with arclength parameter; then γ is harmonic 1-type if and only if either;*

i) γ is a straight line, or

ii)γ is a plane curve with $k_1(s)=b_1\cos(\sqrt{c}s)+b_2\sin(\sqrt{c}s)$ (or $k_1(s)=b_1e^{\sqrt{c}s}+b_2e^{-\sqrt{c}s}$)

iii) γ is a space curve with

$$k_1(s)=\pm\frac{\sqrt{c}}{(4c_1)^{\frac{1}{4}}}\sqrt{\frac{e^{4s-2c_2}+1}{e^{2s-c_2}}}, k_2(s)=2\sqrt{\frac{e^{2s-c_2}}{e^{4s-2c_2}+1}}.$$

c): Curves satisfying $(\Delta H)^{\perp}=0$

In [1] K. Arslan and A. West studied with the isometric immersions of AW(k) type. Further K. Arslan and C. Ozgür considered the immersed curves in $\mathbb{E}^n$ of type AW(k) [2].

Definition 2.1. Frenet curves γ are

i) of type AW(1) if they satisfy $\gamma^{iv}(s)^{\perp}=0$

ii) of type AW(2) if $\gamma^{iv}(s)^{\perp}$ and $\gamma^{'''}(s)^{\perp}$ are linearly dependent,

iii) of type AW(3) if $\gamma^{iv}(s)^{\perp}$ and $\gamma^{''}(s)^{\perp}$ are linearly dependent, where $\gamma^{iv}(s)^{\perp}$, $\gamma^{'''}(s)^{\perp}$ and $\gamma^{''}(s)^{\perp}$ the normal parts of fourth, third, and second derivatives of γ.

The Laplacian Δ of an immersed unit speed curve becomes

$$\Delta=-\frac{d^2}{ds^2}.$$

So we get

$$\Delta H=-\Delta^2\gamma=-\frac{d^4\gamma}{ds^4}.$$

Therefore the condition $\gamma^{iv}(s)^{\perp}=0$ is equivalent to the biminimality condition $(\Delta H)^{\perp}=0$.

Theorem 2.3. *(*[2],[14]*) Let $\gamma: I\subset\mathbb{R}\rightarrow\mathbb{E}^n$ be an immersed curve given with arclength parameter; then γ is of AW(1) type if and only if either;*

i) γ is a straight line, or

ii)γ is a logarithmic spiral with $k_1(s)=\frac{\sqrt{2}}{s}$, or

iii) γ is a space curve with

$$\ddot{k_1}(s)=k_1^3(s)+\frac{1}{k_1^3(s)}, k_2(s)=\frac{c}{k_1(s)}.$$

Remark 2.3. The parametric representation of a planar curve can be obtained from the theorem of Gray (see [7], p 111):

$$\gamma(s)=\left(\int\cos\theta(s)ds+c,\int\sin\theta(s)ds+d\right) \tag{2.10}$$

where $\theta(s) = \int k_1(s)ds + \theta_0$, c, d, θ_0 are constants of integrations. So, the parametric representation of logarithmic spiral given in the previous theorem becomes;
(2.11)

$$\gamma(s) = \frac{s}{3}\left(\cos(\sqrt{2}\log s) + \sqrt{2}\sin(\sqrt{2}\log s), -\sqrt{2}\cos(\sqrt{2}\log s) + \sin(\sqrt{2}\log s)\right)$$

(see [14]).

d) Curves satisfying $(\Delta \mathbf{H})^{\perp} = \lambda \mathbf{H}$

By Definition 2.1 the immersed curves of AW(3) type satisfies $\gamma^{iv}(s)^{\perp} = \lambda\gamma''(s)^{\perp}$. Therefore, by the use of (2.4)and (2.2) we can say that every immersed curves in $\mathbb{E}^n$ of AW(3) type are $\lambda-$biminimal. For more details of curves AW(3) type see [9], [2] and [16].

e) Curves satisfying $(\Delta \mathbf{H})^{\top} = \mathbf{0}$

In [17] H. Pottman and M. Hofer characterized the counterpart on surfaces, C^2 cubic splines.

They give an example; whereas the minimizers of the L^2 norm of the second derivative have cubic segments (vanishing fourth derivative), the corresponding splines on surfaces have segments with vanishing tangential component of the fourth derivative. The authors call such segment "*tangentially cubic*". In the same paper it has been shown that tangentially cubic curves are important for the interpolating cubic spline curves on surfaces. In view of the importance of tangentially cubic curves some explicit representations of such curves on special surfaces, namely certain cylinder surfaces.

Definition 2.2. Let γ be a regular curve in $\mathbb{E}^n$. If the fourth derivative $\gamma'^v(s)$ of γ is orthogonal to $\gamma'(s)$, then γ is called a tangentially cubic curve ($T.C$-curve) of $\mathbb{E}^n$.

Hence the condition for a tangentially cubic parametrization becomes

$$0 = <\gamma'^v(s), \gamma'(s)> = -3k_1(s)k_1'(s). \tag{2.12}$$

So, by (2.12) we can say that the arclength parametrization of a curve γ in $\mathbb{E}^n$ is tangentially cubic if and only if the curve possesses constant curvature k_1 [17]. In the plane we get only circles and straight lines. However, in $\mathbb{E}^n$ this family is relatively rich, since the torsion $\tau = k_2$ can be arbitrary. Parametric representations of special space curves with constant curvature k_1 have been given by E. Salkowski [18], whose formulae for $a \neq \pm\frac{1}{2}$, $b \in \mathbb{R}\setminus\{0\}$:

$$\begin{aligned} x(s) &= \frac{-1}{\sqrt{1+b^2}}\left(\frac{1-a}{4(1+2a)}\sin(1+2a)s + \frac{1+a}{4(1-2a)}\sin(1-2a)s + \frac{1}{2}\sin s\right), \\ y(s) &= \frac{1}{\sqrt{1+b^2}}\left(\frac{1-a}{4(1+2a)}\cos(1+2a)s + \frac{1+a}{4(1-2a)}\cos(1-2a)s + \frac{1}{2}\cos s\right), \\ z(s) &= \frac{1}{4b\sqrt{1+b^2}}\cos 2as. \end{aligned}$$

For more details on $T.C$-curves see also [11].

Remark 2.4. In [8] Jiang constructed an ad-hoc $(0,2)$ tensor S_2 such that $divS_2 = <\tau_2(\gamma), \gamma'(s)>$. So $T.C$-curves are the characterization of curves with vanishing $divS_2$.

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