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IRREDUCIBILITY OF THE SET OF ALL TWO-DIMENSIONAL SCROLLS WITH VERY FEW SECTIONS

E. BALLICO

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Abstract. Let C be a smooth Petri curve of genus $g \geq 2$. Fix integers k, d. Let $S(C; 2, d, k)$ denote the set of all pairs (E, V), where E is a rank 2 vector bundle on C with degree 2 and V is a k-dimensional linear subspace of $H^0(C, E)$ spanning E. Set $\Gamma(C; 2, d, k) := \{(E, V) \in S(C; 2, d, k) : \mathcal{O}_C$ is a factor of $E\}$. Here we prove that $S(C; 2, d, 3)\backslash\Gamma(C; 2, d, 3)$ is irreducible if $d > (2g+6)/2$ and (under some assumptions on d) that E is stable for a general element (E, V) of any irreducible component of $S(C; 2, d, 4)\backslash\Gamma(C; 2, d, 4)$ such that $h^0(C, E)$ is large.

1. Introduction

Let C be a smooth and connected projective curve of genus $g \geq 2$. As in [1] C is called a Petri curve if the Petri map $p_L : H^0(C, L) \otimes H^0(C, \omega_C \otimes L^*) \to H^0(C, \omega_C)$ is injective for all $L \in \mathrm{Pic}(C)$.

A *coherent system* of type (n, d, k) on C is a pair (E, V) such that E is a rank n vector bundle on C such that $\deg(E) = d$ and V is a k-dimensional linear subspace of $H^0(C, E)$ ([1]). The coherent system (E, V) is said to be *spanned* if V spans E. Let $S(C; 2, d, k)$ denote the set of all spanned coherent systems (E, V) of type $(2, d, k)$. We do not claim that $S(C; 2, d, k)$ is an algebraic set, but we will speak about its irreducible components in the following sense. A subset Γ of $S(C; 2, d, k)$ is said to be an irreducible x-dimensional family of $S(C; 2, d, k)$ if there are an x-dimensional integral variety T, a non-empty open subset T' of T and a flat family $\{(E_t, V_t)\}_{t \in T}$ of coherent systems of type $(2, d, k)$ on C such that each element of Γ appears in this family and the associated set-theoretic map $T' \to \Gamma$ is finite-to-one. Γ is said to be an irreducible component if no dense open subset of it (i.e. of T) is contained in an irreducible family of spanned coherent systems with larger dimension. There is a " stupid " irreducible component: the one parametrizing cones, i.e. spanned coherent systems (E, V) such that $E \cong L \oplus \mathcal{O}_C$ for some $L \in \mathrm{Pic}^d(C)$ (see Remark 1.1). If $k \geq 3$ and $d = 2g-2+k$ there is an irreducible family parametrizing the non-special linearly normal scrolls, i.e. the pairs $(E, H^0(C, E))$

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with E spanned and $h^1(C,E)=0$ (see [2] for many results on this component). If C has very low gonality, then it is easy to construct other irreducible components. If C is general, $g \ge 4$ and either $k \ge 13$ or $k = 12$ and $g \equiv 0, 1 \pmod 4$, then there is at least another component ([2], Example 5.12). It is natural to ask if there are other components when k is low and C is general ([2], Remark. 5.13). It is also intersting to say if there is $(E,V) \in S(C;d,g,k)$ such that E is stable. This is related to the Brill-Noether theory of stable vector bundles on C. To get the existence of (E,V) with E stable we use the theory of coherent systems (see [1] for its use for the Brill-Noether theory of stable vector bundles).

Fix $\alpha \in \mathbb{R}$, $\alpha \ge 0$. The α-slope of the coherent system (E,V) of type (n,d,k) is the real number $\mu_\alpha(E,V) := d/n + k\alpha/n$. A coherent subsystem (F,W) of (E,V) is given by a subsheaf F of E and a linear subspace W of V such that $W \subseteq V \cap H^0(C,F)$. The coherent system (E,V) is said to be α-stable (resp. α-semistable) if $\mu_\alpha(F,W) < \mu_\alpha(E,V)$ (resp. $\mu_\alpha(F,W) \le \mu_\alpha(E,V)$) for all coherent subsystems (F,W) of (E,V). Let $G_\alpha(C;n,d,k)$ denote the moduli space of all α-stable coherent systems on C with type (n,d,k). This is an algebraic scheme and each of its irreducible components has dimension at least $\beta(g,n,d,k) := n^2(g-1)+1-k(k-d+n(g-1))$ (the Petri number for rank n vector bundles with degree d and at least k sections) ([1], Cor. 3.6). In this paper (as in [2] and [3]) we will only consider rank two vector bundles. For any fixed d,k there is $\alpha_{2,d,k} \in \mathbb{R}$, $\alpha_{2,d,k} \ge 0$ such that $G_\alpha(C;2,d,k) = G_\beta(C;2,d,k)$ for all $\alpha > \alpha_{2,d,k}$ and $\beta > \alpha_{2,d,k}$ ([1], Prop. 4.6). We will denote with $G_\infty(C;2,d,k)$ any $G_\alpha(C;2,d,k)$ with $\alpha > \alpha_{2,d,k}$. Notice that $(E,V) \in G_\infty(C;2,d,k)$ if and only if (E,V) has no coherent subsystem (M,W) with M a line bundle and either $\dim(W) > k/2$ or k is even, $\dim(W) = k/2$ and $\deg(M) \ge d/2$. Notice that it is sufficient to check these conditions for all rank 1 saturated subsheaves M of E, i.e. the subsheaves M of E such that $E/M \in \mathrm{Pic}(C)$.

To state our results we need to introduce the " stupid " component $\Gamma(C;2,d,k)$ of $S(C;2,d,k)$.

Remark 1.1. Fix a smooth curve C and integers $d \ge 2$ and $k \ge 3$ such that there is a spanned line bundle L on C with degree d and $h^0(C,L) \ge k-1$. For all integers $k \ge 3$ let $\Gamma(C;2,d,k)$ denote the set of all spanned coherent systems $(L \oplus \mathcal{O}_C, V)$ with $L \in \mathrm{Pic}^d(C)$, L spanned and $V = W \oplus H^0(C,\mathcal{O}_C)$ with $\dim(W) = k-1$ and W spanning L. There is a natural bijection between $\Gamma(C;2,d,k)$ and the pairs (L,W) with $L \in \mathrm{Pic}^d(C)$, and W a $(k-1)$-dimensional linear subspace of $H^0(C,L)$ spanning L. If (as in [2]) $d \ge 2g-2+k$, L is always spanned. Hence if $d \ge 2g-2+k$ there is a bijection of $\Gamma(C;2,d,k)$ with the product of $\mathrm{Pic}^d(C)$ with a non-empty open subset of the Grassmannian $G(k-1,k+g-1)$ of the $(k-1)$-dimensional linear subspaces of $\mathbb{K}^{\oplus(k+g-1)}$. If $d \ge g-1+k$, then $\Gamma(C;2,d,k)$ contains a non-empty open subset of a product of $\mathrm{Pic}^d(C)$ with an appropriate Grassmannian, but we do not claim that there are no other irreducible components in $\Gamma(C;2,d,k)$. Notice that $(L \oplus \mathcal{O}_C, V) \notin G_\infty(C;2,d,k)$ for each $k \ge 2$ and each k-dimensional linear subspace V of $H^0(C, L \oplus \mathcal{O}_C)$. Hence we will always miss the cones when we will look at spanned coherent systems $(E,V) \in G_\infty(C;2,d,k)$.

We will use [1], Theorem 9.2, and $G_\infty(C;2,d,k)$ to prove the following results.

Theorem 1.1. *Fix integers $g \ge 2$ and $d > (2g+6)/3$. Let C be a smooth Petri curve of genus g. Then $S(C;2,d,3)\backslash\Gamma(C;2,d,3)$ is irreducible.*

Theorem 1.2. *Let C be a smooth Petri curve of genus $g \ge 2$.*

(i) Fix an integer $d > 0$. If $d \ge 2\cdot\lfloor (g+3)/2\rfloor$, then $S(C;2,d,4)\backslash\Gamma(C;2,d,4) \ne \emptyset$.

(ii) Fix integers x, d such that $0 < x < g/2$, $0 < d < 2g-2$, and $(x+1)d < xg + (x+1)(x+\lfloor (g+3)/2\rfloor)$. Fix any irreducible component Γ of $S(C;2,2g+2,4)\backslash\Gamma(C;2,d,4)$ and let (E,V) be a general element of Γ. Assume $h^0(C,E) \ge 4+x$. Then E is stable.

In the set-up of Theorem 1.1 we do not claim that $S(C;2,d,3) \ne \Gamma(C;2,d,3)$. In part (ii) of Theorem 1.2 we do not claim that $S(C;2,2g+2,4) \ne \Gamma(C;2,d,4)$. The very easy part (i) of Theorem 1.2 gives a very easy non-emptiness statement. We do not know how to improve it using our elementary computations. For more interesting non-emptiness theorems when C has general moduli, see [4].

As in [1] and [2] we work over an algebraically closed field with $\mathrm{char}(\mathbb{K}) = 0$.

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2. The proofs

Proof of Theorem 1.1. We may assume $S(C;2,d,3) \ne \Gamma(C;2,d,3)$. Fix any irreducible component Γ of the set $S(C;2,d,3)\backslash\Gamma(C;2,d,3)$ and take a general $(E,V) \in \Gamma$. Since $3d-2g-6 > 0$, the case $k = 3$ of [1], Th. 9.2, shows that it is sufficient to prove that $(E,V) \in G_\infty(C;2,d,3)$. By the openness of α-stability it is sufficient to prove that $(E,W) \in G_\infty(C;2,d,3)$ for one 3-dimensional linear subspace W of $H^0(C,E)$. We will see that $(E,W) \in G_\infty(C;2,d,3)$ for all 3-dimensional linear subspaces of $H^0(C,E)$ spanning E. Fix any such W and take any saturated line subbundle M of E. Hence E/M is a line bundle. Set $W_M := H^0(C,M)\cap W$. The vector subspace W/W_M of $H^0(C,E/M)$ spans E/M, because W spans E. Since E is spanned, but it has not $\mathcal{O}_C$ as a factor, E/M is not trivial. Hence $\dim(W/W_M) \ge 2$, i.e. $\dim(W_M) \le 1$. We just checked the criterion for membership in $G_\infty(C;2,d,3)$. □

Proof of Theorem 1.2. Since C is a Petri curve, it has gonality $\lfloor (g+3)/2\rfloor$ and for every integer $t \ge \lfloor (g+3)/2\rfloor$ there is a base point free $L_t \in \mathrm{Pic}^t(C)$. Set $E := L_{\lfloor d/2\rfloor} \oplus L_{\lceil d/2\rceil}$ and use that $h^0(C,E) \ge 4$ and that E is spanned to get part (i).

(a) From now on we take integers d, g, x as in part (ii) of the statement of Theorem 1.2. Fix any irreducible component Γ of $S(C;2,d,4)\backslash\Gamma(C;2,d,4)$ and take a general $(E,V) \in \Gamma$. Assume $z := h^0(C,E) \ge 4+x$. In order to obtain a contradiction we assume that E is not stable. The case $k = 4$ of [1], Th. 9.2, shows that it is sufficient to prove that $(E,V) \in G_\infty(C;2,d,4)$, i.e. that for every line subbundle M or E either $\dim(V\cap H^0(C,M)) \le 1$ or $\dim(V\cap H^0(C,M)) = 2$ and $\deg(M) < d/2$. Let $G(4,H^0(C,E))$ denote the Grassmannian of all 4-dimensional linear subspaces of $H^0(C,E)$. $G(4,H^0(C,E))$ is irreducible and of dimension $4(z-4)$. We assumed that E is not associated to a cone, i.e. it has no trivial factor. Let Σ denote the set of all saturated rank 1 subsheaves M of C such that $h^0(C,M) \ge 2$. If $\Sigma = \emptyset$, then $(E,V) \in G_\infty(C;2,d,4)$ for all 4-dimensional linear subspaces V of $H^0(C,E)$. Hence from now on we assume $\Sigma \ne \emptyset$.

(b) Here we assume that E is not semistable. Hence there is an exact sequence

$$0 \to A_1 \to E \xrightarrow{f} A_2 \to 0 \tag{2.1}$$

with $a_1 > a_2$, where $a_i := \deg(A_i)$, $A_i \in \mathrm{Pic}(C)$, and $a_1 + a_2 = d$. Hence $a_1 > d/2$. Let U denote the image of the map $f_* : H^0(C, E) \to H^0(C, A_2)$. Since E is spanned, U spans A_2. Set $m := \dim(U)$. Hence $z = h^0(C, E) = h^0(C, A_1) + m$. Since E is spanned, but it has no trivial factor, $\deg(A_2) > 0$. Hence $m \geq 2$. The destabilizing exact sequence (2.1) is unique. Since A_1 is unique, the natural map $H^0(C, A_1) \to H^0(C, E)$ is injective and V is a general $(z-4)$-codimensional linear subspace of $H^0(C, E)$, we have $\dim(V \cap H^0(C, A_1)) = \max\{0, h^0(C, A_1) - z + 4\}$. Hence $\dim(V \cap H^0(C, A_1)) = \max\{0, 4 - m\}$ for a general $V \in G(4, H^0(C, E))$. Hence $(A_1, H^0(C, A_1) \cap V)$ α-destabilizes (E, V) for $\alpha \gg 0$ if and only if $m = 2$. Now we check that $m \geq 3$. Assume $m = 2$. Since A_2 is spanned and non-trivial and C is a Petri curve, $a_2 \geq \lfloor (g+3)/2 \rfloor$ and hence $a_1 \leq d - \lfloor (g+3)/2 \rfloor$. Since C is a Petri curve, $d < 2g-2$, $\deg(A_1) = a_1$ and $h^0(C, A_1) \geq 2 + x$, we get $g - (x+1)(g+x-a_1) \geq 0$. Thus $(x+1)(d - \lfloor (g+3)/2 \rfloor) \geq xg + x(x+1)$, contradicting one of our assumptions. Hence $m \geq 3$ and A_1 does not α-destabilizes (E, V) for general V when $\alpha \gg 0$. Take $M \in \Gamma \backslash \{A_1\}$ (if any). Let M' be the saturation of M. Since $H^0(C, M) \subseteq H^0(C, M')$ and $\deg(M') \geq \deg(M)$, $M' \in \Gamma$. If $(M, H^0(C, M) \cap V)$ α-destabilizes (E, V) for $\alpha \gg 0$ and general V, then $(M', H^0(C, M') \cap V)$ α-destabilizes (E, V) for $\alpha \gg 0$ and general V. Hence it is sufficient to handle the rank 1 saturated subsheaves. Hence from now on we assume $M = M'$. We may also assume $M \neq A_1$. Since $\deg(M) \leq a_2 < \deg(E)/2$, $\mu_\alpha(M, V \cap H^0(C, M))) < \mu_\alpha(E, V)$ for $\alpha \gg 0$ if $\dim(V \cap H^0(C, M)) \leq 2$. Fix any 4-dimensional linear subspace V of $H^0(C, E)$ such that V spans E and $\dim(V \cap H^0(C, A_1)) \leq 1$. Since M is saturated in E, E/M is a line bundle. Since V spans E the image V_M of V in E/M spans E. Since E/M is not trivial, $\dim(V_M) \geq 2$. Since $\dim(V \cap H^0(C, M)) = 4 - \dim(V_M)$, we get that M does not α-destabilizes (E, V) for large α. Hence $(E, V) \in G_\infty(C; 2, d, 4)$ for general V.

(c) Here we assume that E is properly semistable, i.e. that it fits in an exact sequence (2.1) with $a_1 = a_2 = d/2$. Hence d is even. We first assume that E is indecomposable. Hence the exact sequence (2.1) is unique and A_1 is the only degree $d/2$ line subbundle of E. Any $M \in \Gamma \backslash \{A_1\}$ has degree at most $d/2 - 1$. Thus as in part (b) we see that $(E, V) \in G_\infty(C; 2, d; 4)$ for every 4-dimensional linear subspace V of $H^0(C, E)$ such that V spans E and $\dim(V \cap H^0(C, A_1)) \leq 1$. Since $a_1 = d/2$, we have $g - (x+1)(x+g-a_1) < 0$. Hence $h^0(C, A_1) \leq x+1$. Hence $m \geq 3$. Hence as in part (b) we see that A_1 does not α-destabilize (E, V) for general V when $\alpha \gg 0$. Now we assume that E is decomposable, say $E \cong A_1 \oplus A_2$ with A_1 not isomorphic to A_2. Any $M \in \Gamma \backslash \{A_1, A_2\}$ has degree $\leq d/2 - 1$. Since $h^0(C, E) = h^0(C, A_1) + h^0(C, A_2)$, the proof in part (b) shows that $(E, V) \in G_\infty(C; 2, d; 4)$ for every 4-dimensional linear subspace V of $H^0(C, E)$ such that V spans E and $\dim(V \cap H^0(C, A_i)) \leq 1$ for $i = 1, 2$. Since $h^0(C, E) = h^0(C, A_1) + h^0(C, A_2)$, the condition "$\dim(V \cap H^0(C, A_i)) \leq 1$ for $i = 1, 2$" is satisfied by a general $V \in G(4, H^0(C, E))$ if $h^0(C, A_i) \geq 3$ for all i. This is true for the following reasons. Notice that $(x+1)d < (x+1)(2g+2x) - 2g$ and hence $g - (x+1)(g+x-d/2) < 0$. We have $h^0(C, A_i) \leq x+1$ for all i, because $a_i = d/2$ and hence $g - (x+1)(g + x - d/2) < 0$. Since $h^0(C, E) = h^0(C, A_1) + h^0(C, E) \geq 4 + x$, we get $h^0(C, A_i) \geq 3$ for all i, as wanted. Now we assume $E \cong A_1^{\oplus 2}$. Set $x_1 := h^0(X, A_1)$. Hence

$z = 2x_1$. Let Δ denote the set of all degree $d/2$ line subbundles of E. Notice that $\Delta \cong \mathbf{P}^1$ and that every $A \in \Delta$ is isomorphic to A_1. For any $A \in \Delta$ set $B_A := \{V \in G(4, H^0(C,E)) : \dim(V \cap H^0(C,A)) \geq 2\}$. Since $h^0(C,A) = z/2 \geq 2$, the Schubert cycle B_A has dimension $2(z/2-2)+2(z/2-2) = 2z-8$. Since $\dim(\Delta) = 1$, $\dim(G(4, H^0(C,E)) = 4(z-4)$, $z \geq 4+x$ and $x \geq 2$, we get $V \notin B_A$ for all $A \in \Delta$ if V is general in $G(4, H^0(C,E))$. Take any $V \in G(4, H^0(C,E))$ spanning E. As in the previous cases we see that $\dim(V \cap H^0(C,M)) \leq 2$ for all $M \in \Gamma\backslash\Delta$. □

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University of Trento, 38123 Povo (TN), ITALY
E-mail address: ballico@science.unitn.it