International Electronic Journal of Geometry
Volume 2 No. 2 pp. 58-62 (2009) ©IEJG

TWO DIMENSIONAL NULL SCROLLS IN $\mathbb{R}_1^n$ AND MASSEY'S THEOREM

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(Communicated by H. Hilmi HACISALİHOĞLU)

Abstract. In this study, we investigated new characteristic properties for two dimensional null scroll in the n-dimensional Lorentzian space $\mathbb{R}_1^n$ and we examined the sufficient and necessary conditions for the null scroll M is to be totally geodesic. We also gave the Massey's Theorem for the two dimensional null scroll in $\mathbb{R}_1^n$ which is well known for the ruled surfaces in the Euclidean 3-space [4, 5].

1. Introduction

Let M be an m-dimensional Lorentzian submanifold of $\mathbb{R}_1^n$. Let $\overline{\nabla}$ be a Levi-Civita connection of $\mathbb{R}_1^n$ and ∇ be a Levi-Civita connection of M. If X, $Y \epsilon\ \chi(\mathrm{M})$ and h is the second fundamental form of M, then we have the Gauss equation

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{1.1}$$

where $\chi(\mathrm{M})$ is the tangent bundle of M.

Let ξ be a unit normal vector field on M. Then the Weingarten equation is

$$\overline{\nabla}_X \xi = -A_\xi(X) + \nabla_X^\perp \xi \tag{1.2}$$

where A_ξ determines at each point a self-adjoint linear map on $T_p\mathrm{M}$ and $\nabla^\perp$ is a metric connection on normal bundle of M. In this paper, A_ξ will be used for the linear map and the corresponding matrix of the linear map. From the equations (1.1) and (1.2), we have

$$\left\langle \overline{\nabla}_X Y, \xi \right\rangle = \langle h(X,Y), \xi \rangle \tag{1.3}$$

and

$$\left\langle \overline{\nabla}_X Y, \xi \right\rangle = \langle A_\xi(X), Y \rangle \,. \tag{1.4}$$

Also, by the equations (1.3) and (1.4), we get

$$\langle h(X,Y), \xi \rangle = \langle A_\xi(X), Y \rangle \,. \tag{1.5}$$

2000 *Mathematics Subject Classification.* 53B30, 53A04.
Key words and phrases. Null scroll, Massey's Theorem.

Let $\{\xi_1, \xi_2, ..., \xi_{n-m}\}$ be an orthonormal basis of $\chi^{\perp}(\mathrm{M})$. Then there exist smooth functions $h^j(X,Y)$, $j = 1, ..., n-m$, such that

$$h(X,Y) = \sum_{j=1}^{n-m} h^j(X,Y)\xi_j \tag{1.6}$$

and furthermore we may define the mean curvature vector field H by

$$H = \sum_{j=1}^{n-m} \frac{traceA_{\xi_j}}{m}\xi_j \tag{1.7}$$

If $H(p) = 0$ for each $p \in$M, then M is said to be minimal [6].

Let ξ be a unit normal vector, then the Lipschitz-Killing curvature in the direction ξ at the point $p \in$M is defined by

$$G(p,\xi) = \det A_\xi(p). \tag{1.8}$$

The Gauss curvature is defined by

$$K(p) = \sum_{j=1}^{n-m} G(p,\xi_j) \tag{1.9}$$

If $K(p) = 0$ for all $p \in$M, then M is called developable. In particular, if the Lipschitz-Killing curvature is zero for each point and each normal direction, then M is developable [7].

Let M be a two dimensional null scroll in $\mathbb{R}_1^n$. If the tangent planes of M are constant along the generators of M,then M is called developable [1].

A normal vector field ξ is called parallel in the normal bundle $\chi^{\perp}(\mathrm{M})$ if we have $\nabla_X^{\perp}\xi = 0$ for each $X \in \chi(\mathrm{M})$ [6].

Massey's Theorem in the 3-dimensional Euclidean space $\mathbb{R}^3$ as follows

"*A complete, connected surface in R^3 is a developable surface if and only if K=0, where K is Gauss curvature.*"

In this paper, we will generalize the Massey's Theorem which is well known for the ruled surfaces in $\mathbb{R}^3$ for the two dimensional null scroll in the $\mathbb{R}_1^n$ as similar to the method in [5].

2. Two dimensional null scrolls in $\mathbb{R}_1^n$ and their characteristic properties

The construction of the $(r+1)-$dimensional generalized null scroll is given by following [1, 2].

Let α be a null curve in the $n-$dimensional Lorentzian space $\mathbb{R}_1^n$ and $\{X, Y, Z_1, ...Z_{n-2}\}$ be a pseudo-orthonormal frame along the null curve α such that $\alpha' = X$ and $< X, X >=< Y, Y >= 0$, $< X, Y >= -1$. Then the $(r+1)-$dimensional generalized null scroll M can be expressed by the following parametric equation

$$\Psi(t, u_0, ..., u_{r-1}) = \alpha(t) + u_0Y(t) + \sum_{i=1}^{r-1} u_iZ_i(t).$$

Now suppose that the base curve $\alpha(t)$ of the two dimensional null scroll M in $\mathbb{R}_1^n$ is a pseudo-orthogonal trajectory of the generators which have the direction of the null vector field $Y(t)$. Then M can be represented locally by

$$\Psi(t,u) = \alpha(t) + uY(t). \tag{2.1}$$

It is easy to check that M is a Lorentzian submanifold.

Definition 2.1. Let M be a two dimensional null scroll in $\mathbb{R}_1^n$ and h be the second fundamental form of M. If $h(W,W)=0$ for all $W\in\chi(\mathrm{M})$ then W is called an asymptotic vector field on M [1].

Definition 2.2. Let W be a null vector field on M. If $\nabla_W W=0$, then W is called null geodesic vector field of M [1].

Definition 2.3. Let M be a two dimensional null scroll in $\mathbb{R}_1^n$ and h be the second fundamental form of M. If $h(U,V)=0$ for all $U,V\in\chi(\mathrm{M})$ then M is called totally geodesic [1].

Theorem 2.1. *Let M be a two dimensional null scroll in $\mathbb{R}_1^n$. Then the null generators of M are asymptotics and null geodesics of M.*

Proof. Since the null generators are the null geodesic of $\mathbb{R}_1^n$,we have $\overline{\nabla}_Y Y=0$. Then from (1.1), we find $\nabla_Y Y=0$ and $h(Y,Y)=0$. Considering the definition 2.1. and definition 2.3., the null generators of M are the asymptotics and null geodesics of M. This completes the proof. □

Now suppose that $\{X,Y\}$ is an orthonormal basis field of the tangent bundle $\chi(\mathrm{M})$ and $\{\xi_1,\xi_2,...,\xi_{n-2}\}$ is an orthonormal basis field of the normal bundle $\chi^{\perp}(\mathrm{M})$. Then we can write the following equations:

$$\overline{\nabla}_Y\xi_j=a_{00}^jX+a_{01}^jY+\sum_{i=1}^{n-2}b_{1i}^j\xi_i,\qquad j=1,...,n-2 \tag{2.2}$$

$$\overline{\nabla}_X\xi_j=a_{10}^jX+a_{11}^jY+\sum_{i=1}^{n-2}b_{2i}^j\xi_i.$$

Thus we have

$$A_{\xi_j}=-\begin{bmatrix} a_{00}^j & a_{01}^j \\ a_{10}^j & a_{11}^j\end{bmatrix},\qquad j=1,...,n-2. \tag{2.3}$$

If we consider the equations (2.2)and theorem2.1, then we find

$$a_{00}^j=b_{1i}^j=b_{2i}^j=0,\qquad j=1,...,n-2.$$

$$a_{10}^j=a_{01}^j. \tag{2.4}$$

Hence

$$A_{\xi_j}=-\begin{bmatrix} 0 & a_{01}^j \\ a_{01}^j & a_{11}^j\end{bmatrix},\qquad j=1,...,n-2.$$

Also we have

$$traceA_{\xi_j}=-a_{11}^j=-\langle h(X,X),\xi_j\rangle,\qquad j=1,...,n-2. \tag{2.5}$$

Theorem 2.2. *Let M be a two dimensional null scroll in $\mathbb{R}_1^n$ and $\{X,Y\}$ be an orthonormal basis of $\chi(M)$. Then the Gauss curvature K is given by*

$$K=\left\langle\overline{\nabla}_YX,\overline{\nabla}_YX\right\rangle \tag{2.6}$$

Proof. The proof is clear from (1.1), (1.6) and (2.2), (See [1]). □

Considering the equation (2.6), we have

$$K = \sum_{j=1}^{n-2} (a^j_{01})^2 \tag{2.7}$$

and

$$H = -\sum_{j=1}^{n-2} \frac{\langle h(X,X), \xi_j \rangle \xi_j}{2} \tag{2.8}$$

Theorem 2.3. *Let M be a two dimensional null scroll in $\mathbb{R}^n_1$ Then M is developable and minimal if and only if M is totally geodesic.*

Proof. We suppose that M is developable and minimal. We can write $Z = aX + bY$ and $W = cX + dY$ for $Z, W \in \chi(\mathrm{M})$. Then we have

$$h(Z,W) = ac\ h(X,X) + (ad + bc)\ h(X,Y) + bd\ h(Y,Y). \tag{2.9}$$

According to Theorem 2.1 and the equation (2.8) we have $h(X,X) = 0$ and $h(Y,Y) = 0$. Moreover, since M is developable $\overline{\nabla}_Y X = 0$. Thus using (1.1), we get $h(Z,W) = 0$ for all $Z, W \in \chi(\mathrm{M})$.

Conversely, let us assume that $h(Z,W) = 0$ for all $Z, W \in \chi(\mathrm{M})$. Then we have $h(X,X) = h(Y,Y) = h(X,Y) = 0$. So from Theorem 2.1. we get $\langle \overline{\nabla}_Y X, X \rangle = 0$ and $\langle \overline{\nabla}_Y X, Y \rangle = 0$. This means that $\overline{\nabla}_Y X$ is a normal vector field or $\overline{\nabla}_Y X = h(X,Y)$. Therefore we have $\overline{\nabla}_Y X = 0$. This implies that M is developable and $h(X,X) = 0$, that is M is minimal. □

3. The Massesy's Theorem for two dimensional null scroll in $\mathbb{R}^n_1$

Let M be a two dimensional null scroll in $\mathbb{R}^n_1$ and $\{X, Y\}$ be an orthonormal basis of $\chi(\mathrm{M})$. Let $\{\xi_1, \xi_2, ..., \xi_{n-2}\}$be the spacelike orthonormal basis of $\chi^{\perp}(\mathrm{M})$. Then $\{X, Y, \xi_1, \xi_2, ..., \xi_{n-2}\}$ is the pseudo-orthonormal basis of $\chi(\mathbb{R}^n_1)$ at each point p of $\mathbb{R}^n_1$.

We have also the equations of covariant derivative of the pseudo-orthonormal basis $\{X, Y, \xi_1, \xi_2, ..., \xi_{n-2}\}$ of $\chi(\mathbb{R}^n_1)$ as follow

$$\begin{bmatrix} \overline{\nabla}_X X \\ \overline{\nabla}_X Y \\ \overline{\nabla}_X \xi_1 \\ \overline{\nabla}_X \xi_2 \\ \vdots \\ \overline{\nabla}_X \xi_{n-2} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 & c_{13} & 0 & \cdots & 0 \\ 0 & -c_{22} & -c_{23} & -c_{24} & \cdots & 0 \\ c_{31} & c_{32} & 0 & c_{34} & \cdots & 0 \\ c_{41} & 0 & -c_{43} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{n-2} \end{bmatrix}. \tag{2.10}$$

[1].

Theorem 3.1. *Let M be a two dimensional null scroll in $\mathbb{R}^n_1$ and $\{X, Y\}$ be an orthonormal basis of χ(M). Suppose that $\alpha(t)$ is a pseudo-orthogonal trajectory of the generators of M. Then the following propositions are equivalent*

(i) *M is developable.*

(ii) *The Lipschitz-Killing curvature $G(p, \xi_j) = 0,\ j = 1, 2, ...n-2$.*

(iii) *The Gauss curvature $K = 0$.*

(iv) *In the equation (2.10), $c_{31} = c_{41} = 0$.*

(v) *$A_{\xi_j}(X) = 0,\ j = 1, 2, ...n-2$.*

(vi) $\overline{\nabla}_X Y \in \chi(M)$.

Proof. **(i)** $\Longrightarrow$**(ii)** Suppose that M is developable. Then we have $a^j_{00} = 0$ and $a^j_{10} = a^j_{01}$ Since M is developable, we get $G(p, \xi_j) = 0,\ j = 1, 2, ...n - 2$.

(ii) $\Longrightarrow$**(iii)** Let $G(p, \xi_j) = 0,\ j = 1, 2, ...n - 2$. We know that $K(p) =_{j=1}^{n-2} G(p, \xi_j)$, $\forall p \in$M, so it is clear that $K = 0$, $\forall p \in$M.

(iii) $\Longrightarrow$**(iv)** Assume that $K = 0, \forall p \in$M. Then from (2.7) we have $a^j_{01} = 0,\ j = 1, 2, ...n - 2$. Hence $\overline{\nabla}_X \xi_j$ has no component in the direction X. Thus from (2.10) we find $c_{31} = c_{41} = 0$.

(iv) $\Longrightarrow$**(v)** Let $c_{31} = c_{41} = 0$. This means that $\overline{\nabla}_X \xi_j$ has no component in the direction X. Therefore we get $a^j_{01} = 0,\ j = 1, 2, ...n - 2$. We also find

$$A_{\xi_j} = -\begin{bmatrix} 0 & 0 \\ 0 & a^j_{11} \end{bmatrix}, \qquad j = 1, ..., n - 2$$

and this implies that $A_{\xi_j}(X) = 0,\ j = 1, 2, ...n - 2$.

(v) $\Longrightarrow$**(vi)** Suppose that $A_{\xi_j}(X) = 0,\ j = 1, 2, ...n - 2$. Then from the Weingarten equation we have $a^j_{00} = a^j_{01} = 0,\ j = 1, 2, ...n - 2$. Moreover we find $\left\langle \overline{\nabla}_X Y, \xi_j \right\rangle = 0$, that means $\overline{\nabla}_X Y \in \chi$(M).

(vi) $\Longrightarrow$**(i)** Let $\overline{\nabla}_X Y \in \chi$(M). Then $\left\langle \overline{\nabla}_X Y, \xi_j \right\rangle = -a^j_{01} = 0$. We also have $\left\langle \overline{\nabla}_Y X, X \right\rangle = 0$ and $\left\langle \overline{\nabla}_Y X, Y \right\rangle = 0$, this implies that $\overline{\nabla}_Y X \in \chi$(M) and since $\overline{\nabla}_Y X = \sum_{j=1}^{n-2} a^j_{01} \xi_j$, we get $\overline{\nabla}_Y X = 0$. This means that the tangent planes of M are constant along the generator Y of M, i.e., M is developable.

□

Corollary 3.1. *Let M be a two dimensional null scroll in* $\mathbb{R}^n_1$ *with Gauss curvature being zero. If M is minimal then* $c_{13} = c_{31} = c_{41} = 0$.

Proof. It can be easily done from the equations (1.1), (2.8), (2.10) and Theorem 3.1. □

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