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A NEW CHARACTERIZATION FOR INCLINED CURVES BY THE HELP OF SPHERICAL REPRESENTATIONS

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Abstract. In this work, arc lengths of spherical representations of tangent vector field T, principal normal vector field N, binormal vector field B and the vector field $\overrightarrow{C}=\frac{\overrightarrow{W}}{\left\|\overrightarrow{W}\right\|}$, where $\overrightarrow{W}=\tau\overrightarrow{T}+\kappa\overrightarrow{B}$ is the Darboux vector field of a space curve α in $E^3$are calculated. Let us denote the spherical representation of $\overrightarrow{T},\overrightarrow{N},\overrightarrow{B}$ and $\overrightarrow{C}$ by $\left(\overrightarrow{T}\right),\left(\overrightarrow{N}\right),\left(\overrightarrow{B}\right)and\left(\overrightarrow{C}\right)$, respectively.

The arc element ds$_c$ of the spherical representation $\left(\overrightarrow{C}\right)$ expressed in terms of the harmonic curvature $H=\frac{\kappa}{\tau}$. Thus the following characterization is given.

The curve $\alpha\subset E^3$ is an inclined curve if and only if the arc length s_c of the Darboux spherical representation $\left(\overrightarrow{C}\right)$ of α is constant.

1. Introduction

In recent years, many important and intensive studies are seen about inclined curves. Papers in [1] , [2] , ..., [21] show that how important field of interest inclined curves have. Let κ and τ be the curvatures of a curve in E^3 İn the generalization to E^n, $n\rangle 3$, they consider the following cases:

(a) $\kappa=e^{te}$ and $\tau=e^{te}$,

(b) $\kappa\neq e^{te}$ and $\tau\neq e^{te}$, but $H=\frac{\kappa}{\tau}=e^{te}$.

The case (a) for the generalization to E^n is not seen to be interesting.

However, by generalizing the harmonic curvature $H=\frac{\kappa}{\tau}$ to E^n, the works in (b) are more interesting [13] , [18] , [19]. For this reason, we have given a new characterization for the inclined curves which satisfy the case (b). This comes into light by means of spherical representations of α.

2. Characterizations for Ordinary Helices and Inclined Curves

2.1. The arc length of tangentian representation of the curve $\alpha\subset E^3$. Let $T=T(s)$ be the tangent vector field of the curve

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$$\alpha : \quad I \subset R \to E^3$$
$$s \to \alpha(s).$$

The spherical curve $\alpha_T = T$ on S^2 is called I.st spherical representation of the tangents of α.

Let s be the arc length parameter of α. If we denote the arc length of the curve α_T by s_T, then we may write

$$\alpha_T(s_T) = T(s).$$

Letting $\frac{d\alpha_T}{ds_T} = T_T$ we have $T_T = \kappa \overrightarrow{N} \frac{ds}{ds_T}$.Hence we obtain $\frac{ds_T}{ds} = \kappa$. Thus we give the following result. If κ is the first curvature of the curve $\alpha : I \to E^3$, then the arc length s_T of the tangentian representation α_T of α is

$$s_T = \int \kappa ds + c.$$

If the harmonic curvature of α is $H = \frac{\kappa}{\tau}$, we get

$$ds_T = \int \tau H ds + c$$

where c is an integral constant. Thus we have the following theorem.

Theorem 2.1. *$\alpha \subset E^3$ is an ordinary helix if and only if*

$$s_T = \tau H s + c.$$

2.2. **The Arc Length of the Principal Normal Representation of the Curve $\alpha \subset E^3$.** Let $\overrightarrow{N} = \overrightarrow{N}(s)$ be the principal normal vector field of the curve

$$\alpha : \quad I \subset R \to E^3$$
$$s \to \alpha(s).$$

The spherical curve $\alpha_N = \overrightarrow{N}$ on S^2 is called II.nd spherical representation for α or is called the spherical representation of the principal normals of α. Let $s \in I$ be the arc length of α. If we denote the arc length of α_N by s_N, we may write

$$\alpha_N(s_N) = \overrightarrow{N}(s).$$

Moreover letting $\frac{d\alpha_N}{ds_N} = T_N$, we obtain

$$T_N = (-\kappa \overrightarrow{T} + \tau \overrightarrow{B}) \frac{ds}{ds_N}$$

Hence we have

$$\frac{ds_N}{ds} = \sqrt{\kappa^2 + \tau^2}.$$

Note that $\sqrt{\kappa^2 + \tau^2}$ is the total curvature function of α. Therefore we reach the following result:

$$s_N = \int \sqrt{\kappa^2 + \tau^2} ds + c$$

or in terms of $H = \frac{\kappa}{\tau}$,

$$s_N = \int \tau \sqrt{1 + H^2} ds + c.$$

Thus we have the following theorem:

Theorem 2.2. *$\alpha \subset E^3$ is an ordinary helix if and only if*

$$s_N = \tau\sqrt{1+H^2}s + c.$$

2.3. **The Arc Length of Binormal Representation of the Curve $\alpha \subset E^3$.** Let $\overrightarrow{B} = \overrightarrow{B}(s)$ be the binormal vector field of the curve

$$\alpha : \quad I \subset R \rightarrow E^3$$
$$s \rightarrow \alpha(s).$$

The spherical curve $\alpha_B = \overrightarrow{B}$ on S^2 is called III.rd spherical representation for α or is called the spherical representation of the binormals of α.

Let $s \in I$ be the arc length parameter of α. If we denote the arc length parameter of α_B by s_B, we may write

$$\alpha_B(s_B) = \overrightarrow{B}(s).$$

Moreover letting $\frac{d\alpha_B}{ds_B} = T_B$, we obtain $T_B = -\tau N \frac{ds}{ds_B}$. Hence we have $\frac{ds_B}{ds} = \tau$ and $s_B = \int \tau ds + c$ or in terms of the harmonic curvature of α we obtain

$$s_B = \int \frac{\kappa}{H} ds + c.$$

Thus we give the following theorem:

Theorem 2.3. *$\alpha \subset E^3$ is an ordinary helix if and only if $s_B = \frac{\kappa}{H} ds + c$.*

2.4. **The Arc Length of Darboux Spherical Representation of the Curve $\alpha \subset E^3$.** Let $\overrightarrow{w} = \tau\overrightarrow{T} + \kappa\overrightarrow{B}$ be the Darboux vector field of the curve

$$\alpha : \quad I \subset R \rightarrow E^3$$
$$s \rightarrow \alpha(s).$$

Let us define the curve $\alpha_C = \overrightarrow{C}$ on S^2 by the help of the vector field $\overrightarrow{C} = \frac{\overrightarrow{W}}{\left\|\overrightarrow{W}\right\|}$.This curve is called IV.th spherical representation of α or is called the Darboux representation of α. Let s_C be the arc length of α_C. Then we have $\alpha_C = \overrightarrow{C}(s_C) = \frac{\overrightarrow{W}}{\left\|\overrightarrow{W}\right\|}$. Let us denote the angle between $\overrightarrow{W}$ and $\overrightarrow{T}$ by φ (see Figure 1).

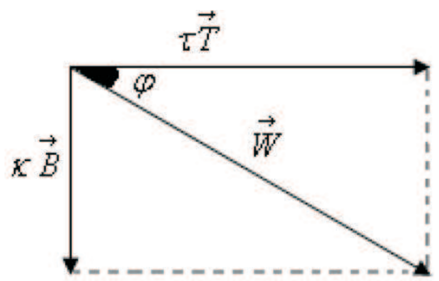

FIGURE 1

Hence

$$\kappa = \left\|\overrightarrow{W}\right\| \sin\varphi \quad \text{and} \quad \tau = \left\|\overrightarrow{W}\right\| \cos\varphi. \tag{1}$$

Therefore we may write

$$\overrightarrow{C} = \cos\varphi \overrightarrow{T} + \sin\varphi \overrightarrow{B}.$$

From this last equality we get

$$\frac{d\overrightarrow{C}}{ds} = \frac{d\overrightarrow{C}}{ds} . \frac{ds}{ds_C}$$

or

$$\frac{ds_C}{ds} = \left\|\frac{d\overrightarrow{C}}{ds}\right\|$$

or

$$\begin{aligned}\frac{d\overrightarrow{C}}{ds} &= (\cos\varphi)\overrightarrow{T}' + (\sin\varphi)'\overrightarrow{B} \\ &= (-\sin\varphi \overrightarrow{T} + \cos\varphi \overrightarrow{B})\frac{d\varphi}{ds}.\end{aligned}$$

Hence we have

$$\left\|\frac{d\overrightarrow{C}}{ds}\right\| = \frac{d\varphi}{ds} = \frac{ds_C}{ds}. \tag{2}$$

From this equations, in (1) we obtain

$$\frac{\kappa}{\tau} = \tan\varphi. \tag{3}$$

Therefore, differentiating with respect to s we have

$$\left(\frac{\kappa}{\tau}\right)' = (1 + \tan^2\varphi)\frac{d\varphi}{ds}$$

or

$$\left(\frac{\kappa}{\tau}\right)' = \left[1 + \left(\frac{\kappa}{\tau}\right)^2\right]\frac{d\varphi}{ds}.$$

From (3), since we have

$$\frac{d\varphi}{ds} = \frac{\left(\frac{\kappa}{\tau}\right)'}{1 + \left(\frac{\kappa}{\tau}\right)^2}$$

and since we have $H = \frac{\kappa}{\tau}$, we get

$$\frac{d\varphi}{ds} = \frac{H'}{1 + H^2}.$$

Hence from (2), we obtain

$$\frac{ds_C}{ds} = \frac{H'}{1 + H^2}$$

or hence

$$ds_C = \frac{H'}{1 + H^2} ds$$

$ds_C = \frac{H'}{1+H^2} ds$ implies that

$$s_C = \int \frac{H'}{1 + H^2} ds + c.$$

Since $H' = \frac{dH}{ds}$ implies $H'ds = dH$, then we have

$$s_C = Arc\tan H + c.$$

Thus we give the following theorem:

Theorem 2.4. *The curve $\alpha \subset E^3$ is an inclined curve if and only if $s_C = const.$*

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