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MULTIPLY WARPED PRODUCT SUBMANIFOLDS IN SASAKIAN SPACE FORMS

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Abstract. Recently, B. Y. Chen and F. Dillen established general sharp inequalities for multiply warped product submanifolds in arbitrary Riemannian manifolds. As applications, they obtained obstructions to minimal isometric immersions of multiply warped products into Riemannian manifolds.

In the present paper, we obtain inequalities for multiply warped products isometrically immersed in Sasakian space forms. Some applications are derived.

1. **Introduction**

Let N_1,...,N_k be Riemannian manifolds and let $N = N_1 \times ... \times N_k$ be the Cartesian product of N_1,...,N_k. For each i, denote by $\pi_i : N \to N_i$ the canonical projection of N onto N_i. When there is no confusion, we identify N_i with the horizontal lift of N_i in N via π_i.

If $\sigma_2, ..., \sigma_k : N_1 \to \mathbb{R}_+$ are positive-valued functions, then

$$< X, Y >=< \pi_{1*}X, \pi_{1*}Y > + \sum_{I=2}^{K} (\sigma_i \circ \pi_1)^2 < \pi_{i*}X, \pi_{i*}Y > \tag{1.1}$$

defines a Riemannian metric g on N called a multiply warped product metric. The product manifold N endowed with this metric is denoted by $N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$.

For a multiply warped product manifold $N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$, let $\mathcal{D}_i$ denote the distributions obtained from the vectors tangent to N_i (or more precisely, vectors tangent to the horizontal lifts of N_i).

Assume that

$$x : N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k \to \widetilde{M}$$

is an isometric immersion of a multiply warped product $N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ into a Riemannian manifold $\widetilde{M}$. Denote by h the second fundamental form of x. Then the immersion x is called *mixed totally geodesic* if $h(\mathcal{D}_i, \mathcal{D}_j) = \{0\}$ holds for distinct $i, j \in \{1, ..., k\}$.

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Let $\Psi : N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k \to \widetilde{M}$ denote an isometric immersion of a multiply warped product $N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ into an arbitrary Riemannian manifold $\widetilde{M}$.

Denote by trace h_i the trace of h restricted to N_i, that is

$$\text{trace}h_i = \sum_{\alpha=1}^{n} h(e_\alpha, e_\alpha)$$

for some orthonormal frame fields $e_1, ..., e_{n_i}$ of $\mathcal{D}_i$.

In [3], B. Y. Chen and F. Dillen established the following general inequality for arbitrary isometric immersions of multiply warped product manifolds in arbitrary riemannian manifolds.

Theorem 1.1. *Let $x : N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k \to \widetilde{M}^m$ be an isometric immersion of a multiply warped product $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ into an arbitrary Riemannian m-manifold. Then we have*

$$\sum_{j=2}^{k} n_j \frac{\Delta \sigma_j}{\sigma_j} \le \frac{n^2}{4} ||H||^2 + n_1 (n - n_1) \max \widetilde{K}, \quad n = \sum_{j=1}^{n} n_j, \tag{1.2}$$

where $\max \widetilde{K}(p)$ denotes the maximum of the sectional curvature function of $\widetilde{M}^m$ restricted to 2-planes sections of the tangent space T_pN of N at $p = (p_1, ..., p_k)$.

The equality of (1.2) holds identically if and only if the following two statements hold:

(1) *x is a mixed totally geodesic immersion satisfying*

$$\text{trace}h_1 = ... = \text{trace}h_k$$

(2) *at each point $p \in N$, the sectional curvature function $\widetilde{K}$ of $\widetilde{M}^m$ satisfies $\widetilde{K}(u, v) = \max \widetilde{K}(p)$ for each unit vector u in $T_{p_1}(N_1)$ and each unit vector v in $T_{(p_2,...,p_k)}(N_2 \times ... \times N_k)$.*

We prove a similar inequality for multiply warped product submanifolds of a Sasakian space form.

In the following, a multiply warped product $N_\top \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ in a Sasakian space form $\widetilde{M}(c)$ is called a *multiply CR-warped product* if $N_\top$ is an invariant submanifold tangent to ξ and $N_\perp :=_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ is a C-totally real submanifold of $\widetilde{M}(c)$.

In [3], B. Y. Chen and F. Dillen also proved that for any multiply CR-warped product $N_\top \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ in an arbitrary Kaehler manifold $\widetilde{M}$ the second fundamental form h and the warping functions $\sigma_2, ..., \sigma_k$ satisfy:

$$||h||^2 \ge 2 \sum_{i=2}^{k} n_i ||\nabla (\ln \sigma_i)||^2, \tag{1.3}$$

where $n_i = \dim N_i$ $(i = 2, ..., k)$ and $\nabla (\ln \sigma_i)$ is the gradient of $\ln \sigma_i$ $(i = 2, ..., k)$.

The second purpose of this article is to obtain a similar inequality for multiply CR-warped products in Sasakian space forms.

2. **Preliminares**

In this section, we recall some definitions and basic formulas which we will use later.

A $(2m+1)$-dimensional Riemannian manifold $\left(\widetilde{M},g\right)$ is said to be an *almost contact metric manifold* if there exist on $\widetilde{M}$ a $(1,1)$-tensor field ϕ, a vector field ξ (called the structure vector field) and a 1-form η such that $\eta(\xi)=1$, $\phi^2(X)=-X+\eta(X)\xi$ and $g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y)$, for any vector fields X,Y on $\widetilde{M}$. In particular, on an almost contact metric manifold we also have $\phi\xi=0$ and $\eta\circ\phi=0$.

We denote an almost contact metric manifold by $\left(\widetilde{M},\phi,\xi,\eta,g\right)$.

Such a manifold is said to be a *contact metric manifold* if $d\eta=\Phi$, where $\Phi(X,Y)=g(\phi X,Y)$ is called the fundamental 2-form of $\widetilde{M}$.

On the other hand, the almost contact metric structure of $\widetilde{M}$ is said to be normal if $[\phi,\phi](X,Y)=-2d\eta(X,Y)\xi$, for any X,Y, where $[\phi,\phi]$ denotes the Nijenhuis torsion of ϕ, given by

$$[\phi,\phi](X,Y)=\phi^2[X,Y]+[\phi X,\phi Y]-\phi[\phi X,Y]-\phi[X,\phi Y].$$

A normal contact metric manifold is called a *Sasakian manifold.*

A plane section π in $T_p\widetilde{M}$ is called a *ϕ-section* if it is spanned by X and ϕX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a ϕ-section is called a ϕ-sectional curvature. A Sasakian manifold with constant ϕ-sectional curvature c is said to be a *Sasakian space form* and is denoted by $\widetilde{M}(c)$.

In such a case, its Riemann curvature tensor is given by equation

$$\widetilde{R}(X,Y)Z=\frac{c+3}{4}\{g(Y,Z)X-g(X,Z)Y\}+$$

$$+\frac{c-1}{4}\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z\}+ \tag{2.1}$$

$$+\frac{c-1}{4}\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\}.$$

Let M be an n-dimensional submanifold in an almost contact metric manifold $\left(\widetilde{M},\phi,\xi,\eta,g\right)$.

We denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi\subset T_pM$, $p\in M$, and ∇ the Riemannian connection of M, respectively. Also, let h be the second fundamental form and R the Riemann curvature tensor of M.

Then the equation of Gauss is given by

$$\widetilde{R}(X,Y,Z,W)=R(X,Y,Z,W)+g(h(X,W),h(Y,Z))-g(h(X,Z),h(Y,W)) \tag{2.2}$$

for any vectors X, Y, Z, W tangent to M.

Let $p\in M$ and $\{e_1,...,e_n,e_{n+1},...,e_{2m+1}\}$ an orthonormal basis of the tangent space $T_p\widetilde{M}$, such that $e_1,...,e_n$ are tangent to M at p. We denote by H the mean curvature vector, that is

$$H=\frac{1}{n}\sum_{i=1}^{n}h(e_i,e_i). \tag{2.3}$$

As is known, M is said to be minimal if H vanishes identically.

Also, we set

$$h_{ij}^r=g(h(e_i,e_j),e_r),\ i,j\in\{1,...,n\},\ r\in\{n+1,...,2m+1\} \tag{2.4}$$

the coefficients of the second fundamental form h with respect to $\{e_1, ..., e_n, e_{n+1}, ..., e_{2m+1}\}$, and

$$||h||^2 = \sum_{i,j=1}^{n} g\left(h\left(e_i, e_j\right), h\left(e_i, e_j\right)\right). \tag{2.5}$$

Let M be a Riemannian n-manifold and $\{e_1, ..., e_n\}$ be an orthonormal basis of M. For a differentiable function f on M, the Laplacian Δf of f is defined by

$$\Delta f = \sum_{j=1}^{n} \{\left(\nabla_{e_j} e_j\right) f - e_j e_j f\}. \tag{2.6}$$

Recall that if M is compact, every eigenvalue of Δ is non-negative.

A *warped product immersion* is defined as follows: Let $M_1 \times_{\rho_2} M_2 \times ... \times_{\rho_k} M_k$ be a warped product and let $\Psi_i : N_i \to M_i$, $i = 1, ..., k$, be isometric immersions, and define $\sigma_i = \rho_i \circ \Psi_1 : N_1 \to R_+$ for $i = 2, ..., k$. Then the map

$$\Psi : N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k \to M_1 \times_{\rho_2} M_2 \times ... \times_{\rho_k} M_k$$

given by

$$\Psi\left(x_1, ..., x_k\right) = \left(\Psi_1\left(x_1\right), ..., \Psi_k\left(x_k\right)\right)$$

is an isometric immersion, which is called a *warped product immersion* [9].

A submanifold M normal to ξ in a Sasakian manifold $\widetilde{M}$ is said to be a C-totally real submanifold. It follows that ϕ maps any tangent space of M into the normal space, that is $\phi\left(T_p M\right) \subset T_p^{\perp} M$, for every $p \in M$.

A submanifold M tangent to ξ in a Sasakian space form $\widetilde{M}$ is called an invariant submanifold if ϕ preserves any tangent space of M, that is $\phi\left(T_p M\right) \subset T_p M$, for every $p \in M$.

Let n be a natural number ≥ 2 and let $n_1, ..., n_k$ be k natural numbers. If $n_1 + ... + n_k = n$, then $(n_1, ..., n_k)$ is called a partition of n.

We recall the following general algebraic lemma from [2] for later use.

Lemma 2.1. *Let $a_1, ..., a_n$ be n real numbers and let k be an integer in $[2, n-1]$. Then, for any partition $(n_1, ..., n_k)$ of n, we have*

$$\sum_{1 \leq i_1 < j_1 \leq n_1} a_{i_1} a_{j_1} + \sum_{n_1+1 \leq i_2 < j_2 \leq n_1+n_2} a_{i_2} a_{j_2} + ... + \sum_{n_1+...+n_{k-1}+1 \leq i_1 < j_1 \leq n} a_{i_k} a_{j_k} \geq$$

$$\geq \frac{1}{2k}\left[\left(a_1 + ... + a_n\right)^2 - k\left(a_1^2 + ... + a_n^2\right)\right],$$

with the equality holding if and only if

$$a_1 + ... + a_{n_1} = ... = a_{n_1+...+n_{k-1}+1} + ... + a_n.$$

3. C-totally real multiply warped product submanifolds in Sasakian space forms

Recently, Bang-Yen Chen and Franki Dillen established a sharp relationship between the warping functions $\sigma_2, ..., \sigma_k$ of a multiply warped product $N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ isometrically immersed in an arbitrary Riemannian manifold and the squared mean curvature $||H||^2$ (see [3], [6]).

We prove a similar inequality for multiply warped product submanifolds of a Sasakian space form.

In this section, we investigate C-totally real multiply warped product submanifolds in a Sasakian space form $\widetilde{M}(c)$.

Theorem 3.1. *Let x be a C-totally real isometric immersion of an n-dimensional multiply warped product $N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ into an $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$. Then:*

$$\sum_{j=2}^{k} n_j \frac{\Delta\sigma_j}{\sigma_j} \leq \frac{n^2}{4}||H||^2 + n_1(n-n_1)\frac{c+3}{4}, \quad n = \sum_{j=1}^{n} n_j. \tag{3.1}$$

The equality sign of (3.1) *holds identically if and only if x is a mixed totally geodesic immersion and*

$$\mathrm{trace} h_1 = ... = \mathrm{trace} h_k \tag{3.2}$$

holds, where trace h_i *denotes the trace of h restricted to N_i.*

Proof. Let $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ be the Riemannian product of the Riemannian manifolds $N_1, ..., N_k$.

We know (see [3]) that the sectional curvature function of the multiply warped product $N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ satisfies

$$K(X_1 \wedge X_i) = \frac{1}{\sigma_i}\left((\nabla_{X_1} X_1)\sigma_i - X_1^2 \sigma_i\right), \tag{3.3}$$

$$K(X_i \wedge X_j) = -\frac{g(\nabla\sigma_i, \nabla\sigma_i)}{\sigma_i \sigma_j}, \ i, j = 2, ..., k, \tag{3.4}$$

for each unit vector X_i tangent to N_i, where $\nabla\sigma$ denotes the gradient of σ.

In particular, (3.3) implies that, for each $i = 2, ..., k$, we have

$$\Delta\sigma_i = \sigma_i \sum_{j=1}^{n_1} K(e_j \wedge X_i), \tag{3.5}$$

for any unit vector X_i tangent to N_i, where $\{e_1, ..., e_{n_1}\}$ is an orthormal basis of $T_{\pi_1(p)} N_1$.

From the equation of Gauss, we have

$$2\tau = n^2||H||^2 - ||h||^2 + n(n-1)\frac{c+3}{4}, \tag{3.6}$$

where $n_i = \dim N_i$, $n = n_1 + ... + n_k$, τ is the scalar curvature, H is the mean curvature and h is the second fundamental form of N in $\widetilde{M}(c)$.

Let us put

$$\eta = 2\tau - n(n-1)\frac{c+3}{4} - n^2\left(1 - \frac{1}{k}\right)||H||^2. \tag{3.7}$$

Then it follows from (3.6) and (3.7) that

$$n^2||H||^2 = k\left(\eta + ||h||^2\right). \tag{3.8}$$

Let us also put

$$\Delta_1 = \{1, ..., n_1\}, ..., \Delta_k = \{n_1 + ... + n_{k-1} + 1, ..., n_1 + ... + n_k\}.$$

For a given point $p \in N$ we choose an orthonormal basis $e_1, ..., e_{2m+1}$ at p such that, for each $j \in \Delta_i$, e_j is tangent to N_i for $i = 1, ..., k$. Moreover, we choose the normal vector e_{n+1} in the direction of the mean curvature vector at p (when the mean curvature vanishes at p, e_{n+1} can be chosen to be any unit normal vector at p) and $e_{2m+1}=\xi$.

Then we get from (3.8) that

$$\left(\sum_{A=1}^{n} a_A\right)^2 - k\sum_{A=1}^{n}(a_A)^2 = k\left[\eta + \sum_{A\neq B}\left(h_{AB}^{n+1}\right)^2 + \sum_{r=n+2}^{2m}\sum_{A,B=1}^{n}\left(h_{AB}^{r}\right)^2\right], \tag{3.9}$$

where $a_A = h_{AA}^{n+1}$ and $h_{AB}^r =< h(e_A,e_B), e_r >$ with $1\le A,B\le n$ and $n+1\le r\le 2m$.

Because $(n_1,...,n_k)$ is a partition of n, we may apply Lemma 2 to (3.9). From this we obtain

$$\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1} + \sum_{\alpha_2<\beta_2} a_{\alpha_2}a_{\beta_2} + ... + \sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k} \ge$$

$$\ge \frac{\eta}{2} + \sum_{A<B}\left(h_{AB}^{n+1}\right)^2 + \frac{1}{2}\sum_{r=n+2}^{2m}\sum_{A,B=1}^{n}\left(h_{AB}^{r}\right)^2, \tag{3.10}$$

where α_i, $\beta_i\in\Delta_i$, $i=1,...,k$.

On the other hand, from the equation of Gauss and (3.5), we find

$$\sum_{i=2}^{k} n_i\frac{\Delta\sigma_i}{\sigma_i} = \sum_{j\in\Delta_1}\sum_{\beta\in\Delta_2\cup...\cup\Delta_k} K(e_j\wedge e_\beta) =$$

$$= \tau - \sum_{1\le j_1<j_2\le n_1} K(e_{j_1}\wedge e_{j_2}) - \sum_{n_1+1\le\alpha<\beta\le n} K(e_\alpha\wedge e_\beta) =$$

$$= \tau - \frac{n_1(n_1-1)(c+3)}{8} - \sum_{r=n+1}^{2m}\sum_{1\le j_1<j_2\le n_1}\left(h_{j_1j_1}^r h_{j_2j_2}^r - \left(h_{j_1j_2}^r\right)^2\right) -$$

$$-\frac{n_1(n-n_1-1)(c+3)}{8} - \sum_{r=n+1}^{2m}\sum_{n_1+1\le\alpha<\beta<n}\left(h_{\alpha\alpha}^r h_{\beta\beta}^r - \left(h_{\alpha\beta}^r\right)^2\right). \tag{3.11}$$

Therefore, by combining (3.7), (3.10) and (3.11), we obtain

$$\sum_{i=2}^{k} n_i\frac{\Delta\sigma_i}{\sigma_i} \le \tau - \frac{n(n-1)(c+3)}{8} + \frac{n_1(n-n_1)(c+3)}{4} - \frac{\eta}{2} -$$

$$-\frac{1}{2}\sum_{r=n+2}^{2m}\sum_{A,B=1}^{n}\left(h_{AB}^r\right)^2 - \sum_{\substack{1\le j\le n_1\\ n_1+1\le\alpha<n}}\left(h_{j\alpha}^{n+1}\right)^2 +$$

$$+\sum_{r=n+2}^{2m}\sum_{1\le j_1<j_2\le n_1}\left(\left(h_{j_1j_2}^r\right)^2 - h_{j_1j_1}^r h_{j_2j_2}^r\right) +$$

$$+\sum_{r=n+2}^{2m}\sum_{n_1+1\le\alpha<\beta<n}\left(\left(h_{\alpha\beta}^r\right)^2 - h_{\alpha\alpha}^r h_{\beta\beta}^r\right) =$$

$$= \tau - \frac{n(n-1)(c+3)}{8} + \frac{n_1(n-n_1)(c+3)}{4} - \frac{\eta}{2} -$$

$$-\sum_{r=n+1}^{2m}\sum_{1\leq j\leq n_1}\sum_{n_1+1\leq\alpha\leq n}\left(h_{j\alpha}^r\right)^2-$$

$$-\frac{1}{2}\sum_{r=n+2}^{2m}\left(\sum_{1\leq j\leq n_1}h_{jj}^r\right)^2-\frac{1}{2}\sum_{r=n+2}^{2m}\left(\sum_{n_1+1\leq\alpha\leq n}h_{\alpha\alpha}^r\right)^2\leq$$

$$\leq\tau-\frac{n\left(n-1\right)\left(c+3\right)}{8}+\frac{n_1\left(n-n_1\right)\left(c+3\right)}{4}-\frac{\eta}{2}=$$

$$=\frac{n^2}{4}||H||^2+\frac{n_1\left(n-n_1\right)\left(c+3\right)}{4}, \tag{3.12}$$

which proves the inequality (3.1).

If the equality sign of (3.1) holds, then all of inequalities in (3.10) and (3.12) become equalities. Hence, by applying Lemma 2, we know that the equality sign of (3.1) holds if and only if the immersion is mixed totally geodesic and trace $h_1=...=$ trace h_k hold identically.

The converse statement is straightforward. ■

As applications, we derive certain obstructions to the existence of minimal C-totally real multiply warped product submanifolds in Sasakian space forms.

Corollary 3.2. *If $\sigma_2,...,\sigma_k$ are harmonic functions on N_1, then:*

(i) *$N_1\times_{\sigma_2}N_2\times...\times_{\sigma_k}N_k$ admits no minimal C-totally real immersion into a Sasakian space form $\widetilde{M}\left(c\right)$ with $c<-3$.*

(ii) *Every minimal C-totally real immersion of $N_1\times_{\sigma_2}N_2\times...\times_{\sigma_k}N_k$ in the standard Sasakian space form $\mathbb{R}^{2m+1}$ is a warped product immersion.*

Proof. Assume $\sigma_2,...,\sigma_k$ are harmonic functions on N_1 and $N_1\times_{\sigma_2}N_2\times...\times_{\sigma_k}N_k$ admits a minimal C-totally real immersion into a Sasakian space form $\widetilde{M}\left(c\right)$.

Then, the inequality (3.1) becomes $c\geq-3$.

If $c=-3$, the equality case of (3.1) holds. By Theorem 3.1, it follows that $N_1\times_{\sigma_2}N_2\times...\times_{\sigma_k}N_k$ is mixed totally geodesic. Hence, by applying a well-known result of Nölker [9], we know that the immersion is a warped product immersion. ■

Corollary 3.3. *If $\sigma_2,...,\sigma_k$ are eigenfunctions of the Laplacian Δ on N_1 with nonnegative eigenvalues, then $N_1\times_{\sigma_2}N_2\times...\times_{\sigma_k}N_k$ admits no minimal C-totally real immersion into a Sasakian space form $\widetilde{M}\left(c\right)$ with $c\leq-3$.*

Example 3.4. *Let $f:M^2\rightarrow S^{n+3}\left(1\right)$ be a minimal isometric immersion. Then the following warped product immersion*

$$x:\left(0,\frac{\pi}{2}\right)\times_{\cos t}M^2\times_{\sin t}S^{n-3}\left(1\right)\rightarrow S^{2n+1}\left(1\right)$$

$$\left(t,p,q\right)\mapsto\cos tf\left(p\right)+\sin tq$$

is a minimal C-totally real immersion which satisfies the equality case of (3.1).

Therefore, inequality (3.1) *is optimal.*

4. An inequality for the squared norm of the second fundamental form

Recently, Bang-Yen Chen and Franki Dillen established a sharp relationship between the warping functions $\sigma_2, ..., \sigma_k$ of a multiply warped product $N_\top \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ and the squared norm of the second fundamental form $||h||^2$ (see [3]).

Later, the author obtained a similar inequality for doubly CR-warped products isometrically immersed in Sasakian space forms (see [8]).

In the present section, we will give an interesting inequality for the squared norm of the second fundamental form (an extrinsic invariant) in terms of the warping functions (intrinsic invariants) for multiply CR-warped products isometrically immersed in Sasakian manifolds.

Theorem 4.1. *Let $N = N_\top \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ be an n-dimensional multiply CR-warped product in a Sasakian manifold $\widetilde{M}$, such that $N_\top$ is tangent to ξ. Then the second fundamental form h and the warping functions $\sigma_2, ..., \sigma_k$ satisfy*

$$||h||^2 \geq 2\sum_{i=2}^{k} n_i[||\nabla (\ln \sigma_i)||^2 + 1], \tag{4.1}$$

where $n_i = \dim N_i$ $(i = 2, ..., k)$ and $\nabla (\ln \sigma_i)$ is the gradient of $\ln \sigma_i$ $(i = 2, ..., k)$.

The equality sign of (4.1) holds identically if and only if the following statements hold:

1. *$N_\top$ is a totally geodesic submanifold of $\widetilde{M}$;*
2. *For each $i \in \{2, ..., k\}$, N_i is a totally umbilical submanifold of $\widetilde{M}$;*
3. *$N_\perp :=_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ is immersed as mixed totally geodesic submanifold in $\widetilde{M}$;*
4. *For each $p \in N$, the first normal space Imh_p is a subspace of $\phi(T_pN_\perp)$;*
5. *N is a minimal submanifold of $\widetilde{M}$.*

Proof. Let $N = N_\top \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ be an n-dimensional multiply CR-warped product of a Sasakian manifold $\widetilde{M}$, such that $N_\top$ is an invariant submanifold tangent to ξ and $N_\perp :=_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ is a C-totally real submanifold of $\widetilde{M}$.

Let $\mathcal{D}_\top, \mathcal{D}_2, ..., \mathcal{D}_k, \mathcal{D}_\perp$ denote the distributions obtained from vectors tangent to $N_\top, N_2, ..., N_k, N_\perp$, respectively.

Let $\widehat{\nabla}$, ∇ denote the Levi-Civita connections of the Riemannian product $N_\top \times N_2 \times ... \times N_k$ and of the multiply warped product $N_\top \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$.

If we put $H_i = -\nabla((\ln \sigma_i) \circ \pi_1)$, then we have (cf. [9])

$$\nabla_X Y - \widehat{\nabla}_X Y = \sum_{i=2}^{k} \left(g\left(X^i, Y^i\right) H_i - g\left(H_i, X\right) Y^i - g\left(H_i, Y\right) X^i \right), \tag{4.2}$$

where X^i denotes the N_i-component of X.

Since $N_\top \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ is a multiply warped product, (4.2) implies that $N_\top$ is totally geodesic in N. Thus we have

$$g\left(\nabla_X Z, Y\right) = g\left(\nabla_X Y, Z\right) = 0,$$

for any vector fields X, Y in $\mathcal{D}_\top$ and Z in $\mathcal{D}_\perp$.

(4.2) also implies that

$$\nabla_X Z = \sum_{i=2}^{k} \left(X\left(\ln \sigma_i\right)\right) Z^i, \tag{4.3}$$

for any vector fields X in $\mathcal{D}_\top$ and Z in $\mathcal{D}_\perp$, where Z^i denotes the N_i-component of Z.

By applying (4.3) we find

$$\begin{aligned} g\left(h\left(\phi X, Z\right), \phi W\right) &= g\left(\widetilde{\nabla}_Z \phi X, \phi W\right) = g\left(\phi \widetilde{\nabla}_Z X, \phi W\right) = \\ &= g\left(\widetilde{\nabla}_Z X, W\right) = g\left(\nabla_Z X, W\right) = \sum_{i=2}^{k} \left(X\left(\ln \sigma_i\right)\right) g(Z^i, W^i), \end{aligned} \tag{4.4}$$

for any vector fields X in $\mathcal{D}_\top$ and Z, W in $\mathcal{D}_\perp$.

On the other hand, since the ambient manifold $\widetilde{M}$ is Sasakian, it is easily seen that

$$h\left(\xi, Z\right) = \phi Z = \sum_{i=2}^{k} \phi Z^i. \tag{4.5}$$

For a given point $p \in N$ we may choose an orthonormal basis $e_1, ..., e_n$ at p such that e_α is tangent to N_i for each $\alpha \in \Delta_i$, $i = 2, ..., k$. For each $i \in \{2, ..., k\}$, (4.4) implies that

$$\sum_{\alpha \in \Delta_i} g\left(h\left(\phi X, e_\alpha\right), \phi e_\alpha\right) = n_i \sum_{i=2}^{k} X\left(\ln \sigma_i\right). \tag{4.6}$$

Now, inequality (4.1) follows from (4.5) and (4.6).

It follows from (4.6) that the equality sign of (4.1) holds identically if and only if we have

$$h\left(\mathcal{D}_\top, \mathcal{D}_\top\right) = \{0\},\ h\left(\mathcal{D}_\perp, \mathcal{D}_\perp\right) = \{0\},\ h\left(\mathcal{D}_\top, \mathcal{D}_\perp\right) \subset \phi\mathcal{D}_\perp. \tag{4.7}$$

Because $N_\top$ is totally geodesic in $N = N_\top \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ the first condition in (4.7) implies that $N_\top$ is totally geodesic in $\widetilde{M}$. This gives statement (1).

From (4.2) we know that, for any $2 \leq i \neq j \leq k$, and any vector field Z_i in $\mathcal{D}_i$ and Z_j in $\mathcal{D}_j$ we have $\nabla_{Z_i} Z_j = 0$. This yields

$$g\left(\nabla_{Z_i} W_i, Z_j\right) = 0.$$

Thus, if $\widehat{h}_i$ denotes the second fundamental form of N_i in N we have

$$\widehat{h}_i\left(\mathcal{D}_i, \mathcal{D}_i\right) \subset \mathcal{D}_\top. \tag{4.8}$$

From (4.4) and (4.8) we find

$$\widehat{h}_i\left(Z_i, W_i\right) = -\left(X\left(\ln \sigma_i\right)\right) g\left(Z_i, W_i\right),$$

for Z_i, W_i tangent to N_i. Therefore, by combining the first condition in (4.7) and (4.8) we obtain statement (2).

Statement (3) follows immediately from (4.2) and the second condition in (4.7).

Statement (4) follows from (4.7).

Moreover, by (4.7), it follows that N is a minimal submanifold of $\widetilde{M}$. ∎

Corollary 4.2. *Let $\widetilde{M}(c)$ be a $(2m+1)$-dimensional Sasakian space form of constant ϕ-sectional curvature c and $N = N_{\top} \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ an n-dimensional non-trivial multiply warped product submanifold, such that $N_{\top}$ is an invariant submanifold tangent to ξ and $N_{\perp} :=_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ is a C-totally real submanifold of $\widetilde{M}(c)$ satisfying*

$$||h||^2 = 2\sum_{i=2}^{k} n_i[||\nabla(\ln \sigma_i)||^2 + 1].$$

Then, we have:

1. *$N_{\top}$ is a totally geodesic invariant submanifold of $\widetilde{M}(c)$. Hence $N_{\top}$ is a Sasakian space form of constant ϕ-sectional curvature c.*
2. *$N_{\perp} :=_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ is a totally umbilical C-totally real submanifold of $\widetilde{M}(c)$. Hence $N_{\perp}$ is a real space form of sectional curvature $\varepsilon \geq \frac{c+3}{4}$.*

Proof. Statement (1) follows from Theorem 4.1.

Also, we know that $N_{\perp}$ is a totally umbilical submanifold of $\widetilde{M}(c)$. Gauss equation implies that $N_{\perp}$ is a real space form of constant sectional curvature $\varepsilon \geq \frac{c+3}{4}$. ∎

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