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A LEVEL SET METHOD-BASED DERIVATION OF DIFFERENTIAL EQUATION FOR DEVELOPABLE SURFACES

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Abstract. In this paper, an approach to derive the partial differential equation of the developable surfaces in $\Re^3$ based on Level Set Methods as originally proposed by Osher and Sethian is described. The formulations are based on first converting the developable surface into a collection of planar level curves, which determine a Curve Evolution Problem. Then using the Level Set Method argument we obtain Eikonal Equation the differential equation of developable surfaces.

1. Introduction

Level set methods have received lot of attention in the recent literature in numerical analysis and other allied fields. These methods are powerful numerical techniques for analyzing and computing interface motion in a host of settings. They rely on a fundamental shift in how one views moving boundary or curve evolution problems, rethinking the natural geometric Lagrangian perspective and replacing it with an Eulerian partial differential equations perspective[3]. In this paper, we will use this technique to give a derivation of the partial differential equation corresponding to the developable surfaces. Given the well known parametric formulation of the developable surface we will convert its representation into the Eulerain form. It is also known that the developable surface is represented by the Eikonal Equation[1].

Here we describe an approach to derive the Eikonal equations of the developable surfaces in $\Re^3$ based on Level Set Methods. The formulations are based on first converting the developable surface into a collection of level curves, which determine a Curve Evolution Problem. Then using the Level set method argument the Eikonal Equation will be obtained.

One of the main features of the proposed derivation is its simplicity. Moreover, by the conversion of a surface to curve evolution problem we can also discuss the behavior of the weak solutions in the sense of viscosity solutions associated with the partial differential equation and also the points of singularity on the surface[4].

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We dedicate this paper to Bhagavan Sri Satya Sai Baba.

The article is organized as follows. In the section 2 we state the necessary definitions and relevant results for the derivation in the later section. Intuition behind the derivation is also discussed in this section. In Section 3, the derivation is presented. We follow the notation as in [2].

2. Developable Surface and Level set method

2.1. Developable Surfaces.

Definition 2.1. Developable Surface A surface having the Gaussian curvature to be zero at every point is called a Developable Surface.

Given a parametrized space curve $\alpha : I \to \Re^3$ we define $\phi(r,s) = \alpha(s) + r(\sin\theta\vec{n} + \cos\theta\vec{b})$, where α will be called the base curve, r belongs to an open interval containing zero (with out loss of generality we can assume $(-\epsilon, \epsilon)$) and θ is the angle between $\vec{n}(s)$, the normal of the curve at $\alpha(s)$ and $\bar{N}(\alpha(s))$, the normal to the surface at $\phi(0,s)$.

Theorem 2.1. *The surface $\phi(r,s)$ is developable iff $\alpha(s)$ is planar.*

Proof. By a simple computation we can obtain the following first fundamental forms:

$$\begin{aligned} E &= 1 \\ F &= 0 \\ G &= (1 - r\kappa\sin\theta)^2 + r^2\tau^2 \end{aligned}$$

Also the second fundamental forms:

$$\begin{aligned} L &= \phi_{rr} \cdot \bar{N} = 0 \\ M &= \phi_{rs} \cdot \bar{N} = \frac{-\tau}{[(1 - r\kappa\sin\theta)^2 + r^2\tau^2]^{\frac{1}{2}}} \\ N &= \frac{A + B - C}{[(1 - r\kappa\sin\theta)^2 + r^2\tau^2]^{\frac{1}{2}}} \end{aligned}$$

where A, B and C are

$$\begin{aligned} A &= (r\tau\kappa\cos\theta - r\sin\theta\frac{d\kappa}{ds})r\tau \\ B &= (\kappa - r\kappa^2\sin\theta - r\cos\theta\frac{d\tau}{ds} - r\tau^2\sin\theta)(1 - r\kappa\sin\theta)\cos\theta \\ C &= (r\sin\theta\frac{d\tau}{ds} - r\tau^2\cos\theta)(1 - r\kappa\sin\theta)\sin\theta \end{aligned}$$

Therefore, the Gaussian curvature $K(r,s)$ is,

$$K = \frac{LN - M^2}{EG - F^2} = -\frac{\tau^2}{[(1 - r\kappa\sin\theta)^2 + r^2\tau^2]^2}$$

Clearly, $K(r,s) = 0\ \forall r \in (-\epsilon, \epsilon)\ \forall s \in I$ iff $\tau(s) = 0\ \forall s \in I$, i.e. α is planar. This completes the proof.

Hence, parametrized representation of the developable surface is given by the $\phi(r,s)$.

As discussed in [1] the developable surface can be obtained by evaluating the distance function of some planar curve C, which is in fact a planar problem. That is, if D is the developable surface of constant slope $a = \tan\vartheta$, where $0 < \vartheta < \frac{\pi}{2}$ passing through C. The slope of D's tangent planes equals a and so D can be generated as the envelope of planes with slope a passing through tangent lines of C. The surface D is the graph of the function,

$$f(\mathbf{x}) = z_0 \pm dist(\mathbf{x}, \mathbf{p}') \tan\vartheta, \tag{2.1}$$

where $\mathbf{x} = (x, y, 0)$ is an arbitrary query point in the xy-plane, $\mathbf{p} = (p_1, p_2, z_0)$ is the closest point to $\mathbf{x}$ and θ with $0 < \vartheta < \pi/2$ is the angle between D's tangent plane and xy-plane. The projection of $\mathbf{p} \in C$ onto $z = 0$ is denoted by $\mathbf{p}' = (p_1, p_2, 0) \in C'$. The function $f(\mathbf{x})$ satisfies

$$\|\nabla f\| = \tan\vartheta, \tag{2.2}$$

the Eikonal equation.

2.2. **Level Set Method and Curve evolution Problem.** We will consider a boundary, a curve in two dimensions, separating one region from another. Assume that this curve moves in a direction normal to itself with a known speed function F. The goal is to track the motion of this interface as it evolves. We are concerned only with the motion of the interface in its normal direction; throughout, we shall ignore motions of the interface in its tangential direction. This type of evolution is called **Geometric Curve Evolution** or Interface Propagation Problem.

Let F be the speed function, which may depend on many factors like the curvature, area bounded by the contour etc.

If $F > 0 (or < 0)$ and it does not change its sign then it can be shown as given in [3] that the time taken T by the front is given by

$$|\nabla T| F = 1. \tag{2.3}$$

In this case, the level sets of T correspond to the wavefronts at different times; that is, the set $\{x : T(x) = t\}$ is the wavefront at time t. The derivation of this equation is straightforward: let $\sigma(x)$ represent the differential path of the contour from point x at time t, which reaches the new wavefront at time $t + dt$. This path is perpendicular to the wavefront it starts from, and has length $|F(x)dt|$, so

$$dt = \left| \frac{dT}{d\sigma}(x) \right| = |\langle \nabla T(x), \sigma \rangle| = |\nabla T(x)| \cdot |\sigma(x)| = |\nabla T(x)| \cdot |F(x)| \cdot dt$$

$$\Rightarrow |\nabla T(x)| = \frac{1}{|F(x)|}$$

yields the Eikonal equation.

Note that if F=1 identically and T=0 on the boundary of a region Ω, then for x outside Ω, the solution $T(x)$ to the eikonal equation measures the distance from x to Ω.

2.3. **Intuition behind the derivation.** The key idea of the derivation is in converting the Lagrangian framework to the Eulerian perspective. This is achieved by representing the surface as a collection of level curves. These curves can be formulated in terms of the curve evolution problem. The figures illustrate this.

The developable surface is considered as collection of level curves corresponding to the heights of each curve i.e. collapsing the surface onto the $xy-$plane. Mathematically, this can be achieved by removing the binormal component of $\phi(r,s)$. As for a planar curve the normal component is in the $xy-$plane and binormal component is perpendicular, for an unit increase in r, $\phi(r,s)$ gains a height of $\cos\theta$ and the contribution of the normal component to the height is zero. The next section gives the proof in detail.

3. Derivation

Lemma 3.1. *Angle made by the tangent plane of the parametrized surface ϕ is $\frac{\pi}{2}-\theta$ with respect to the xy-plane.*

Proof.

$$\begin{aligned} E_1(r,s)\cdot\vec{n}(s) &= \left(\frac{\partial\phi}{\partial r}\right)\cdot\vec{n}(s) \\ &= \left(\sin\theta\vec{n}(s)+\cos\theta\vec{b}(s)\right)\cdot\vec{n}(s) \\ &= \sin\theta=\cos(\frac{\pi}{2}-\theta) \end{aligned}$$

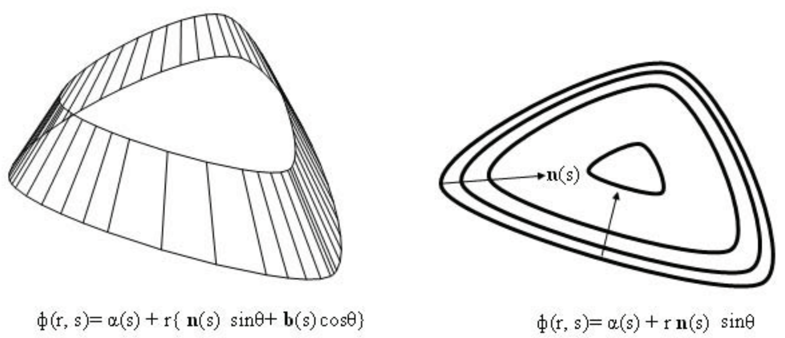
This proves the lemma.

Let us denote the angle made by the tangent plane of the parameterized surface and $xy-$plane by $\vartheta=\frac{\pi}{2}-\theta$.

Lemma 3.2. *Graph of a function f is a level set. i.e, $D=\{(x,y,z)):z=f(x,y)\quad x,y\in\Re^2)\}$ is a level set.*

Proof: Set $g(x,y,z)=f(x,y)-z$ Hence,

$$D=\{(x,y,z):g(x,y,z)=0\}$$



(a) (b)

Figure 1. (a) A developable surface generated from a closed curve. (b) Converting developable surface to curve evolution problem. Collection of level curves of the above developable surface.

is the level set, $[g^{-1}(c)]$, such that

$$\nabla g = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \right\rangle \neq 0$$

Thus graph of the function f is a surface.

From the above derivation and the section 2, it can also be noticed that g satisfies Eikonal equation i.e.,

$$\|\nabla g\| = \sec \vartheta. \tag{3.1}$$

Theorem 3.1. *The level set form of the developable surface,*

$$\phi(s,t) = \alpha(s) + r(\sin\theta\vec{n} + \cos\theta\vec{b})$$

satisfies the Eikonal equation.

Proof. Given,

$$\phi(r,s) = \alpha(s) + r(\sin\theta\vec{n} + \cos\theta\vec{b}),$$

define,

$$C^r(s) = \alpha(s) + r(\sin\theta\vec{n}), \tag{3.2}$$

for each $r \in [0, \epsilon)$.

Differentiating the above equation (3.2) w.r.t r both side we get,

$$\frac{\partial C^r}{\partial r} = \vec{n}\sin\theta$$

and $C^0(s) = \alpha(s)$.

Considering r as a 'time' parameter the above equation states that as time evolves the curve shrinks/expands by a factor of $\sin\theta$ in normal direction. This is a geometric curve evolution problem with $F = \cos\theta$ and $T(x,y) = r_0\cos\theta$ representing the level curve $\phi(r_0, s) = \alpha(s) + r_0(\vec{n}\sin\theta + \vec{b}\cos\theta)$. That is,

$$C^{r_0} = \{(x,y) : T(x,y) = r_0\cos\theta\}$$

Let $g(x,y,z) = (x,y,T(x,y))$, $F = \sin\theta$ and $\vartheta = \frac{\pi}{2} - \theta$ we obtain, by Eq.(2.3),

$$\begin{aligned} \|\nabla g\|\cos\vartheta &= 1 \\ \Rightarrow \quad \|\nabla g\| &= \sec\vartheta, \end{aligned}$$

which is the required equation. Hence g satisfies the Eikonal equation given in Eq.(3.1).

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