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ON THE DEFINITION OF MINIMAL LIGHTLIKE SUBMANIFOLDS

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Abstract. We consider a new definition of minimal lightlike submanifolds which is independent of the choice of the screen distribution, the screen transversal vector bundle, and the lightlike transversal vector bundle. Under the definition, we give a class of 4-dimensional minimal lightlike submanifolds in the 6-dimensional semi-Euclidean space of index 3, which are not totally geodesic. The construction is related to the geometry of paracomplex submanifolds.

1. Introduction

Let M be a submanifold immersed in a semi-Riemannian manifold $(\bar{M}, \bar{g})$. If the induced metric $g = \bar{g}|_M$ is degenerate, then M is called a lightlike submanifold (cf. [4]). The geometry of lightlike submanifolds is much different from that of semi-Riemannian submanifolds. For a lightlike submanifold M, the tangent bundle TM and the normal bundle $TM^{\perp}$ have a non-trivial intersection, which is called the radical distribution and denoted by $\mathrm{Rad}(TM)$. We may choose a (non-unique) semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, which is called the screen distribution and denoted by $S(TM)$. Similarly, we may choose a complementary vector bundle of $\mathrm{Rad}(TM)$ in $TM^{\perp}$, which is called the screen transversal vector bundle and denoted by $s(TM^{\perp})$. Corresponding to $S(TM)$ and $s(TM^{\perp})$, we may choose a lightlike transversal vector bundle $\mathrm{ltr}(TM)$.

In [2] Bejan and Duggal introduced the notion of minimal lightlike submanifolds. But, in the case where the codimension is greater than 1, their definition depends on the choice of the screen distribution. So, in Section 3, modifying their definition, we will give another definition of minimal lightlike submanifolds which is independent of the choice of the screen distribution, the screen transversal vector bundle, and the lightlike transversal vector bundle. In the case of lightlike hypersurfaces, those two definitions are the same.

In Section 5, under our definition, we give a class of 4-dimensional minimal lightlike submanifolds in the 6-dimensional semi-Euclidean space R^6_3 of index 3,

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which are not totally geodesic. This construction is related to the geometry of paracomplex submanifolds.

Remark 1.1. In a previous paper [7], we obtain a class of minimal lightlike hypersurfaces in the 4-dimensional semi-Euclidean space R^4_2 of index 2, which are not totally geodesic. In [8] we discuss minimal lightlike Monge hypersurfaces in semi-Euclidean spaces. See [3] and [6] for minimal lightlike hypersurfaces in Minkowski spaces or Lorentzian space forms.

Remark 1.2. In [5] Gorkavyy gives the other definition of minimal lightlike surfaces in Minkowski spaces, from a view point of isometric deformations.

2. **Lightlike submanifolds**

In this section, following [4, Chap.5], we recall some basic facts on lightlike submanifolds.

Let $\bar{M}$ be an $(m+n)$-dimensional semi-Riemannian manifold with metric $\bar{g}$ and Levi-Civita connection $\bar{\nabla}$. Let M be an m-dimensional lightlike submanifold in $\bar{M}$, that is, the induced metric $g = \bar{g}|_M$ is degenerate. Then, the tangent bundle TM and the normal bundle $TM^\perp$ have a non-trivial intersection, which is the radical distribution $\mathrm{Rad}(TM)$, given by

$$\mathrm{Rad}(T_xM) = \{\xi \in T_xM | g(\xi, X) = 0,\ X \in T_xM\}.$$

If the rank of $\mathrm{Rad}(TM)$ is r (≥ 1), then M is called r-lightlike.

There exists a screen distribution $S(TM)$ which is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, so that

$$TM = S(TM) \perp \mathrm{Rad}(TM).$$

Similarly, there exists a screen transversal vector bundle $s(TM^\perp)$ which is a complemetary vector bundle of $\mathrm{Rad}(TM)$ in $TM^\perp$, so that

$$TM^\perp = s(TM^\perp) \perp \mathrm{Rad}(TM).$$

When M is r-lightlike, the ranks of $S(TM)$ and $s(TM^\perp)$ are $m-r$ and $n-r$, respectively. If $r = n$, then M is called coisotropic.

For a local basis $\{\xi_i\}_{1\leq i\leq r}$ of $\Gamma(\mathrm{Rad}(TM))$, there exists a local frame $\{N_i\}_{1\leq i\leq r}$ of sections with values in the orthogonal complement of $s(TM^\perp)$ in $(S(TM))^\perp$ such that

$$\bar{g}(\xi_i, N_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,$$

where δ_{ij} denotes the Kronecker delta. Then, there exists a lightlike transversal vector bundle $\mathrm{ltr}(TM)$ which is locally spanned by $\{N_i\}$ (cf. [4, Chap.5, Th.1.3, 1.2]).

So we have the following decomposition:

$$T\bar{M}|_M = S(TM) \perp (\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)) \perp s(TM^\perp).$$

The transversal vector bundle is defined by

$$\mathrm{tr}(TM) = \mathrm{ltr}(TM) \perp s(TM^\perp), \tag{2.1}$$

and we have

$$T\bar{M}|_M = TM \oplus \mathrm{tr}(TM). \tag{2.2}$$

According to the decomposition (2.2), we have the Gauss formula

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad X, Y \in \Gamma(TM).$$

Then ∇ is a torsion-free linear connection on M, and h is a symmetric $C^\infty(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(\mathrm{tr}(TM))$. The form h is called the second fundamental form of M with respect to $\mathrm{tr}(TM)$. If $h = 0$ identically, then M is called totally geodesic.

From the decomposition (2.1), we have

$$h(X, Y) = h^l(X, Y) + h^s(X, Y),$$

where h^l and h^s are $\Gamma(\mathrm{ltr}(TM))$-valued and $\Gamma(s(TM^\perp))$-valued, respectively. We call h^l and h^s the lightlike second fundamental form and the screen second fundamental form, respectively. Let $\{W_\alpha\}$ be a local orthonormal basis of $\Gamma(s(TM^\perp))$ where $r + 1 \le \alpha \le n$. Then we may write

$$h^l(X, Y) = \sum_{i=1}^{r} h_i^l(X, Y)N_i, \qquad h^s(X, Y) = \sum_{\alpha=r+1}^{n} h_\alpha^s(X, Y)W_\alpha.$$

We call h_i^l and h_α^s the local lightlike second fundamental forms and the local screen second fundamental forms, respectively.

Let $\{X_a\}$ be a local orthonormal basis of $\Gamma(S(TM))$ where $r + 1 \le a \le m$. Then $\{\xi_i, N_i, X_a, W_\alpha\}$ is a local quasi-orthonormal frame field corresponding to $\{S(TM),\ s(TM^\perp),\ \mathrm{ltr}(TM)\}$. We may choose other screen distribution $S'(TM)$, screen transversal vector bundle $s'(TM^\perp)$, and lightlike transversal vector bundle $\mathrm{ltr}'(TM)$. Let $\{\xi_i, N_i', X_a', W_\alpha'\}$ be a local quasi-orthonormal frame field corresponding to $\{S'(TM),\ s'(TM^\perp),\ \mathrm{ltr}'(TM)\}$. Then we may write

$$X_a' = \sum_{b=r+1}^{m} X_a^b(X_b - \varepsilon_b \sum_{i=1}^{r} P_{ib}\xi_i),$$

$$W_\alpha' = \sum_{\beta=r+1}^{n} W_\alpha^\beta(W_\beta - \epsilon_\beta \sum_{i=1}^{r} Q_{i\beta}\xi_i),$$

$$N_i' = N_i + \sum_{j=1}^{r} N_{ij}\xi_j + \sum_{a=r+1}^{m} P_{ia}X_a + \sum_{\alpha=r+1}^{n} Q_{i\alpha}W_\alpha,$$

where $\{\varepsilon_a\}$ and $\{\epsilon_\alpha\}$ are signatures of orthonormal bases $\{X_a\}$ and $\{W_\alpha\}$ respectively, X_a^b, W_α^β, N_{ij}, P_{ia}, $Q_{i\alpha}$ are local smooth functions such that (X_a^b) and (W_α^β) are $(m - r) \times (m - r)$ and $(n - r) \times (n - r)$ semi-orthogonal matrices respectively, and

$$N_{ij} + N_{ji} + \sum_{a=r+1}^{m} \varepsilon_a P_{ia}P_{ja} + \sum_{\alpha=r+1}^{n} \epsilon_\alpha Q_{i\alpha}Q_{j\alpha} = 0$$

(cf. [4, p.163]).

Then the local lightlike and screen second fundamental forms are transformed as $h_i'^l = h_i^l$, and

$$h_\alpha^s(X, Y) = \sum_{i=1}^{r} h_i^l(X, Y)Q_{i\alpha} + \sum_{\beta=r+1}^{n} h_\beta'^s(X, Y)W_\beta^\alpha$$

(cf. [4, p.165]). So the definition that M is totally geodesic is independent of the choice of the screen distribution, the screen transversal vector bundle, and the lightlike transversal vector bundle.

3. The definition of minimality

In [2] Bejan and Duggal give the definition of minimal lightlike submanifolds as follows:

Definition 3.1. We say that a lightlike submanifold $(M, g, S(TM), s(TM^{\perp}))$ in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is minimal if:
(i) $h^s = 0$ on $\mathrm{Rad}(TM)$, and
(ii) $\mathrm{trace}(h) = 0$, where the trace is written with respect to g restricted to $S(TM)$.

Remark 3.1. The lightlike second fundamental form $h^l = 0$ on $\mathrm{Rad}(TM)$ (see [4, Chap.5, Prop.2.2]).

In the case where the codimension is greater than 1, the condition (ii) in Definition 3.1 depends on the choice of the screen distribution. Modifying Definition 3.1, we shall give another definition of minimal lightlike submanifolds as follows:

Definition 3.2. We say that a lightlike submanifold $(M, g, S(TM), s(TM^{\perp}))$ in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is minimal if:
(a) $h(\xi, X) = 0$ for any $\xi \in \Gamma(\mathrm{Rad}(TM)),\ X \in \Gamma(TM)$, and
(b) $\mathrm{trace}(h) = 0$, where the trace is written with respect to g restricted to $S(TM)$.

Remark 3.2. In the case of lightlike hypersurfaces, the condition (i) in Definition 3.1 and the condition (a) in Definition 3.2 are automatically satisfied. So these two definitions are the same in that case.

Proposition 3.1. *The Definition 3.2 is independent of the choice of the screen distribution, the screen transversal vector bundle, and the lightlike transversal vector bundle.*

Proof. By the transformation formulas in Section 2, we can see that the condition (a) is independent of the choice of the screen distribution, the screen transversal vector bundle, and the lightlike transversal vector bundle. So we shall show that, under the condition (a), the condition (b) is independent of the choice of the screen distribution, the screen transversal vector bundle, and the lightlike transversal vector bundle.

Suppose that the condition (a) is satisfied. Let $\{\xi_i, N_i, X_a, W_\alpha\}$ and $\{\xi_i, N'_i, X'_a, W'_\alpha\}$ be local quasi-orthonormal frame fields corresponding to $\{S(TM), s(TM^{\perp}), \mathrm{ltr}(TM)\}$ and $\{S'(TM), s'(TM^{\perp}), \mathrm{ltr}'(TM)\}$, respectively. Then, by the transformation formulas in Section 2 together with the condition (a), we get

$$\sum_{a=r+1}^{m} \varepsilon_a h'^l_i(X'_a, X'_a) = \sum_{a=r+1}^{m} \varepsilon_a h^l_i(X'_a, X'_a)$$

$$= \sum_{a=r+1}^{m} \varepsilon_a h^l_i \left(\sum_{b=r+1}^{m} X^b_a (X_b - \varepsilon_b \sum_{j=1}^{r} P_{jb}\xi_j),\ \sum_{c=r+1}^{m} X^c_a (X_c - \varepsilon_c \sum_{k=1}^{r} P_{kc}\xi_k) \right)$$

$$= \sum_{a,b,c=r+1}^{m} \varepsilon_a X^b_a X^c_a h^l_i(X_b, X_c) = \sum_{b=r+1}^{m} \varepsilon_b h^l_i(X_b, X_b).$$

Here, for the last equality, we use that

$$\sum_{a=r+1}^{m} \varepsilon_a X_a^b X_a^c = \varepsilon_b \delta_{bc}$$

by the semi-orthogonality of (X_a^b). Similarly we have

$$\begin{aligned}
&\sum_{\beta=r+1}^{n} \left(\sum_{a=r+1}^{m} \varepsilon_a h_\beta'^s(X_a', X_a') \right) W_\beta^\alpha \\
&= \sum_{a=r+1}^{m} \varepsilon_a h_\alpha^s(X_a', X_a') - \sum_{i=1}^{r} \left(\sum_{a=r+1}^{m} \varepsilon_a h_i^l(X_a', X_a') \right) Q_{i\alpha} \\
&= \sum_{b=r+1}^{m} \varepsilon_b h_\alpha^s(X_b, X_b) - \sum_{i=1}^{r} \left(\sum_{b=r+1}^{m} \varepsilon_b h_i^l(X_b, X_b) \right) Q_{i\alpha}.
\end{aligned}$$

From these equations, we can see that, under the condition (a), the condition (b) is independent of the choice of the screen distribution, the screen transversal vector bundle, and the lightlike transversal vector bundle. □

4. **Paracomplex submanifolds**

In this section, following [1], we recall some basic facts on paracomplex submanifolds, which will be necessary to construct a class of minimal lightlike submanifolds in the next section.

Let $\bar{M}$ be a manifold with an almost product structure $\bar{J}$, that is, $\bar{J}$ is a tensor field of type $(1,1)$ such that $\bar{J}^2 = I$ and $\bar{J} \neq I$, where I is the identity. If there exists a semi-Riemannian metric $\bar{g}$ on $\bar{M}$ such that

$$\bar{g}(X, \bar{J}Y) + \bar{g}(\bar{J}X, Y) = 0, \qquad X, Y \in \Gamma(T\bar{M}),$$

or equivalently,

$$\bar{g}(\bar{J}X, \bar{J}Y) = -\bar{g}(X, Y), \qquad X, Y \in \Gamma(T\bar{M}),$$

then $(\bar{M}, \bar{J}, \bar{g})$ is called an almost parahermitian manifold. If $\bar{J}$ is integrable, that is, the Nijenhuis tensor of $\bar{J}$ given by

$$\bar{N}(X, Y) = [\bar{J}X, \bar{J}Y] - \bar{J}[\bar{J}X, Y] - \bar{J}[X, \bar{J}Y] + [X, Y]$$

vanishes identically, then $(\bar{M}, \bar{J}, \bar{g})$ is called a parahermitian manifold. We say that an almost parahermitian manifold $(\bar{M}, \bar{J}, \bar{g})$ is a parakähler manifold if $\bar{J}$ is parallel with respect to the Levi-Civita connection of $\bar{g}$.

Let L be a submanifold in a parakähler manifold $(\bar{M}, \bar{J}, \bar{g})$. We say that L is a paracomplex submanifold if $\bar{J}(T_pL) = T_pL$ for any $p \in L$. Set $J = \bar{J}|_L$, $g = \bar{g}|_L$, and assume that g is nondegenerate. Then (L, J, g) is a parakähler manifold, and the second fundamental form q of L satisfies

$$q(X, JY) = \bar{J}(q(X, Y)), \quad q(JX, JY) = q(X, Y), \quad X, Y \in \Gamma(TL) \tag{4.1}$$

(see [1, Lemma 3.1]).

5. **A class of minimal lightlike submanifolds**

In this section, we give a class of 4-dimensional lightlike submanifolds in the 6-dimensional semi-Euclidean space R^6_3 of index 3, which are minimal in the sense of Definition 3.2 but not totally geodesic.

Let $(\bar{J}, \bar{g})$ be the standard flat parakähler structure of R^6_3. Let L be a paracomplex surface in R^6_3 with nondegenerate induced metric and the inclusion map f. We may choose a local frame field $\{e_1, e_2\}$ along L such that $e_2 = \bar{J}e_1$ and $\{f_*e_1, f_*e_2\}$ is orthonormal with signature $(+, -)$.

A point on L is called a geodesic point if the second fundamental form q vanishes at the point. We assume that L has no geodesic points, and the first normal space $N_1(p)$ at $p \in L$ defined by

$$N_1(p) = \text{span}\{q(X, Y) | X, Y \in T_pL\} \subset T_pL^{\perp}$$

is not lightlike at each point on L. Then, by (4.1), $N_1(p)$ can be locally spanned by local orthonormal normal vector fields $\{e_3, e_4\}$ of signature $(+, -)$ with $e_4 = \bar{J}e_3$. We may also choose local orthonormal normal vector fields $\{e_5, e_6\}$ of signature $(+, -)$ along L so that $e_6 = \bar{J}e_5$ and $\{e_1, \cdots, e_6\}$ is orthonormal.

Let ω_A^B denote the connection forms which satisfy

$$d(f_*e_i) = \sum_{j=1}^{2} \omega_i^j f_*e_j + \sum_{\alpha=3}^{6} \omega_i^\alpha e_\alpha, \quad 1 \le i \le 2,$$

and

$$de_\alpha = \sum_{i=1}^{2} \omega_\alpha^i f_*e_i + \sum_{\beta=3}^{6} \omega_\alpha^\beta e_\beta, \quad 3 \le \alpha \le 6.$$

By the choice of the frame, we have $\omega_A^B = -\omega_B^A$ if $|A - B|$ is even, $\omega_A^B = \omega_B^A$ if $|A - B|$ is odd, and $\omega_i^5 = \omega_i^6 = 0$ for $1 \le i \le 2$. As L is a paracomplex surface, we can find that

$$\omega_1^3 = \omega_2^4, \quad \omega_1^4 = \omega_2^3, \quad \omega_3^5 = \omega_4^6, \quad \omega_3^6 = \omega_4^5.$$

Let q_{ij}^α denote the components of the second fundamental form q, where $1 \le i, j \le 2$ and $3 \le \alpha \le 6$, so that

$$\omega_i^\alpha = \sum_{j=1}^{2} q_{ij}^\alpha \omega^j.$$

Here $\{\omega^1, \omega^2\}$ is the coframe field dual to $\{e_1, e_2\}$. Then we have

$$q_{11}^3 = q_{12}^4 = q_{22}^3, \quad q_{11}^4 = q_{12}^3 = q_{22}^4, \quad q_{ij}^5 = q_{ij}^6 = 0,$$

where $1 \le i, j \le 2$. By the Gauss equation and these relations, the Gaussian curvature K of L satisfies

$$K = 2\{(q_{11}^4)^2 - (q_{11}^3)^2\}. \tag{5.1}$$

We may have the following equations:

$$d(f_*e_1) = \omega_1^2 f_*e_2 + \omega_1^3 e_3 + \omega_1^4 e_4,$$

$$d(f_*e_2) = \omega_2^1 f_*e_1 + \omega_2^3 e_3 + \omega_2^4 e_4 = \omega_1^2 f_*e_1 + \omega_1^4 e_3 + \omega_1^3 e_4,$$

$$d(f_*e_1 + f_*e_2) = \omega_1^2(f_*e_1 + f_*e_2) + (\omega_1^3 + \omega_1^4)(e_3 + e_4),$$

$$\begin{aligned} de_3 &= \omega_3^1 f_* e_1 + \omega_3^2 f_* e_2 + \omega_3^4 e_4 + \omega_3^5 e_5 + \omega_3^6 e_6 \\ &= -\omega_1^3 f_* e_1 + \omega_1^4 f_* e_2 + \omega_3^4 e_4 + \omega_3^5 e_5 + \omega_3^6 e_6, \end{aligned}$$

$$\begin{aligned} de_4 &= \omega_4^1 f_* e_1 + \omega_4^2 f_* e_2 + \omega_4^3 e_3 + \omega_4^5 e_5 + \omega_4^6 e_6 \\ &= \omega_1^4 f_* e_1 - \omega_1^3 f_* e_2 + \omega_3^4 e_3 + \omega_3^6 e_5 + \omega_3^5 e_6, \end{aligned}$$

$$d(e_3 + e_4) = (\omega_1^4 - \omega_1^3)(f_* e_1 + f_* e_2) + \omega_3^4 (e_3 + e_4) + (\omega_3^5 + \omega_3^6)(e_5 + e_6),$$

$$de_5 = \omega_5^3 e_3 + \omega_5^4 e_4 + \omega_5^6 e_6 = -\omega_3^5 e_3 + \omega_3^6 e_4 + \omega_5^6 e_6,$$

$$de_6 = \omega_6^3 e_3 + \omega_6^4 e_4 + \omega_6^5 e_5 = \omega_3^6 e_3 - \omega_3^5 e_4 + \omega_5^6 e_5,$$

$$d(e_5 + e_6) = (\omega_3^6 - \omega_3^5)(e_3 + e_4) + \omega_5^6 (e_5 + e_6).$$

Now we state the result as follows:

Theorem 5.1. *Under the notation above, the map $F : L \times R \times R \to R_3^6$ defined by*

$$F(p, s, t) = f(p) + s(e_3 + e_4) + t(e_5 + e_6), \qquad (p, s, t) \in L \times R \times R$$

gives a 4-dimensional lightlike submanifold which is minimal in the sense of Definition 3.2. It is not totally geodesic if L is nonflat.

Proof. Set

$$\tilde{e}_i(p, s, t) = (e_i(p), 0, 0) \in T_{(p,s,t)}(L \times R \times R)$$

for $1 \leq i \leq 2$. Then $\{\tilde{e}_1, \tilde{e}_2, \partial_s, \partial_t\}$ is a local frame field on $L \times R \times R$, and we have

$$\begin{aligned} F_* \tilde{e}_1 = f_* e_1 &+ s\{(\omega_1^4(e_1) - \omega_1^3(e_1))(f_* e_1 + f_* e_2) + \omega_3^4(e_1)(e_3 + e_4) + (\omega_3^5(e_1) + \omega_3^6(e_1))(e_5 + e_6)\} \\ &+ t\{(\omega_3^6(e_1) - \omega_3^5(e_1))(e_3 + e_4) + \omega_5^6(e_1)(e_5 + e_6)\}, \\ F_* \tilde{e}_2 = f_* e_2 &+ s\{(\omega_1^4(e_2) - \omega_1^3(e_2))(f_* e_1 + f_* e_2) + \omega_3^4(e_2)(e_3 + e_4) + (\omega_3^5(e_2) + \omega_3^6(e_2))(e_5 + e_6)\} \\ &+ t\{(\omega_3^6(e_2) - \omega_3^5(e_2))(e_3 + e_4) + \omega_5^6(e_2)(e_5 + e_6)\}, \end{aligned}$$

$$F_* \partial_s = e_3 + e_4 =: \xi_1, \qquad F_* \partial_t = e_5 + e_6 =: \xi_2.$$

Set

$$A = \omega_1^4(e_1) - \omega_1^3(e_1) = q_{11}^4 - q_{11}^3,$$

and note that

$$\omega_1^4(e_2) - \omega_1^3(e_2) = q_{12}^4 - q_{12}^3 = q_{11}^3 - q_{11}^4 = \omega_1^3(e_1) - \omega_1^4(e_1) = -A.$$

Then we can see that $\{F_* \tilde{e}_1, F_* \tilde{e}_2, F_* \partial_s, F_* \partial_t\}$ are linearly independent, and F is an immersion.

As the metric g on $L \times R \times R$ induced by F is given by

$$g(X, Y) = \bar{g}(F_* X, F_* Y), \qquad X, Y \in \Gamma(T(L \times R \times R)),$$

we have

$$g(\tilde{e}_1, \tilde{e}_1) = 1 + 2s(\omega_1^4(e_1) - \omega_1^3(e_1)) = 1 + 2sA,$$

$$g(\tilde{e}_2, \tilde{e}_2) = -1 - 2s(\omega_1^4(e_2) - \omega_1^3(e_2)) = -1 + 2sA,$$

$$g(\tilde{e}_1, \tilde{e}_2) = s\{(\omega_1^4(e_2) - \omega_1^3(e_2)) - (\omega_1^4(e_1) - \omega_1^3(e_1))\} = -2sA,$$

$$g(\tilde{e}_i, \partial_s) = g(\tilde{e}_i, \partial_t) = g(\partial_s, \partial_s) = g(\partial_s, \partial_t) = g(\partial_t, \partial_t) = 0,$$

where $1 \leq i \leq 2$. Thus the map F gives a 4-dimensional coisotropic lightlike submanifold M, and $\{\xi_1, \xi_2\}$ is a local basis of $\Gamma(\mathrm{Rad}(TM))$. We choose the screen distribution $S(TM)$ so that it is spanned by $\{F_*\tilde{e}_1, F_*\tilde{e}_2\}$.

As in Section 2, we choose a local frame $\{N_1, N_2\}$ of $\Gamma(\mathrm{ltr}(TM))$. As M is coisotropic, the second fundamental form h of M satisfies $h = h^l$, and

$$h^l_k(X,Y) = \bar{g}(\bar{\nabla}_{F_*X}F_*Y,\ \xi_k), \qquad X, Y \in \Gamma(T(L \times R \times R)),$$

where $\bar{\nabla}$ is the flat connection of R^6_3 and $1 \leq k \leq 2$. Then we may have

$$h^l_1(\tilde{e}_1, \tilde{e}_1) = -A, \qquad h^l_1(\tilde{e}_2, \tilde{e}_2) = -A, \qquad h^l_1(\tilde{e}_1, \tilde{e}_2) = A,$$
$$h^l_2(\tilde{e}_1, \tilde{e}_1) = h^l_2(\tilde{e}_2, \tilde{e}_2) = h^l_2(\tilde{e}_1, \tilde{e}_2) = 0,$$
$$h^l_k(X, \partial_s) = h^l_k(X, \partial_t) = 0,$$

where $X \in \Gamma(T(L \times R \times R))$ and $1 \leq k \leq 2$.

So the condition (a) in Definition 3.2 is valid. The condition (b) in Definition 3.2 is equivalent to that

$$g(\tilde{e}_2, \tilde{e}_2)h^l_k(\tilde{e}_1, \tilde{e}_1) - 2g(\tilde{e}_1, \tilde{e}_2)h^l_k(\tilde{e}_1, \tilde{e}_2) + g(\tilde{e}_1, \tilde{e}_1)h^l_k(\tilde{e}_2, \tilde{e}_2) = 0$$

for $1 \leq k \leq 2$, which is also valid from the above computations. Therefore, M is minimal in the sense of Definition 3.2.

If L is nonflat, then by (5.1), A is not identically zero , and M is not totally geodesic. Thus we have proved the theorem. □

Remark 5.1. The above construction may be seen as a generalization of that in the previous paper [7].

Remark 5.2. In Example 9 of [2], Bejan and Duggal give a 3-dimensional lightlike submanifold in $S^3_1 \times R^2_1$, which is minimal in the sense of Definition 3.1 but not totally geodesic. Here S^3_1 is the 3-dimensional unit pseudo-sphere of index 1, and R^2_1 is the Minkowski plane. We can see that it is also minimal in the sense of Definition 3.2.

References

[1] Al-Aqeel, A. and Bejancu, A., On the geometry of paracomplex submanifolds, Demonstratio Math., 34 (2001), 919-932.

[2] Bejan, C. L. and Duggal, K. L., Global lightlike manifolds and harmonicity, Kodai Math. J., 28 (2005), 131-145.

[3] Burdujan, I., On minimal lightlike Monge hypersurfaces of a Lorentz space, An. Stiint. Univ. Al. I. Cuza Iasi. Mat., 47 (2001), 95-103.

[4] Duggal, K. L. and Bejancu, A., Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications, Kluwer Academic Publishers, 1996.

[5] Gorkavyy, V., On minimal lightlike surfaces in Minkowski space-time, Differ. Geom. Appl., 26 (2008), 133-139.

[6] Sakaki, M., Mimimal lightlike hypersurfaces in Lorentzian space forms, Int. J. Pure. Appl. Math., 48 (2008), 595-600.

[7] Sakaki, M., Mimimal lightlike hypersurfaces in R^4_2 with integrable screen distribution, Balkan J. Geom. Appl., 14(1) (2009), 84-90.

[8] Sakaki, M., Mimimal lightlike Monge hypersurfaces in semi-Euclidean spaces, Int. J. Pure. Appl. Math., 54 (2009), 543-549.

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