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SOME PROPERTIES OF A KENMOTSU MANIFOLD WITH A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. The aim of this paper is to study generalized recurrent, generalized Ricci-recurrent, weakly symmetric and weakly Ricci-symmetric Kenmotsu manifolds with respect to the semi-symmetric metric connection.

1. Introduction

A. Friedmann and J. A. Schouten [6] introduced the idea of a semi-symmetric linear connection on a Riemannian manifold. The definition of semi-symmetric metric connection was given by H. A. Hayden [7]. Later, K. Yano [17] initiated studying of a semi-symmetric metric connection on a Riemannian manifold. He showed that a Riemannian manifold with respect to the semi-symmetric metric connection has vanishing curvature tensor if and only if it is conformally flat. Then this result was generalized for vanishing Ricci tensor of the semi-symmetric metric connection by T. Imai ([8], [9]). Moreover, he gave some properties of a hypersurface of a Riemannian manifold admitting a semi-symmetric metric connection and obtained the formulas of Gauss curvature and Codazzi-Mainardi. In [18], A. Yücesan studied submanifolds of a semi-Riemannian manifold with a semi-symmetric metric connection. In a recent paper, M. M. Tripathi [16] studied semi-symmetric metric connection in a Kenmotsu manifold. As a result of these circumstances in this paper, we consider generalized recurrent, generalized Ricci-recurrent, weakly symmetric and weakly Ricci-symmetric Kenmotsu manifolds with semi-symmetric metric connection.

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The paper is organized as follows: Section 2 is devoted to preliminaries. In section 3, we prove that $\beta = 2\alpha$ on generalized recurrent and generalized Ricci-recurrent Kenmotsu manifolds with respect to the semi-symmetric metric connection. In the last section, we show that there is no weakly symmetric or weakly Ricci-symmetric Kenmotsu manifolds admitting a semi-symmetric metric connection, $n > 3$, unless $\alpha + \sigma + \gamma$ or $\rho + \mu + \upsilon$ is everywhere zero, respectively.

2. **Preliminaries**

Let M be an n-dimensional almost contact metric manifold [1] with an almost contact metric structure (φ, ξ, η, g) consisting of a $(1,1)$ tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g on M satisfying

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \tag{2}$$

for all vector fields X, Y on M. If an almost contact metric manifold satisfies [10]

$$(\nabla_X \varphi)Y = g(\varphi X, Y) - \eta(Y)\varphi X \tag{3}$$

and

$$\nabla_X \xi = X - \eta(X)\xi, \tag{4}$$

then M is called a *Kenmotsu manifold*, where ∇ is Levi-Civita connection of g. From above equations it follows that

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y). \tag{5}$$

Moreover, the curvature tensor R and the Ricci tensor S satisfy [10]

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X \tag{6}$$

and

$$S(X, \xi) = -(n-1)\eta(X). \tag{7}$$

A linear connection $\widetilde{\nabla}$ in M is called *semi-symmetric connection* ([13], [17]) if the torsion tensor T of the connection $\widetilde{\nabla}$

$$T(X, Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X, Y] \tag{8}$$

satisfies

$$T(X, Y) = \eta(Y)X - \eta(X)Y. \tag{9}$$

If ∇ is the Levi-Civita connection of an almost contact metric manifold M, the *semi-symmetric metric connection* $\widetilde{\nabla}$ in M is denoted by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi. \tag{10}$$

Let R and $\widetilde{R}$ be the curvature tensors of ∇ and $\widetilde{\nabla}$ of an almost contact metric manifold, respectively. Then R and $\widetilde{R}$ are related by ([13], [17])

$$\widetilde{R}(X,Y)Z = R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y - g(Y,Z)AX + g(X,Z)AY, \tag{11}$$

where α is a $(0,2)$ tensor field denoted by

$$\alpha(X,Y) = (\nabla_X\eta)Y - \eta(X)\eta(Y) + \frac{1}{2}\eta(\xi)g(X,Y) \tag{12}$$

and

$$g(AX,Y) = \alpha(X,Y). \tag{13}$$

Since M is a Kenmotsu manifold using (5) it follows that

$$\alpha(X,Y) = \frac{3}{2}g(X,Y) - 2\eta(X)\eta(Y) \tag{14}$$

and

$$AX = \frac{3}{2}X - 2\eta(X)\xi. \tag{15}$$

In view of (14) and (15) in (11) we have

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z - 3g(Y,Z)X + 3g(X,Z)Y + 2\eta(Y)\eta(Z)X \\ &\quad -2\eta(X)\eta(Z)Y + 2g(Y,Z)\eta(X)\xi - 2g(X,Z)\eta(Y)\xi.\end{aligned} \tag{16}$$

In view of (16) we get

$$\widetilde{S}(X,Y) = S(X,Y) - (3n-5)g(X,Y) + 2(n-2)\eta(X)\eta(Y), \tag{17}$$

where $\widetilde{S}$ and S are Ricci tensors of M with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ and Levi-Civita connection ∇, respectively.

Lemma 2.1. [5] *In a Kenmotsu manifold M with semi-symmetric metric connection we have*

$$\widetilde{R}(X,Y)\xi = 2R(X,Y)\xi \tag{18}$$

and

$$\widetilde{S}(X,\xi) = 2S(X,\xi). \tag{19}$$

3. Generalized Recurrent Kenmotsu Manifolds

A non-flat n-dimensional differentiable manifold M, $n > 3$, is called *generalized recurrent* [4] if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)W = \alpha(X)R(Y,Z)W + \beta(X)[g(Z,W)Y - g(Y,W)Z], \tag{20}$$

where ∇ is the Levi-Civita connection and α and β are two 1-forms, $(\beta \neq 0)$, defined by

$$\alpha(X) = g(X,A), \quad \beta(X) = g(X,B) \tag{21}$$

and A, B are vector fields related with 1-forms α and β, respectively.

A non-flat n-dimensional differentiable manifold M, $n > 3$, is called *generalized Ricci-recurrent* [4] if its Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y,Z) = \alpha(X)S(Y,Z) + (n-1)\beta(X)g(Y,Z), \tag{22}$$

where α and β are defined as in (21).

Similarly, a non-flat n-dimensional differentiable manifold M, $n > 3$, is called *generalized recurrent with respect to the semi-symmetric metric connection* if its curvature tensor $\widetilde{R}$ satisfies the condition

$$(\widetilde{\nabla}_X \widetilde{R})(Y,Z)W = \alpha(X)\widetilde{R}(Y,Z)W + \beta(X)[g(Z,W)Y - g(Y,W)Z], \tag{23}$$

where $\widetilde{\nabla}$ is the semi-symmetric metric connection and $\widetilde{R}$ is the curvature tensor of $\widetilde{\nabla}$.

A non-flat n-dimensional differentiable manifold M, $n > 3$, is called *generalized Ricci-recurrent with respect to the semi-symmetric metric connection* if its Ricci tensor $\widetilde{S}$ satisfies the condition

$$(\widetilde{\nabla}_X \widetilde{S})(Y,Z) = \alpha(X)\widetilde{S}(Y,Z) + (n-1)\beta(X)g(Y,Z). \tag{24}$$

In [12], C. Özgür considered generalized recurrent Kenmotsu manifolds and it was obtained the following results:

Theorem 3.1. [12] *Let M be a generalized recurrent Kenmotsu manifold. Then $\beta = \alpha$ holds on M.*

Theorem 3.2. [12] *Let M be a generalized Ricci-recurrent Kenmotsu manifold. Then $\beta = \alpha$ holds on M.*

Now we consider generalized recurrent and generalized Ricci-recurrent Kenmotsu manifolds with respect to the semi-symmetric metric connection. We begin with the following theorem:

Theorem 3.3. *If a generalized recurrent Kenmotsu manifold M admits a semi-symmetric metric connection, then $\beta = 2\alpha$ holds on M.*

Proof. Suppose that M is a generalized recurrent Kenmotsu manifold admitting a semi-symmetric metric connection. Then from (23), it can be written as

$$(\widetilde{\nabla}_X \widetilde{R})(Y,Z)W = \alpha(X)\widetilde{R}(Y,Z)W + \beta(X)[g(Z,W)Y - g(Y,W)Z] \tag{25}$$

for all vector fields X, Y, Z, W. Taking $Y = W = \xi$ in (25) we have

$$(\widetilde{\nabla}_X \widetilde{R})(\xi,Z)\xi = \alpha(X)\widetilde{R}(\xi,Z)\xi + \beta(X)[\eta(Z)\xi - Z].$$

By making use of (6) and (18) in the above equation we get

$$(\widetilde{\nabla}_X \widetilde{R})(\xi,Z)\xi = [\beta(X) - 2\alpha(X)]\{\eta(Z)\xi - Z\}. \tag{26}$$

On the other hand, it is obvious that

(27)
$$(\widetilde{\nabla}_X\widetilde{R})(\xi,Z)\xi = \widetilde{\nabla}_X\widetilde{R}(\xi,Z)\xi - \widetilde{R}(\widetilde{\nabla}_X\xi,Z)\xi - \widetilde{R}(\xi,\widetilde{\nabla}_X Z)\xi - \widetilde{R}(\xi,Z)\widetilde{\nabla}_X\xi.$$

Then in view of (6) and (18), the equation (27) yields to

$$(\widetilde{\nabla}_X\widetilde{R})(\xi,Z)\xi = 0. \tag{28}$$

Hence comparing the right hand sides of the equations (26) and (28) we obtain

$$[\beta(X) - 2\alpha(X)]\{\eta(Z)\xi - Z\} = 0,$$

which implies that $\beta(X) = 2\alpha(X)$ for any vector field X. Thus our theorem is proved. □

Theorem 3.4. *Let M be a generalized Ricci-recurrent Kenmotsu manifold admitting a semi-symmetric metric connection. Then $\beta = 2\alpha$ holds on M.*

Proof. Assume that M is a generalized Ricci-recurrent Kenmotsu manifold admitting a semi-symmetric metric connection. Then from (24), we can write

$$(\widetilde{\nabla}_X\widetilde{S})(Y,Z) = \alpha(X)\widetilde{S}(Y,Z) + (n-1)\beta(X)g(Y,Z)$$

for all vector fields X, Y, Z on M. Putting $Z = \xi$ in the above equation we get

$$(\widetilde{\nabla}_X\widetilde{S})(Y,\xi) = \alpha(X)\widetilde{S}(Y,\xi) + (n-1)\beta(X)\eta(Y).$$

Then by virtue of (7) and (19), it can be easily seen that

$$(\widetilde{\nabla}_X\widetilde{S})(Y,\xi) = (n-1)[\beta(X) - 2\alpha(X)]\eta(Y). \tag{29}$$

On the other hand, by making use of the definition of covariant derivative of $\widetilde{S}$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$, it is well-known that

$$(\widetilde{\nabla}_X\widetilde{S})(Y,\xi) = \widetilde{\nabla}_X\widetilde{S}(Y,\xi) - \widetilde{S}(\widetilde{\nabla}_X Y,\xi) - \widetilde{S}(Y,\widetilde{\nabla}_X\xi). \tag{30}$$

By the use of (4), (7), (10) and (19) in (30) we get

$$(\widetilde{\nabla}_X\widetilde{S})(Y,\xi) = 2(n-3)g(X,Y) - 4(n-2)\eta(X)\eta(Y) - 2S(X,Y). \tag{31}$$

Thus comparing the right hand sides of the equations (29) and (31) we can write

$$\begin{aligned} &(n-1)[\beta(X) - 2\alpha(X)]\eta(Y) \\ = \; &2(n-3)g(X,Y) - 4(n-2)\eta(X)\eta(Y) - 2S(X,Y). \end{aligned} \tag{32}$$

Taking $Y = \xi$ in (32) and using (7) we obtain $\beta(X) = 2\alpha(X)$ for any vector field X on M. Hence the proof of the theorem is completed. □

4. **Weakly Symmetric Kenmotsu Manifolds**

A non-flat n-dimensional differentiable manifold M, $n > 3$, is called *pseudosymmetric* if there is a 1-form α on M such that

$$(\nabla_X R)(Y,Z,V) = 2\alpha(X)R(Y,Z)V + \alpha(Y)R(X,Z)V + \alpha(Z)R(Y,X)V + \alpha(V)R(Y,Z)X + g(R(Y,Z)V,X)A,$$

where ∇ is the Levi-Civita connection and X, Y, Z, V are vector fields on M. $A \in TM$ is the vector field associated with 1-form α which is defined by $g(X, A) = \alpha(X)$ in [2].

A non-flat n-dimensional differentiable manifold M, $n > 3$, is called *weakly symmetric* ([14], [15]) if there are 1-forms α, β, γ and σ such that

$$(\nabla_X R)(Y,Z,V) = \alpha(X)R(Y,Z)V + \beta(Y)R(X,Z)V + \gamma(Z)R(Y,X)V + \sigma(V)R(Y,Z)X + g(R(Y,Z)V,X)P, \tag{33}$$

for all vector fields X, Y, Z, V on M. A weakly symmetric manifold M is pseudosymmetric if $\beta = \gamma = \sigma = \frac{1}{2}\alpha$ and $P = A$, locally symmetric if $\alpha = \beta = \gamma = \sigma = 0$ and $P = 0$. A weakly symmetric manifold is said to be proper if at least one of the 1-forms α, β, γ and σ is not zero or $P \neq 0$.

A non-flat n-dimensional differentiable manifold M, $n > 3$, is called *weakly Ricci-symmetric* ([14], [15]) if there are 1-forms ρ, μ, υ such that

$$(\nabla_X S)(Y,Z) = \rho(X)S(Y,Z) + \mu(Y)S(X,Z) + \upsilon(Z)S(X,Y), \tag{34}$$

for all vector fields X, Y, Z on M. If $\rho = \mu = \upsilon$ then M is called *pseudo Ricci-symmetric* (see [3]).

If M is weakly symmetric, from (33), we have

$$(\nabla_X S)(Z,V) = \alpha(X)S(Z,V) + \beta(R(X,Z)V) + \gamma(Z)S(X,V) + \sigma(V)S(X,Z) + p(R(X,V)Z), \tag{35}$$

where p is defined by $p(X) = g(X, P)$ for any $X \in TM$ in [15].

Similarly, a non-flat n-dimensional differentiable manifold M, $n > 3$, is called *weakly symmetric with respect to the semi-symmetric metric connection* if there are 1-forms α, β, γ and σ such that

$$(\widetilde{\nabla}_X \widetilde{R})(Y,Z)V = \alpha(X)\widetilde{R}(Y,Z)V + \beta(Y)\widetilde{R}(X,Z)V + \gamma(Z)\widetilde{R}(Y,X)V + \sigma(V)\widetilde{R}(Y,Z)X + g(\widetilde{R}(Y,Z)V,X)P, \tag{36}$$

for all vector fields X, Y, Z, V on M.

A non-flat n-dimensional differentiable manifold M, $n > 3$, is called *weakly Ricci-symmetric with respect to the semi-symmetric metric connection* if there are 1-forms ρ, μ, υ such that

$$(\widetilde{\nabla}_X \widetilde{S})(Y,Z) = \rho(X)\widetilde{S}(Y,Z) + \mu(Y)\widetilde{S}(X,Z) + \upsilon(Z)\widetilde{S}(X,Y), \tag{37}$$

for all vector fields X, Y, Z on M.

If M is weakly symmetric with respect to the semi-symmetric metric connection, by a contraction from (36), we have

$$(\widetilde{\nabla}_X \widetilde{S})(Z,V) = \alpha(X)\widetilde{S}(Z,V) + \beta(\widetilde{R}(X,Z)V) + \gamma(Z)\widetilde{S}(X,V) + \sigma(V)\widetilde{S}(X,Z) + p(\widetilde{R}(X,V)Z). \tag{38}$$

In [11], C. Özgür studied weakly symmetric and weakly Ricci-symmetric Kenmotsu manifolds and it was obtained the following results:

Theorem 4.1. [11] *There is no weakly symmetric Kenmotsu manifold M, $n > 3$, unless $\alpha + \sigma + \gamma$ is everywhere zero.*

Theorem 4.2. [11] *There is no weakly Ricci-symmetric Kenmotsu manifold M, $n > 3$, unless $\rho + \mu + \upsilon$ is everywhere zero.*

Now we consider weakly symmetric and weakly Ricci-symmetric Kenmotsu manifolds with respect to the semi-symmetric metric connection. We begin with the following theorem:

Theorem 4.3. *There is no weakly symmetric Kenmotsu manifold M admitting a semi-symmetric metric connection, $n > 3$, unless $\alpha + \sigma + \gamma$ is everywhere zero.*

Proof. Let M be a weakly symmetric Kenmotsu manifold with semi-symmetric metric connection $\widetilde{\nabla}$. By the covariant differentiation of the Ricci tensor $\widetilde{S}$ of the semi-symmetric metric connection with respect to X, we have

$$(\widetilde{\nabla}_X \widetilde{S})(Z,V) = \widetilde{\nabla}_X \widetilde{S}(Z,V) - \widetilde{S}(\widetilde{\nabla}_X Z, V) - \widetilde{S}(Z, \widetilde{\nabla}_X V). \tag{39}$$

Putting $V = \xi$ in (39) and using (4), (7), (10) and (19), it follows that

$$(\widetilde{\nabla}_X \widetilde{S})(Z,\xi) = 2(n-3)g(X,Z) - 4(n-2)\eta(X)\eta(Z) - 2S(X,Z). \tag{40}$$

On the other hand, replacing V with ξ in (38), we get

$$(\widetilde{\nabla}_X \widetilde{S})(Z,\xi) = \alpha(X)\widetilde{S}(Z,\xi) + \beta(\widetilde{R}(X,Z)\xi) + \gamma(Z)\widetilde{S}(X,\xi) + \sigma(\xi)\widetilde{S}(X,Z) + p(\widetilde{R}(X,\xi)Z). \tag{41}$$

By the use of (6), (7), (17), (18) and (19) in (41) we have

$$\begin{aligned}(\widetilde{\nabla}_X \widetilde{S})(Z,\xi) &= 2(1-n)\alpha(X)\eta(Z) + 2\eta(X)\beta(Z) - 2\eta(Z)\beta(X) \\ &+2(1-n)\gamma(Z)\eta(X) + \sigma(\xi)S(X,Z) \\ &-(3n-5)\sigma(\xi)g(X,Z) + 2(n-2)\sigma(\xi)\eta(X)\eta(Z) \\ &+2g(X,Z)p(\xi) - 2\eta(Z)p(X).\end{aligned} \tag{42}$$

Thus, comparing the right hand sides of the equations (40) and (42) we obtain

$$\begin{aligned} & 2(n-3)g(X,Z)-4(n-2)\eta(X)\eta(Z)-2S(X,Z) \\ = & 2(1-n)\alpha(X)\eta(Z)+2\eta(X)\beta(Z)-2\eta(Z)\beta(X) \\ & +2(1-n)\gamma(Z)\eta(X)+\sigma(\xi)S(X,Z)-(3n-5)\sigma(\xi)g(X,Z) \\ & +2(n-2)\sigma(\xi)\eta(X)\eta(Z)+2g(X,Z)p(\xi)-2\eta(Z)p(X). \end{aligned} \tag{43}$$

Then taking $X = Z = \xi$ in (43) and using (1) and (7) we get

$$2(1-n)[\alpha(\xi)+\gamma(\xi)+\sigma(\xi)] = 0,$$

which implies that (since $n > 3$)

$$\alpha(\xi)+\gamma(\xi)+\sigma(\xi) = 0 \tag{44}$$

holds on M. Now we shall prove that $\alpha + \gamma + \sigma = 0$ for all vector fields holds on M. In (38) replacing Z with ξ it follows that

$$\begin{aligned} & 2(n-3)g(X,V)-4(n-2)\eta(X)\eta(V)-2S(X,V) \\ = & 2(1-n)\alpha(X)\eta(V)+2g(X,V)\beta(\xi)-2\eta(V)\beta(X) \\ & +\gamma(\xi)S(X,V)-(3n-5)\gamma(\xi)g(X,V)+2(n-2)\gamma(\xi)\eta(X)\eta(V) \\ & +2(1-n)\sigma(V)\eta(X)+2\eta(X)p(V)-2\eta(V)p(X). \end{aligned} \tag{45}$$

Taking $V = \xi$ in (45) and in view of (1) and (7) we get

$$\begin{aligned} 0 = & 2(1-n)\alpha(X)+2\eta(X)\beta(\xi)-2\beta(X) \\ & +2(1-n)\gamma(\xi)\eta(X)+2(1-n)\sigma(\xi)\eta(X) \\ & +2\eta(X)p(\xi)-2p(X). \end{aligned} \tag{46}$$

Replacing X with ξ in (45) we have

$$\begin{aligned} 0 = & 2(1-n)\alpha(\xi)\eta(V)+2(1-n)\gamma(\xi)\eta(V) \\ & +2(1-n)\sigma(V)+2p(V)-2\eta(V)p(\xi). \end{aligned} \tag{47}$$

Now taking $V = X$ in (47) and summing with (46), by virtue of (44) we find

$$\begin{aligned} 0 = & 2(1-n)\alpha(X)+2\eta(X)\beta(\xi)-2\beta(X) \\ & +2(1-n)\gamma(\xi)\eta(X)+2(1-n)\sigma(X). \end{aligned} \tag{48}$$

Replacing X with ξ in (43) we have

$$\begin{aligned} 0 = & 2(1-n)\alpha(\xi)\eta(Z)+2\beta(Z)-2\eta(Z)\beta(\xi) \\ & +2(1-n)\gamma(Z)+2(1-n)\sigma(\xi)\eta(Z). \end{aligned} \tag{49}$$

Finally taking $Z = X$ in (49) and adding (48) and (49) we have

$$\begin{aligned} 0 &= 2(1-n)\eta(X)[\alpha(\xi)+\gamma(\xi)+\sigma(\xi)] \\ &\quad +2(1-n)[\alpha(X)+\gamma(X)+\sigma(X)]. \end{aligned}$$

Since $\alpha(\xi)+\gamma(\xi)+\sigma(\xi) = 0$, so we obtain $\alpha(X)+\gamma(X)+\sigma(X) = 0$ for any X on M. Thus the proof of the theorem is completed. □

Theorem 4.4. *There is no weakly Ricci-symmetric Kenmotsu manifold M with respect to the semi-symmetric metric connection, $n > 3$, unless $\rho+\mu+\upsilon$ is everywhere zero.*

Proof. Assume that M is a weakly Ricci-symmetric Kenmotsu manifold with a semi-symmetric metric connection $\widetilde{\nabla}$. Taking $Z = \xi$ in (37) and using (17) and (19) we have

$$\begin{aligned} (\widetilde{\nabla}_X \widetilde{S})(Y,\xi) &= 2\rho(X)S(Y,\xi)+2\mu(Y)S(X,\xi)+\upsilon(\xi)S(X,Y) \\ &\quad -(3n-5)\upsilon(\xi)g(X,Y)+2(n-2)\upsilon(\xi)\eta(X)\eta(Y). \end{aligned} \tag{50}$$

On the other hand from (40) and (50) we can write

$$\begin{aligned} &2(n-3)g(X,Y)-4(n-2)\eta(X)\eta(Y)-2S(X,Y) \\ &= 2\rho(X)S(Y,\xi)+2\mu(Y)S(X,\xi)+\upsilon(\xi)S(X,Y) \\ &\quad -(3n-5)\upsilon(\xi)g(X,Y)+2(n-2)\upsilon(\xi)\eta(X)\eta(Y). \end{aligned} \tag{51}$$

Putting $X = Y = \xi$ in (51) and by making use of the relations (1) and (7) we find

$$2(1-n)[\rho(\xi)+\mu(\xi)+\upsilon(\xi)] = 0,$$

which implies that (since $n > 3$)

$$\rho(\xi)+\mu(\xi)+\upsilon(\xi) = 0. \tag{52}$$

Then taking $X = \xi$ in (51) we obtain

$$2(1-n)\eta(Y)[\rho(\xi)+\upsilon(\xi)]+2(1-n)\mu(Y) = 0.$$

So in view of (52), the above equation turns into

$$2(1-n)[-\mu(\xi)\eta(Y)+\mu(Y)] = 0,$$

which shows that (since $n > 3$)

$$\mu(Y) = \mu(\xi)\eta(Y). \tag{53}$$

Similarly putting $Y = \xi$ in (51) we also have

$$\rho(X)+\eta(X)[\mu(\xi)+\upsilon(\xi)] = 0.$$

Then by virtue of (52) we get from the last equation

$$\rho(X) = \rho(\xi)\eta(X). \tag{54}$$

Since $(\widetilde{\nabla}_\xi \widetilde{S})(\xi, X) = 0$, then from (37), we find

$$\eta(X)[\rho(\xi) + \mu(\xi)] + \upsilon(X) = 0,$$

which means that

$$\upsilon(X) = \upsilon(\xi)\eta(X). \tag{55}$$

Thus replacing Y with X in (53) and summing of the equations (53), (54) and (55) we get

$$\rho(X) + \mu(X) + \upsilon(X) = [\rho(\xi) + \mu(\xi) + \upsilon(\xi)]\eta(X)$$

and so by virtue of the equation (52) it is clear that

$$\rho(X) + \mu(X) + \upsilon(X) = 0,$$

for any vector field X holds on M, which means that $\rho + \mu + \upsilon = 0$. Hence, our theorem is proved. □

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