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ON DOUBLY WARPED AND DOUBLY TWISTED PRODUCT SUBMANIFOLDS

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ABSTRACT. In the present note we study the existence or non-existence of doubly warped and doubly twisted product CR-submanifolds in nearly Kaehler manifolds.

1. Introduction

Bishop and O'Neill introduced the notion of warped product manifolds.These manifolds are generalization of Riemannian product manifolds and occur naturally (e.g. surface of revolution is a warped product manifold). With regard to the physical applications of these manifolds, one may realize that the space time around a massive star or a black hole can be modeled on a warped product manifold for instance, the relativistic model of Schwarzschild. Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds of dimensions m and n, respectively and let $\pi_1 : N_1 \times N_2 \to N_1$ and $\pi_2 : N_1 \times N_2 \to N_2$ be the canonical projections. Also let $f_1 : N_1 \times N_2 \to \mathbb{R}^+$, $f_2 : N_1 \times N_2 \to \mathbb{R}^+$ smooth functions. Then the Doubly twisted product ([5], [10]) of (N_1, g_1) and (N_2, g_2) with twisting functions f_1 and f_2 is defined to be the product manifold $M = N_1 \times N_2$ with metric tensor $g = f_2^2 g_1 + f_1^2 g_2$. The twisted product manifold $(N_1 \times N_2, g)$ is denoted by ${}_{f_2}N_1 \times_{f_1} N_2$. If X is tangent to N_1 and Z is tangent to N_2, then from Proposition 1 of [5], we have

$$\nabla_X Z = \nabla_Z X = (Z \ln f_2)X + (X \ln f_1)Z, \tag{1.1}$$

where ∇ denotes the Levi-Civita connection of the doubly twisted product ${}_{f_2}N_1 \times {}_{f_1}N_2$ of (N_1, g_1) and (N_2, g_2). In particular, if $f_1 = 0$, then ${}_{f_2}N_1 \times N_2$ is called the twisted product of (N_1, g_1) and (N_2, g_2) with twisting function f_2. The notion of twisted products was introduced by Chen in [2]. If $M = N_1 \times {}_f N_2$ is a twisted product manifold, then (1.1) becomes

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z \tag{1.2}$$

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for all $X \in TN_1$ and $Z \in TN_2$. Moreover, if f only depends on the points of N_1, then $N_1 \times {}_f N_2$ is called warped product of (N_1, g_1) and (N_2, g_2) with warping function f. In this case, for $X \in TN_1$ and $Z \in TN_2$, from Lemma 7.3 of [1], we have

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z \tag{1.3}$$

where f depends on the points of N_1 only.

As a generalization of the warped product of two Riemannian manifolds, ${}_{f_2}N_1 \times {}_{f_1} N_2$ is called the doubly warped product of Riemannian manifolds (N_1, g_1) and (N_2, g_2) with warping functions f_1 and f_2 if only depend on the points of N_1 and N_2, respectively.

The study of differential geometry of warped product manifolds are intensified after the impulse given by B.Y. Chen's work on warped product CR-submanifolds of Kaehler manifolds (cf., [3],[4]). In the present paper we show that there do not exist doubly warped and doubly twisted product CR-submanifolds in nearly Kaehler manifolds.

2. **Preliminaries**

Let $(\bar{M}, J, g)$ be a nearly Kaehler manifold with an almost complex structure J and Hermitian metric g and a Levi-Civita connection $\bar{\nabla}$ such that

$$J^2 = -I, \tag{2.1}$$

$$g(JX, JY) = g(X, Y), \tag{2.2}$$

$$(\bar{\nabla}_X J)X = 0. \tag{2.3}$$

for all vector fields X and Y on $\bar{M}$.

Let $\bar{M}$ be a nearly Kaehler manifold with an almost complex structure J and Hermitian metric g and M, a Riemannian manifold isometrically immersed in $\bar{M}$. Then M is called *holomorphic (complex)* if $J(T_pM) \subset T_pM$, for every $p \in M$ where T_pM denotes the tangent space to M at the point p. M is called *totally real* if $J(T_pM) \subset T_p^\perp M$ for every $p \in M$, where $T_p^\perp M$ denotes the no rmal space to M at the point p. A submanifold M is called a *CR-submanifold* if there exist on M a differentiable distribution $D : p \to D_p \subset T_pM$ such that D is invariant with respect to J and its orthogonal complement $D^\perp$ is totally real distribution, i.e., $J(D^\perp) \subseteq T_p^\perp M$. Obviously holomorphic and totally real submanifolds are CR-submanifolds having $D = T_pM$ and $D = 0$, respectively. A CR-submanifold is called proper if it is neither holomorphic nor totally real.

Let M be a submanifold of $\bar{M}$. Then the induced Riemannian metric on M is denoted by the same symbol g and the induced connection on M is denoted by the symbol ∇. If $T\bar{M}$ and TM denote the tangent bundle on $\bar{M}$ and M respectively and $T^\perp M$, the normal bundle on M, then the Gauss and Weingarten formulae are respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.4}$$

(2.5) $$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N,$$

for $X, Y \in TM$ and $N \in T^{\perp}M$ where $\nabla^{\perp}$ denotes the connection on the normal bundle $T^{\perp}M$. h and A_N are the second fundamental forms and the shape operator of the immersions of M into $\bar{M}$ corresponding to the normal vector field N. They are related as

(2.6) $$g(A_N X, Y) = g(h(X, Y), N).$$

For any $X \in TM$ and $N \in T^{\perp}M$ we write

(2.7) $$JX = PX + FX,$$

(2.8) $$JN = tN + fN,$$

where PX and tN are the tangential components of JX and JN respectively and FX and fN are the normal components of JX and JN respectively.

The covariant differentiation of the tensors P and F are defined respectively as

(2.9) $$(\bar{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y,$$

(2.10) $$(\bar{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y,$$

Furthermore, for any $X, Y \in TM$, let us decompose $(\bar{\nabla}_X J)Y$ into tangential and normal parts as

(2.11) $$(\bar{\nabla}_X J)Y = P_X Y + Q_X Y.$$

By making use of equations (2.4)–(2.10), we may obtain that

(2.12) $$P_X Y = (\bar{\nabla}_X P)Y - A_{FY}X - th(X, Y),$$

(2.13) $$Q_X Y = (\bar{\nabla}_X F)Y + h(X, PY) - fh(X, Y).$$

The following properties of P and Q are used in our subsequent sections and can be verified through a straightforward computation

$$(p_1)\ (i)\ P_{X+Y}W = P_X W + P_Y W,\ (ii)\ Q_{X+Y}W = Q_X W + Q_Y W.$$

$$(p_2)\ (i)\ P_X(Y + W) = P_X Y + P_X W,\ (ii)\ Q_X(Y + W) = Q_X Y + Q_X W.$$

$$(p_3)\ (i)\ g(P_X Y,\ W) = -g(Y, P_X W),\ (ii)\ g(Q_X Y,\ N) = -g(Y, P_X N).$$

$$(p_4)\ P_X JY + Q_X JY = -J(P_X Y + Q_X Y).$$

On a submanifold M of a nearly Kaehler manifold, by equations (2.3) and (2.11) we obtain

(2.14) $$(a)\ P_X Y + P_Y X = 0,\ (b)\ Q_X Y + Q_Y X = 0$$

for any $X,\ Y \in TM$.

3. **Doubly Warped and Doubly Twisted Product** CR**-Submanifolds**

Throughout the section, we consider CR-submanifolds which are either doubly warped or doubly twisted product submanifolds in the form ${}_{f_2}N_T \times {}_{f_1}N_\perp$, where N_T and $N_\perp$ are holomorphic and totally real submanifolds of nearly Kaehler manifolds $\bar{M}$, respectively.

Theorem 3.1. *Let $\bar{M}$ be a nearly Kaehler manifold. Then there do not exist doubly twisted product CR-submanifolds which are not (singly) twisted product CR-submanifolds of the form ${}_{f_2}N_T \times {}_{f_1}N_\perp$ such that N_T is a holomorphic submanifold and $N_\perp$ is a totally real submanifold of $\bar{M}$.*

Proof. From (1.1), we have $g(\nabla_X Z, Y) = (Z \ln f_2)g(X, Y)$ for all $X,\ Y \in TN_T$ and $Z \in TN_\perp$. Since D and $D^\perp$ the corresponding distributions of N_T and $N_\perp$, respectively. These two distributions are orthogonal, we get $-g(\nabla_X Y, Z) = (Z \ln f_2)g(X, Y)$. In particular, on using Gauss formula and (2.3), we obtain

$$-g(\bar{\nabla}_X JX, JZ) = (Z \ln f_2)\|X\|^2.$$

Thus, from (2.4), we drive

$$-g(h(X, JX), JZ) = (Z \ln f_2)\|X\|^2 \tag{3.1}$$

for $X,\ Y \in TN_T$ and $Z \in TN_\perp$. On the other hand, by (2.4), we have

$$g(h(X, JX), JZ) = g(\bar{\nabla}_{JX} X, JZ)$$

for $X,\ Y \in TN_T$ and $Z \in TN_\perp$. On using (2.2), (2.11), (2.14) (a) and (p_4), above equation gives

$$\begin{aligned} g(h(X, JX), JZ) &= -g(J\bar{\nabla}_{JX} X, Z) \\ &= -g(\bar{\nabla}_{JX} JX, Z) + g((\bar{\nabla}_{JX} J)X, Z) \\ &= g(JX, \bar{\nabla}_{JX} Z) + g(P_{JX} X, Z) \\ &= g(JX, \nabla_{JX} Z) - g(P_X JX, Z) \\ &= (Z \ln f_2)g(JX, JX) + g(JP_X X, Z) \\ &= (Z \ln f_2)g(X, X) - g(P_X X, JZ) \\ &= (Z \ln f_2)\|X\|^2. \end{aligned}$$

That is,

$$g(h(X, JX), JZ) = (Z \ln f_2)\|X\|^2. \tag{3.2}$$

Since N_T is Riemannian then the equations (3.1) and (3.2) imply that $Z \ln f_2 = 0$, for all $Z \in TN_\perp$. This means that f_2 is constant on $N_\perp$, i.e., f_2 only depends on the points of N_T. Thus it follow that M is twisted product CR-submanifold of the form $N_T \times {}_{f_1}N_\perp$, (see [2] for twisted product CR-submanifolds). Hence, we see that there are no doubly twisted product CR-submanifolds in nearly Kaehler manifolds, other than twisted product CR-submanifolds. This proves the theorem completely. □

The following corollary is an immediate consequence of the above theorem.

Corollary 3.1. *There do not exist doubly warped product CR-submanifolds* ${}_{f_2}N_T \times {}_{f_1}N_\perp$ *of a nearly Kaehler manifold* $\bar{M}$ *such that* N_T *is a holomorphic submanifold and* $N_\perp$ *is a totally real submanifold of* $\bar{M}$.

Proof. The proof follows from Theorem 3.1. □

Corollary 3.1 says that there exist no doubly warped product CR-submanifolds in nearly Kaehler manifolds other than warped product CR-submanifolds.

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References

[1] Bishop, R.L. and O'Neill, B., Manifolds of negative curvature, Trans. Amer. Math. Soc., 145 (1969), 1-49.

[2] Chen, B.Y., Twisted product CR-submanifolds in Kaehler manifolds, Tamsui Oxf. J. Math. Sci., 16 (2000), 105-121.

[3] Chen, B.Y., Geometry of warped product CR-submanifolds in Kaehler manifolds, Monatsh. Math., 133 (2001), 177-195.

[4] Chen, B.Y., Geometry of warped product CR-submanifolds in Kaehler manifolds II, Monatsh. Math., 134 (2001), 103-119.

[5] Fernandez, L.M., Garcia, R.E., Kupeli, D.N. and Unal, B., A curvature condition for a twisted product to be a warped product, Manuscripta Math., 106 (2001), 213-217.

[6] Gray, A., Nearly Kaehler manifolds, J. Diff. Geom., 4 (1970), 283-309.

[7] Khan, K.A. and Khan, V.A., Totally umbilical CR-submanifolds of a nearly Kaehler manifold, Geom. Dedicata, 50 (1994), 47-51.

[8] Khan, V.A., Khan, K.A. and Uddin S., Warped product CR-submanifolds of a nearly Kaehler manifold, SUT Math. J., 43 (2007), 201-213.

[9] Munteanu, M.I., Doubly warped products CR-submanifolds in locally conformal Kaehler manifolds, Monatsh. Math., 150 (2007), 333-342.

[10] Ponge R. and Reckziegel H., Twisted products in pseudo-Riemannian geometry, Geom. Dedicata, 48 (1993), 15-25.

[11] Unal, B., Doubly warped products, PhD Thesis, University of Missouri-Columbia, 2000.

[12] Unal, B., Doubly warped products, Diff. Geom. Appl., 15 (2001), 253-263.

[13] Yano, K. and Kon, M., Structures on manifolds, Series in Pure Mathematics, World Scientific Publishing co., Singapore, 1984.

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