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SOME CHARACTERIZATIONS OF PSEUDO AND PARTIALLY NULL OSCULATING CURVES IN MINKOWSKI SPACE-TIME

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Abstract. In this paper, we characterize pseudo and partially null osculating curves of the first and second kind in Minkowski space-time $\mathbb{E}_1^4$ in terms of their curvature functions. We give the necessary and sufficient conditions for pseudo and partially null curves to be osculating curves. In particular, we show that there exists a simple relationship between pseudo and partially null osculating curves and pseudo and partially null normal and rectifying curves. Finally, we obtain some explicit parameter equations of pseudo and partially null osculating curves in $\mathbb{E}_1^4$.

1. Introduction

In the Euclidean space $\mathbb{E}^3$, rectifying curves are defined in [2] as the curves whose rectifying planes always contain a fixed point. Such curves have many interesting geometrical properties. For example, there exists a simple relationship between rectifying curves and centrodes (Darboux vectors), which play some important roles in mechanics, kinematics as well as in differential geometry. In Minkowski space-time, rectifying curves are characterized in [7].

Analogously, timelike normal curves in Minkowski 3-space $\mathbb{E}_1^3$ are defined as the curves whose normal planes always contain a fixed point. Therefore, the position vector of such curves (with respect to some chosen origin), always lies in its normal plane ([3]). In particular, timelike normal curves lie in pseudosphere in $\mathbb{E}_1^3$.

In elementary differential geometry, it is well-known that if osculating planes of the curve α in $\mathbb{E}^3$ always contain a fixed point, then α is planar curve. The curves whose osculating planes always contain a fixed point are called osculating curves in [5].

In Minkowski space-time $\mathbb{E}_1^4$, osculating curves are defined in [6] as the curves whose the position vector (with respect to some chosen origin) always lies in its osculating space. If osculating space is orthogonal complement of the second binormal vector field B_2 of the curve α, then α is called osculating curve of the first

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kind. On the other hand, if osculating space is orthogonal complement of the first binormal vector field B_1, then α is called osculating curve of the second kind. In $\mathbb{E}_1^4$, timelike osculating curves and spacelike osculating curves whose the Frenet frame contains only non-null vectors are characterized in [6].

In this paper, we characterize pseudo and partially null osculating curves of the first and second kind in $\mathbb{E}_1^4$ in terms of their curvature functions. We give the necessary and sufficient conditions for pseudo and partially null curves to be osculating curves. In particular, we show that there exists a simple relationship between pseudo and partially null osculating curves and pseudo and partially null normal and rectifying curves. Finally, we obtain some explicit parameter equations of pseudo and partially null osculating curves in $\mathbb{E}_1^4$.

2. Preliminary

The Minkowski space-time $\mathbb{E}_1^4$ is the Euclidean 4-space $\mathbb{E}^4$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{E}_1^4$. Recall that arbitrary vector $v \in \mathbb{E}_1^4 \backslash \{0\}$ can be *spacelike* if $g(v,v) > 0$, *timelike* if $g(v,v) < 0$ and *null (lightlike)* if $g(v,v) = 0$ and $v \neq 0$. In particular, the vector $v = 0$ is a spacelike. The norm of a vector v is given by $||v|| = \sqrt{|g(v,v)|}$. Two vectors v and w are said to be orthogonal if $g(v,w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^4$ can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null. A spacelike or a timelike curve $\alpha(s)$ has unit speed, if $g(\alpha'(s), \alpha'(s)) = \pm 1$ [8]. A spacelike curve in $\mathbb{E}_1^4$ is called pseudo null or partially null curve, if respectively its principal normal vector is null or its first binormal vector is null [1]. A null curve α has unit speed, if $g(\alpha''(s), \alpha''(s)) = \pm 1$.

Let $\{T, N, B_1, B_2\}$ be the moving Frenet frame along a curve α in $\mathbb{E}_1^4$, consisting of the tangent, the principal normal, the first binormal and the second binormal vector fields respectively. If α is pseudo null curve, the Frenet formulas are ([1,9]):

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & -k_2 \\ -k_1 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.1}$$

where the first curvature $k_1(s) = 0$, if α is straight line, or $k_1(s) = 1$ in all other cases. Such curve has two curvatures $\kappa_2(s)$ and $\kappa_3(s)$ and the following conditions are satisfied

$$g(T,T) = g(B_1, B_1) = 1, \quad g(N,N) = g(B_2, B_2) = 0,$$

$$g(T,N) = g(T,B_1) = g(T,B_2) = g(N,B_1) = g(B_1,B_2) = 0, \quad g(N,B_2) = 1.$$

If α is partially null curve, the Frenet formulas read ([1,9]):

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & -k_2 & 0 & -k_3 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.2}$$

where the third curvature $\kappa_3(s) = 0$ for each s. Such curve has two curvatures $\kappa_1(s)$ and $\kappa_2(s)$ and lies fully in a lightlike hyperplane of $\mathbb{E}^4_1$. In particular, the following equations hold

$$g(T,T) = g(N,N) = 1, \quad g(B_1,B_1) = g(B_2,B_2) = 0,$$

$$g(T,N) = g(T,B_1) = g(T,B_2) = g(N,B_1) = g(N,B_2) = 0, \quad g(B_1,B_2) = 1.$$

Recall that arbitrary curve α in $\mathbb{E}^4_1$ is called *osculating curve of the first or second kind*, if its position vector (with respect to some chosen origin) always lies in the orthogonal complement $B_2^{\perp}$ or $B_1^{\perp}$, respectively ([6]).

If α is pseudo null curve, the orthogonal complement $B_2^{\perp}$ is a lightlike (degenerate) hyperplane of $\mathbb{E}^4_1$, spanned by $\{T, B_1, B_2\}$ and $B_1^{\perp}$ is non-degenerate hyperplane of $\mathbb{E}^4_1$, spanned by $\{T, N, B_2\}$. Consequently, the position vector of pseudo null osculating curve of the first and second kind satisfies respectively the equations

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B_1(s) + \nu(s)B_2(s), \tag{2.3}$$

$$\alpha(s) = \lambda(s)T(s) + \mu(s)N(s) + \nu(s)B_2(s), \tag{2.4}$$

for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ in arclength function s.

If α is partially null osculating curve of the first kind, then its position vector satisfies the condition $g(\alpha(s), B_2(s)) = 0$. It follows that the position vector of partially null osculating curve of the first kind is given by

$$\alpha(s) = \lambda(s)T(s) + \mu(s)N(s), \tag{2.5}$$

where $\lambda = g(\alpha, T)$ and $\mu = g(\alpha, N)$ are arbitrary differentiable functions in arclength function s.

Moreover, if α is partially null osculating curve of the second kind, then its position vector satisfies the equation $g(\alpha(s), B_1(s)) = 0$. Consequently, the position vector of partially null osculating curve of the second kind satisfies the relation

$$\alpha(s) = \lambda(s)T(s) + \mu(s)N(s) + \nu(s)B_1(s), \tag{2.6}$$

where $\lambda = g(\alpha, T)$, $\mu = g(\alpha, N)$ and $\nu = g(\alpha, B_2)$ are arbitrary differentiable functions in arclenght function s.

3. Partially null osculating curves of the first kind in $\mathbb{E}^4_1$

Partially null straight lines are the simplest examples of partially null osculating curves of the first kind. In the next theorem, we show that every planar partially null curve in $\mathbb{E}^4_1$, is partially null osculating curve of the first kind.

Theorem 3.1. *Let α be a unit speed partially null curve in $\mathbb{E}^4_1$ with the first curvature $k_1(s) \neq 0$. Then α is osculating curve of the first kind if and only if its second curvature $k_2(s) = 0$ for each s.*

Proof. First assume that α is partially null osculating curve of the first kind. Then its position vector satisfies relation (2.5). Differentiating relation (2.5) with respect to s and using Frenet equations (2.2), we obtain the system of equations

$$\begin{aligned} &\lambda'(s) - \mu(s)k_1(s) = 1, \\ &\mu'(s) + \lambda(s)k_1(s) = 0, \\ &\mu(s)k_2(s) = 0. \end{aligned} \tag{3.1}$$

The third equation of (3.1) implies $\mu(s) = 0$ or $k_2(s) = 0$. If $\mu(s) = 0$, by using (3.1) we get a contradiction. Hence $\mu(s) \neq 0$ and $k_2(s) = 0$.

Conversely, suppose that the second curvature $k_2(s) = 0$ for each s. By using the Frenet formulas (2.2), we get $\alpha'(s) = T(s)$, $\alpha''(s) = k_1(s)N(s)$, $\alpha'''(s) = k_1'(s)N(s) - k_1^2(s)T(s)$. Moreover, all higher order derivatives of α are linear combinations of vectors T and N. By using *MacLaurin* expansion for α given by

$$\alpha(s) = \alpha(0) + \alpha'(0)s + \alpha''(0)\frac{s^2}{2!} + \alpha'''(0)\frac{s^3}{3!} + \dots,$$

we conclude that α lies fully in spacelike 2-plane spanned by $\{T, N\}$. Hence its position vector satisfies relation (2.5), which proves the theorem. □

Theorem 3.2. *Let α be a unit speed partially null curve in $\mathbb{E}^4_1$ with the first curvature $k_1(s) \neq 0$. Then α is osculating curve of the first kind if and only if its position vector is given by*

$$\alpha(s) = -\frac{\mu'(s)}{k_1(s)}T(s) + \mu(s)N(s), \tag{3.2}$$

where $\mu(s) \neq 0$ is arbitrary differentiable function satisfying differential equation

$$\mu''(s)k_1(s) - \mu'(s)k_1'(s) + \mu(s)k_1^3(s) + k_1^2(s) = 0. \tag{3.3}$$

Proof. First assume that α is partially null osculating curve of the first kind. According to the proof of theorem 3.1, there holds relation (3.1) where $\mu(s) \neq 0$. From the second equation of (3.1) we get

$$\lambda(s) = -\frac{\mu'(s)}{k_1(s)}. \tag{3.4}$$

Substituting (3.4) in (2.5), we obtain that position vector of α is given by (3.2). Moreover, by using the first and second equation of (3.1), we easily get that function $\mu(s)$ satisfies relation (3.3).

Conversely, if position vector of partially null curve α satisfies relation (3.2) we easily get $g(\alpha(s), B_2(s)) = 0$, which means that α is partially null osculating curve of the first kind. This completes the proof of theorem. □

Clearly, the solution of differential equation (3.3) depends on the equation of the first curvature $k_1(s)$. By theorem 3.1, every partially null osculating curve α of the first kind lies fully in spacelike 2-plane of $\mathbb{E}^4_1$, so it is possible to determine its parameter equation by knowing the equation of the first curvature $k_1(s)$. Up to isometries of $\mathbb{E}^4_1$, parameter equation of α can be written as

$$\alpha(s) = \int_0^s T(s)\,ds + C = \int_0^s \frac{1}{k_1(\phi)}(\cos\phi\, e_2 + \sin\phi\, e_3)\,d\phi + C,$$

where $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, $k_1(s) = \phi'(s)$, $C \in \mathbb{E}^4_1$ is constant vector and $\phi(s)$ is the angle between spacelike vectors $T(s)$ and e_2.

Example 3.1. Let us consider partially null curve in $\mathbb{E}^4_1$ given by

$$\alpha(s) = (0, \sqrt{2s}\sin(\sqrt{2s}) + \cos(\sqrt{2s}), \sin(\sqrt{2s}) - \sqrt{2s}\cos(\sqrt{2s}), 0).$$

By using Frenet equations (2.2), we find that Frenet vectors and curvatures of α have the form

$$\begin{aligned}
&T(s) = (0, \cos(\sqrt{2s}), \sin(\sqrt{2s}), 0),\\
&N(s) = (0, -\sin(\sqrt{2s}), \cos(\sqrt{2s}), 0),\\
&B_1(s) = (1, 0, 0, 1),\\
&B_2(s) = (-\frac{1}{2}, 0, 0, \frac{1}{2}),\\
&k_1(s) = \frac{1}{\sqrt{2s}}, \quad k_2(s) = k_3(s) = 0.
\end{aligned}$$

In particular, by using relation (3.3) we obtain $\mu(s) = -\sqrt{2s}$. According to theorem 3.2 the position vector of α can be written as

$$\alpha(s) = T(s) - \sqrt{2s}N(s).$$

The last equation implies $g(\alpha(s), B_2(s)) = 0$, which means that α is partially null osculating curve of the first kind.

Recall that arbitrary curve α in $\mathbb{E}^4_1$ is called normal curve in [4] (rectifying curve in [7]), if its position vector with respect to some chosen origin always lies in the orthogonal complement of the tangential vector field T (principal normal vector field N) of the curve. The next theorem gives the simple relation between partially null osculating curves of the first kind and partially null normal curves.

Theorem 3.3. *Every partially null osculating curve of the first kind with the first curvature $k_1(s) \neq 0$ and zero tangential component $g(\alpha, T)$ is partially null normal curve and hence a circle.*

It is proved in [7] that the only partially null rectifying curves in $\mathbb{E}^4_1$ are partially null straight lines. Hence the next theorem holds.

Theorem 3.4. *Every partially null rectifying curve in $\mathbb{E}^4_1$ is partially null osculating curve of the first kind.*

4. Partially null osculating curves of the second kind in $\mathbb{E}^4_1$

In this section, we characterize partially null osculating curves of the second kind in $\mathbb{E}^4_1$ with the curvature functions $k_1(s) \neq 0$ and $k_2(s) \neq 0$ for each s. The following theorem can be proved in a similar way as theorem 3.2, so we omit its proof.

Theorem 4.1. *Let α be a unit speed partially null curve in $\mathbb{E}^4_1$ with the curvatures $k_1(s) \neq 0$ and $k_2(s) \neq 0$. Then α is osculating curve of the second kind if and only if its position vector is given by*

$$\alpha(s) = -\frac{\mu'(s)}{k_1(s)}T(s) + \mu(s)N(s) - \int \mu(s)k_2(s)\, ds\, B_1(s), \tag{4.1}$$

where $\mu(s) \neq 0$ is arbitrary differentiable function satisfying relation (3.3).

As a consequence, we obtain the next theorem.

Theorem 4.2. *Let $\alpha(s)$ be a unit speed partially null curve in $\mathbb{E}^4_1$ with the curvatures $k_1(s) \neq 0$ and $k_2(s) \neq 0$. If α is osculating curve of the second kind, then the following statements hold:*

(i) the tangential and principal normal component of the position vector are respectively given by

$$g(\alpha(s),T(s)) = -\frac{\mu'(s)}{k_1(s)}, \quad g(\alpha(s),N(s)) = \mu(s),$$

where μ is arbitrary differentiable function satisfying (3.3).

(ii) the tangential and second binormal component of the position vector are respectively given by

$$g(\alpha(s),T(s)) = -\frac{\mu'(s)}{k_1(s)}, \quad g(\alpha(s),B_2(s)) = -\int \mu(s)\,k_2(s)\,ds,$$

where μ is arbitrary differentiable function satisfying (3.3).

(iii) the tangential component of the position vector and the distance function $\rho(s) = ||\alpha(s)||$ are respectively given by

$$g(\alpha(s),T(s)) = -\frac{\mu'(s)}{k_1(s)}, \quad \rho^2(s) = \frac{\mu'^2(s)}{k_1^2(s)} + \mu^2(s),$$

where μ is arbitrary differentiable function satisfying (3.3).

Conversely, if $\alpha(s)$ is unit speed partially null curve in $\mathbb{E}_1^4$ with the curvatures $k_1(s) \neq 0$, $k_2(s) \neq 0$ and one of the statements (i),(ii) or (iii) holds, then $\alpha(s)$ is osculating curve of the second kind.

Proof. Let us first assume that α is partially null osculating curve of the second kind. By using relation (4.1), we easily obtain that statements (i),(ii) and (iii) hold.

Conversely, suppose that statement (i) holds. By taking the derivative of the relation $g(\alpha, N) = \mu$ with respect to s and using (2.2), we get $g(\alpha, B_1) = 0$, which means that α is osculating curve of the second kind. If statement (ii) holds, differentiating the equation $g(\alpha, B_2) = -\int \mu k_2\,ds$ two times with respect to s and using (2.2), we find $g(\alpha, B_1) = 0$. Hence α is osculating curve of the second kind. Finally, if statement (iii) holds, then $g(\alpha, \alpha) = g(\alpha, T)^2 + \mu^2$. By taking the derivative of the last equation two times with respect to s and using (2.2), we get $g(\alpha, B_1) = 0$, which completes the proof of the theorem. □

The following theorem gives the simple relation between partially null osculating curves of the second kind and partially null normal curves.

Theorem 4.3. *Every partially null osculating curve of the second kind in $\mathbb{E}_1^4$ with tangential component of the position vector $g(\alpha, T) = 0$ is partially null normal curve.*

Example 4.1. Let us consider partially null helix with null axes

$$\alpha(s) = (s, s, \frac{1}{c}\cos(cs), \frac{1}{c}\sin(cs)), \quad c \in \mathbb{R}_0^+.$$

we find that Frenet vectors and curvatures of α are given by

$$\begin{aligned}
&T(s) = (1, 1, -\sin(cs), \cos(cs)),\\
&N(s) = (0, 0, -\cos(cs), -\sin(cs)),\\
&B_1(s) = (c, c, 0, 0),\\
&B_2(s) = (-\frac{1}{c}, 0, \frac{1}{c}\sin(cs), -\frac{1}{c}\cos(cs)),\\
&k_1(s) = c, \quad k_2(s) = 1, \quad k_3(s) = 0, \quad c \in \mathbb{R}_0^+.
\end{aligned}$$

It can be easily verified that $g(\alpha(s), B_1(s)) = 0$, so α is partially null osculating curve of the second kind. Moreover, by using theorem 4.1 we find $\mu(s) = -1/c$. Then relation (4.1) imply that the position vector of α can be written as

$$\alpha(s) = -\frac{1}{c}N(s) + \frac{s}{c}B_1(s).$$

Example 4.2. Let us consider partially null curve given by

$$\beta(s) = (e^s, e^s, \frac{1}{c}\cos(cs), \frac{1}{c}\sin(cs)), \quad c \in \mathbb{R}_0^+.$$

We obtain that Frenet frame of β has the form

$$\begin{aligned} T(s) &= (e^s, e^s, -\sin(cs), \cos(cs)), \\ N(s) &= \frac{1}{c}(e^s, e^s, -c\cos(cs), -c\sin(cs)), \\ B_1(s) &= \frac{1+c^2}{c}(1,1,0,0), \end{aligned}$$

$$\begin{aligned} B_2(s) &= (-\frac{e^{2s}}{2c} - \frac{c}{2(1+c^2)}, -\frac{e^{2s}}{2c} + \frac{c}{2(1+c^2)}, \frac{e^s}{1+c^2}(\cos(cs) + c\sin(cs)), \\ &\quad \frac{e^s}{1+c^2}(\sin(cs) - c\cos(cs))), \end{aligned}$$

while the curvature functions are given by

$$k_1(s) = c, \quad k_2(s) = e^s, \quad k_3(s) = 0, \quad c \in \mathbb{R}_0^+.$$

It can be easily checked that $g(\beta(s), B_1(s)) = 0$, which means that β is partially null osculating curve of the second kind. In particular, according to theorem 4.1 the position vector of β can be written as

$$\beta(s) = -\frac{1}{c}N(s) + \frac{e^s}{c}B_1(s), \quad c \in \mathbb{R}_0^+.$$

5. Pseudo null osculating curves of the first kind in $\mathbb{E}_1^4$

Let $\alpha : I \to \mathbb{E}_1^4$ be pseudo null osculating curve of the first kind, with curvatures $k_1(s) = 1$, $k_2(s) \neq 0$ and $k_3(s)$, where the third curvature can be equal to zero or different from zero. Since the position vector of α satisfies relation (2.3), differentiating relation (2.3) with respect to s and by applying (2.1) we obtain the system of equations

$$\begin{aligned} \lambda'(s) - \nu(s) &= 1, \\ \nu'(s) - \mu(s)k_2(s) &= 0, \\ \lambda(s) + \mu(s)k_3(s) &= 0, \\ \mu'(s) - \nu(s)k_3(s) &= 0. \end{aligned} \tag{5.1}$$

We may distinguish two cases: **(A)** $k_3(s) = 0$ and **(B)** $k_3(s) \neq 0$.

(A) $k_3(s) = 0$.

In this case, the system of equations (5.1) becomes

$$\begin{aligned} &\lambda(s)=0,\\ &\mu(s)=0,\\ &\nu(s)=-1,\\ &\mu'(s)=0. \end{aligned} \tag{5.2}$$

Substituting relation (5.2) in relation (2.3), it follows that the position vector of α satisfies the equation

$$\alpha(s)=-B_2(s). \tag{5.3}$$

In this way, the following theorem is obtained.

Theorem 5.1. *Let α be pseudo null curve lying fully in $\mathbb{E}^4_1$ with curvatures $k_1(s)=1$, $k_2(s)\neq 0$ and $k_3(s)=0$. If α is osculating curve of the first kind then the following statements hold:*

(i) the tangential component $g(\alpha,T)$ of the position vector is zero;

(ii) the principal normal component of the position vector is given by $g(\alpha,N)=-1$;

(iii) the first binormal component $g(\alpha,B_1)$ of the position vector is zero;

(iv) α lies in a lightcone.

Conversely, if α is pseudo null curve lying fully in $\mathbb{E}^4_1$ with curvatures $k_1(s)=1$, $k_2(s)\neq 0$, $k_3(s)=0$ and one of statements (i), (ii), (iii) or (iv) holds, then α is osculating curve of the first kind.

Theorem 5.1 can be proved in a similar way as theorem 4.2. Hence we omit its proof.

Example 5.1. Let us consider pseudo null curve with the equation

$$\alpha(s)=(\frac{1}{\sqrt{2c}}\cosh(\sqrt{c}s),\frac{1}{\sqrt{2c}}\sinh(\sqrt{c}s),\frac{1}{\sqrt{2c}}\sin(\sqrt{c}s),-\frac{1}{\sqrt{2c}}\cos(\sqrt{c}s)),\quad c\in\mathbb{R}^+_0.$$

We obtain that Frenet vectors of α have the form

$$\begin{aligned} T(s)&=(\frac{1}{\sqrt{2}}\sinh(\sqrt{c}s),\frac{1}{\sqrt{2}}\cosh(\sqrt{c}s),\frac{1}{\sqrt{2}}\cos(\sqrt{c}s),\frac{1}{\sqrt{2}}\sin(\sqrt{c}s)),\\ N(s)&=(\frac{\sqrt{c}}{\sqrt{2}}\cosh(\sqrt{c}s),\frac{\sqrt{c}}{\sqrt{2}}\sinh(\sqrt{c}s),-\frac{\sqrt{c}}{\sqrt{2}}\sin(\sqrt{c}s),\frac{\sqrt{c}}{\sqrt{2}}\cos(\sqrt{c}s)),\\ B_1(s)&=(\frac{1}{\sqrt{2}}\sinh(\sqrt{c}s),\frac{1}{\sqrt{2}}\cosh(\sqrt{c}s),-\frac{1}{\sqrt{2}}\cos(\sqrt{c}s),-\frac{1}{\sqrt{2}}\sin(\sqrt{c}s)),\\ B_2(s)&=(-\frac{1}{\sqrt{2c}}\cosh(\sqrt{c}s),-\frac{1}{\sqrt{2c}}\sinh(\sqrt{c}s),-\frac{1}{\sqrt{2c}}\sin(\sqrt{c}s),\frac{1}{\sqrt{2c}}\cos(\sqrt{c}s)). \end{aligned}$$

In particular, the curvature functions are given by

$$k_1(s)=1,\quad k_2(s)=c,\quad k_3(s)=0,\quad c\in\mathbb{R}^+_0.$$

It can be easily verified that $g(\alpha(s),B_2(s))=0$, which means that α is pseudo null osculating curve of the first kind.

(B) $k_3(s)\neq 0$.

In this case, the system of equations (5.1) becomes

$$\begin{aligned} &(1+k_3^2(s))\mu'(s)+k_3(s)k_3'(s)\mu(s)+k_3(s)=0,\\ &\lambda(s)=-k_3(s)\mu(s),\\ &\nu'(s)=k_2(s)\mu(s),\\ &\nu(s)=\mu'(s)/k_3(s). \end{aligned} \tag{5.4}$$

The first equation in the system of equations (5.4) is linear differential equation whose general solution is given by

$$\mu(s)=\frac{1}{\sqrt{1+k_3^2(s)}}(c-\int\frac{k_3(s)}{\sqrt{1+k_3^2(s)}}ds),\quad c\in\mathbb{R}. \tag{5.5}$$

By using relations (5.4) and (5.5), we obtain the following theorems.

Theorem 5.2. *Let α be pseudo null curve lying fully in $\mathbb{E}_1^4$ with curvatures $k_1(s)=1$, $k_2(s)\neq 0$ and $k_3(s)\neq 0$. If α is osculating curve of the first kind, then the following statements hold:*

(i) the tangential component of the position vector is given by

$$g(\alpha(s),T(s))=-\frac{k_3(s)}{\sqrt{1+k_3^2(s)}}(c-\int\frac{k_3(s)}{\sqrt{1+k_3^2(s)}}ds),\quad c\in\mathbb{R};$$

(ii) the principal normal component of the position vector is given by

$$g(\alpha(s),N(s))=-\frac{1}{1+k_3^2(s)}-\frac{k_3'(s)}{(1+k_3^2(s))^{\frac{3}{2}}}(c-\int\frac{k_3(s)}{\sqrt{1+k_3^2(s)}}ds),\quad c\in\mathbb{R};$$

(iii) the first binormal component of the position vector is given by

$$g(\alpha(s),B_1(s))=\frac{1}{\sqrt{1+k_3^2(s)}}(c-\int\frac{k_3(s)}{\sqrt{1+k_3^2(s)}}ds),\quad c\in\mathbb{R}.$$

Conversely, if α is pseudo null curve lying fully in $\mathbb{E}_1^4$ with curvatures $k_1(s)=1$, $k_2(s)\neq 0$, $k_3(s)\neq 0$ and one of statements (i), (ii) or (iii) holds, then α is osculating curve of the first kind.

Theorem 5.3. *Let α be pseudo null curve lying fully in $\mathbb{E}_1^4$ with curvatures $k_1(s)=1$, $k_2(s)\neq 0$ and $k_3(s)\neq 0$. Then α is congruent to osculating curve of the first kind if and only if the curvature functions $k_2(s)$ and $k_3(s)$ satisfy differential equation*

$$\begin{aligned} 3k_3(s)k_3'(s) &= \frac{1}{\sqrt{1+k_3^2(s)}}(c-\int\frac{k_3(s)}{\sqrt{1+k_3^2(s)}}ds)[k_2(s)(1+k_3^2(s))^2\\ &\quad -3k_3(s)k_3'^2(s)+k_3''(s)(1+k_3^2(s))],\quad c\in\mathbb{R}. \end{aligned} \tag{5.6}$$

Proof. First assume that α is congruent to osculating curve of the first kind. By using theorem 5.2 and the third equation of the system of equations (5.4), we obtain that curvatures $k_2(s)$ and $k_3(s)$ satisfy relation (5.6).

Conversely, assume that curvature functions $k_2(s)$ and $k_3(s)$ satisfy relation (5.6). Let us consider the vector $X(s)\in\mathbb{E}_1^4$ given by

$$\begin{aligned} X(s) &= \alpha(s)+\frac{k_3(s)}{\sqrt{1+k_3^2(s)}}(c-\int\frac{k_3(s)}{\sqrt{1+k_3^2(s)}}ds)T(s)\\ &- \frac{1}{\sqrt{1+k_3^2(s)}}(c-\int\frac{k_3(s)}{\sqrt{1+k_3^2(s)}}ds)B_1(s)\\ &+ \frac{1}{1+k_3^2(s)}-\frac{k_3'(s)}{(1+k_3^2(s))^{\frac{3}{2}}}(c-\int\frac{k_3(s)}{\sqrt{1+k_3^2(s)}}ds)B_2(s),\quad c\in\mathbb{R}. \end{aligned}$$

By using relations (2.1) and (5.6), we find $X'(s) = 0$. Consequently, $X(s)$ is a constant vector in $\mathbb{E}_1^4$, which means that α is congruent to osculating curve of the first kind. This proves the theorem. □

6. Pseudo null osculating curves of the second kind in $\mathbb{E}_1^4$

Let α be pseudo null osculating curve of the second kind in $\mathbb{E}_1^4$, with curvatures $k_1(s) = 1$, $k_2(s) \neq 0$ and $k_3(s)$, where $k_3(s)$ can be equal to zero or different from zero (as in the section 5). By taking the derivative of relation (2.4) with respect to s and applying (2.1), we obtain the system of equations

$$\begin{aligned} &\lambda'(s) - \nu(s) = 1, \\ &\lambda(s) + \mu'(s) = 0, \\ &\mu(s)k_2(s) - \nu(s)k_3(s) = 0, \\ &\nu'(s) = 0. \end{aligned} \tag{6.1}$$

We may distinguish two cases: **(A)** $k_3(s) = 0$ and **(B)** $k_3(s) \neq 0$.

(A) $k_3(s) = 0$.

In this case, the system of equations (6.1) read

$$\begin{aligned} &\lambda(s) = 0, \\ &\mu(s) = 0, \\ &\nu(s) = -1, \\ &\nu'(s) = 0. \end{aligned} \tag{6.2}$$

Substituting relation (6.2) in relation (2.4), we find that position vector of pseudo null osculating curve α of the second kind satisfies the relation

$$\alpha(s) = -B_2(s),$$

which is equivalent to relation (5.3). In this way, the following theorem is obtained.

Theorem 6.1. *Every pseudo null osculating curve of the second kind with curvatures $k_1(s) = 1$, $k_2(s) \neq 0$ and $k_3(s) = 0$ is pseudo null osculating curve of the first kind and vice versa.*

Corollary 6.1. *Every pseudo null osculating curve of the first or second kind with curvatures $k_1(s) = 1$, $k_2(s) \neq 0$ and $k_3(s) = 0$ is pseudo null normal curve.*

(B) $k_3(s) \neq 0$.

In this case, the last equation of the system of equations (6.1) implies $\nu(s) \in \mathbb{R}$. If $\nu(s) = 0$, by using (6.1) we obtain a contradiction. Hence $\nu(s) = c_0 \in \mathbb{R}_0$. In particular, substituting $\nu(s) = c_0$ in (6.1) we find

$$\begin{aligned} &\lambda(s) = -c_0(k_3(s)/k_2(s))', \\ &\mu(s) = c_0 k_3(s)/k_2(s), \\ &\nu(s) = c_0, \\ &\lambda'(s) - c_0 = 1, \quad c_0 \in \mathbb{R}_0. \end{aligned} \tag{6.3}$$

Next, we may distinguish two cases:

(B.1) $k_3(s)/k_2(s) = constant$; **(B.2)** $k_3(s)/k_2(s) \neq constant$.

(B.1) $k_3(s)/k_2(s) = c_1 \in \mathbb{R}_0$.

Then relation (6.3) implies that the position vector can be written as

$$\alpha(s) = -c_1 N(s) - B_2(s), \quad c_1 \in \mathbb{R}_0. \tag{6.4}$$

In particular, the following theorem holds.

Theorem 6.2. *Let $\alpha(s)$ be unit speed pseudo null curve in $\mathbb{E}^4_1$ with curvatures $k_1(s) = 1$, $k_2(s) \neq 0$, $k_3(s) \neq 0$ and zero tangential component $g(\alpha, T)$ of the position vector. Then α is congruent to osculating curve of the second kind if and only if the ratio of its third and second curvature is a constant function*

$$k_3(s)/k_2(s) = c_1, \quad c_1 \in \mathbb{R}_0. \tag{6.5}$$

Proof. First assume that α is congruent to osculating curve of the second kind. Since $g(\alpha, T) = 0$, by using the first equation of relation (6.3), we obtain that relation (6.5) holds.

Conversely, assume that the third and second curvature satisfy relation (6.5). Let us consider the vector $X \in \mathbb{E}^4_1$ given by

$$X(s) = \alpha(s) + c_1 N(s) + B_2(s), \quad c_1 \in \mathbb{R}_0.$$

By taking the derivative of the last relation with respect to s and using (2.1), we find $X'(s) = 0$. Therefore, $X(s)$ is a constant vector, which means that α is congruent to osculating curve of the second kind. □

By using relation (6.3), we obtain the following relationship between pseudo null osculating curves of the second kind and pseudo null normal curves.

Theorem 6.3. *Every pseudo null osculating curve of the second kind in $\mathbb{E}^4_1$ with curvatures $k_1(s) = 1$, $k_2(s) \neq 0$, $k_3(s) \neq 0$ and $k_3(s)/k_2(s) = constant$ is pseudo null normal curve.*

Example 6.1. Let us consider pseudo null curve with the equation

$$\alpha(s) = \frac{3}{\sqrt{10}}(\frac{1}{9}\cosh(3s), \frac{1}{9}\sinh(3s), \sin(s), -\cos(s)).$$

The Frenet frame of α is given by

$$\begin{aligned} T(s) &= \frac{3}{\sqrt{10}}(\frac{1}{3}\sinh(3s), \frac{1}{3}\cosh(3s), \cos(s), \sin(s)), \\ N(s) &= \frac{3}{\sqrt{10}}(\cosh(3s), \sinh(3s), -\sin(s), \cos(s)), \\ B_1(s) &= \frac{1}{\sqrt{10}}(3\sinh(3s), 3\cosh(3s), -\cos(s), -\sin(s)), \\ B_2(s) &= \frac{5}{3\sqrt{10}}(-\cosh(3s), -\sinh(3s), -\sin(s), \cos(s)). \end{aligned}$$

The curvatures of α read

$$k_1(s) = 1, \quad k_2(s) = 3, \quad k_3(s) = \frac{4}{3}.$$

Since $k_3(s)/k_2(s) = 4/9$, relation (6.4) implies that the position vector satisfies the equation

$$\alpha(s) = -\frac{4}{9}N(s) - B_2(s).$$

By using the last equation, it can be easily verified that $g(\alpha(s), T(s)) = 0$. According to theorem 6.2, α is pseudo null osculating curve of the second kind.

(B.2) $k_3(s)/k_2(s) \neq$ *constant.*

Then relation (6.3) implies that the position vector of the curve α is given by

$$\alpha(s) = -c_0(k_3(s)/k_2(s))'T(s) + c_0(k_3(s)/k_2(s))N(s) + c_0 B_2(s), \quad c_0 \in \mathbb{R}_0.$$

In this case, we obtain the last theorem which can be proved in a similar way as theorem 6.2.

Theorem 6.4. *Let $\alpha(s)$ be unit speed pseudo null curve in $\mathbb{E}_1^4$ with curvatures $k_1(s) = 1$, $k_2(s) \neq 0$, $k_3(s) \neq 0$ and non-zero tangential component $g(\alpha, T)$ of the position vector. Then α is congruent to osculating curve of the second kind if and only if its third and second curvature satisfy the relation*

$$k_3(s)/k_2(s) = as^2 + bs + c,$$

where $a, b, c \in \mathbb{R}$ and a, b not both equal to zero.

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